

Vortex Loops and Majoranas

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We investigate the role that vortex loops play in characterizing eigenstates of interacting Majoranas. We give some general results and then focus on ladder Hamiltonian examples as a test of further ideas. Two methods yield exact results: i.) A mapping of certain spin Hamiltonians to quartic interactions of Majoranas shows that the spectra of these two examples coincide. ii) In cases with reflection-symmetric Hamiltonians, we use reflection positivity for Majoranas to characterize vortices in the ground states. Two additional methods suggest wider applicability of these results: iii.) Numerical evidence suggests similar behavior for certain systems without reflection symmetry. iv.) A perturbative analysis also suggests similar behavior without the assumption of reflection symmetry.

I. INTRODUCTION

I.1. Motivation

The spin systems we study have their origin in the “compass model” introduced by Kugel and Khomskii [25] to describe the Jahn-Teller effect in magnetic insulators. This model has a rich structure, and in a two-dimensional version, known as the honeycomb model, Kitaev found excitations with fractional statistics [24]. There is a long history of fractional-statistics excitations, now called anyons in modern theoretical studies. These arise from braid statistics, see for example [41, 17, 27, 42, 13].

In the honeycomb model (possibly with a magnetic field) both abelian and non-abelian anyons occur for different values of the coupling constants. There has also been extensive study of “ladders” with possibly anyonic excitations [6, 37, 18]. Models with anyons appear of interest in current studies of topological quantum computing. One argues that the degenerate ground-state subspace of such systems is a good place to store and to process quantum information, see for example [23, 22, 1, 9, 30, 10, 38, 35]. The advantage for storage is that the subspace of ground states is stable against a wide class of local perturbations. On the other hand, non-local perturbations (such as braiding of anyons) may implement quantum gates.

In this context, it is important to understand the properties of the ground states, for these states are candidates to encode quantum information. In particular, we are interested in the question whether vortices are present or absent in the ground states, which is intimately connected to our study of Majoranas. In particular for a large system, the location of the vortex should be independent of its energy, and therefore labels a degeneracy. Transitions between the different ground states could be counter-productive to storage of information.

We take advantage here of the equivalence of a quartic Hamiltonian of interest, to a family of Hamiltonians describing quadratic interactions of Majoranas. One naturally comes to the question: which Hamiltonian within the family has the lowest ground-state energy? The answer to this question characterizes the ground states of the original Hamiltonian. Besides being of fundamental interest in its own right, the question of determining which Hamiltonian has minimal ground-state energy turns out to be related to the presence or absence of vortices, and hence to the storage of information as outlined above. Although other methods might allow one to study the spin Hamiltonians directly, we have not investigated that possibility.

I.2. Goals and Results

In this paper we explore some general properties of a family of Majorana interactions on a cubic lattice in d dimensions. We focus on the question of how to identify which Hamiltonians within the family have minimal ground-state energy.

Our first approach is to use reflection positivity to study the ground states of a family of Hamiltonians that are quadratic in Majoranas, $\tilde{H}_u = \sum_{(ij)} J_{(ij)} u_{ij} i c_i c_j$. Here $J_{(ij)}$ is a constant with sign $u_{ij} = \pm 1$, and

c_i is a Majorana, in the notation of §II. Using reflection positivity for Majoranas [19], we give constraints on which Hamiltonians \tilde{H}_u have the lowest ground-state energy. We use these constraints to show that these ground states are vortex-free. While there are results in [28, 31] concerning reflection positivity and vortices for certain interactions of hopping charged particles, our results on Majoranas are new.

The Majorana interactions considered here arise naturally from mapping certain nearest-neighbor quadratic interactions of quantum spins ladders $H = -\sum_{(ij)} J_{(ij)} \sigma_i^{(ij)} \sigma_j^{(ij)}$ into quartic interactions of Majoranas $\tilde{H} = -\sum_{(ij)} J_{(ij)} i b_i^{(ij)} b_j^{(ij)} i c_i c_j$. Our notation for the spins is explained in §III. We use the map $H \mapsto \tilde{H}$ defined by $\sigma_j^\alpha \mapsto \tilde{\sigma}_j^\alpha = i b_j^\alpha c_j$, for Majoranas b_j^α and c_j as explained in §IV.

In §V, Theorem 6, we prove that for open boundary conditions, the spectra of these two Hamiltonians H and \tilde{H} are the same (except for multiplicity). On the other hand, using numerical methods we show that the spectra are different in the case of periodic boundary conditions. In spite of the fact that in general the eigenvalues of H and \tilde{H} do not coincide, numerical evidence suggests that the ground-state energies of these two Hamiltonians are the same. We use the relation between H and \tilde{H} and our results on \tilde{H}_u to characterize the vortex configuration in the ground states of H , when H is reflection-symmetric. We believe this is the first times these facts have been proved.

In the general case when reflection-symmetry of H is absent, we investigate the properties of the ground-state energy using numerical methods to compute low-lying energy levels. This investigation indicates that the ground state remains vortex free (with the relevant restriction on the sign of various couplings) even though our mathematical proof does not apply. Based on this information, we formulate a general *vortex-free ground-state conjecture* for ladders in §IX. We also study spin ladders by perturbation theory. We show that the ground state remains vortex free, and we incorporate these insights into the conjecture in §IX.

I.3. Organization of the Paper

In §II we define a family of Hamiltonians with nearest-neighbor Majorana interactions on a cubic lattice in arbitrary dimension. In this section we assume the existence of a reflection plane leaving the lattice invariant and transforming the Hamiltonians in a simple way. Using reflection-positivity one can characterize vortex loop configurations of the Hamiltonians that minimize the ground state energy within the given family. When all the coupling constants are positive (or negative), the minimal-energy is achieved for a vortex-free ground state. This property of vortex loops is related to results of Lieb [28], and of Macris and Nachtergaele [31] for hopping Hamiltonians.

In §III–§VI we apply these results to spin ladders and their Majorana fermionic representations. While in much of this paper we analyze ladders as an example, most of our results extend in a straightforward way to models defined on a honeycomb lattice with similar trivalent couplings at each site.

In §V we show that the spectrum of an open spin ladder coincides with the spectrum of its Majorana fermionic representation, aside from multiplicity. While the spectrum of a closed spin ladder seems not to have this property, we conjecture that the ground state energies are the same.

In §VII we study certain ladders numerically. These ladders do not possess the symmetry required to use reflection-positivity arguments. Numerical evidence suggests that the ground state energy of a closed spin ladder coincides with the ground state energy of its fermionic representation. Furthermore, the numerical calculations suggest that the ground states remain vortex-free (or vortex-full) as for the spin ladders for which reflection-positivity applies.

In §VIII we use third-order perturbation theory (the lowest non-trivial order) to complement the picture. These results also show that for certain regions of the coupling constants for non-symmetric, open and closed ladders, the ground states are vortex-free (or vortex-full).

II. NEAREST-NEIGHBOR MAJORANA INTERACTIONS ON A CUBIC LATTICE

II.1. The Cubic Lattice

We consider a finite subset Λ of the cubic lattice \mathbb{Z}^d in Euclidean d -space, with an even number $|\Lambda|$ of sites i . We assume Λ to be a rectangular box, with sites $i \in \mathbb{Z}^d$ and bonds (ij) connecting nearest-neighbor sites. The side length of the box along each coordinate axis may be different. We call this an *open* box. We

sometimes *close* the box in one or more coordinate directions. One closes the box in the k^{th} direction by defining sites with minimum and maximum value of the k^{th} coordinate, but the same value of each of the other coordinates, to be nearest-neighbors.

II.2. The Majoranas and The Hilbert Space

A set of Majoranas is a self-adjoint representation of an even-dimensional Clifford algebra,

$$\{c_i, c_j\} = 2\delta_{ij}, \quad \text{where } c_j = c_j^* = c_j^{-1}. \quad (\text{II.1})$$

We assign a Majorana c_j to each site j . Majoranas can be represented on a Fock-Hilbert space $\tilde{\mathcal{H}}_c$ of dimension $2^{|\Lambda|/2}$ and we use this representation. We consider the family of Hamiltonians

$$\tilde{H}_u = \sum_{(ij)} J_{(ij)} u_{ij} i c_i c_j, \quad (\text{II.2})$$

with $J_{(ij)} = J_{(ji)} \geq 0$ and $u_{ij} = -u_{ji} = \pm 1$. In case the subscripts are difficult to distinguish, we write $J_{(i,j)}$ in place of $J_{(ij)}$.

II.3. Vortex Loops

Define a loop \mathfrak{C} of length $|\mathfrak{C}| = \ell$ as an ordered sequence of nearest-neighbor sites $\{i_1, i_2, \dots, i_\ell, i_1\}$ in Λ , starting and ending at the same site. In addition we assume i_1, \dots, i_ℓ are distinct so the loop is not self-intersecting. We identify the loop with a closed, directed path connecting nearest-neighbor sites i_k and i_{k+1} by bonds (i_k, i_{k+1}) . Denote $-\mathfrak{C}$ as the reverse loop which contains the same sites as \mathfrak{C} but the opposite orientation, $\{i_1, i_\ell, i_{\ell-1}, \dots, i_2, i_1\}$. Let $\prod_{(ij) \in \mathfrak{C}} K_{ij}$ denote the ordered product around the loop,

$$\prod_{(ij) \in \mathfrak{C}} K_{ij} = K_{i_1 i_2} K_{i_2 i_3} \cdots K_{i_{\ell-1} i_\ell} K_{i_\ell i_1}. \quad (\text{II.3})$$

In the case where K_{ij} are matrices, the starting point of the loop is important, though the trace $\text{Tr} \left(\prod_{(ij) \in \mathfrak{C}} K_{ij} \right)$ is independent of the cyclic permutation of sites in the loop. The smallest loop contains four sites, which are the corner of a square or plaquette p bounded by the loop $\mathfrak{C} = \partial p$. Define a loop to be *non-degenerate* if the coupling constants on the loop do not vanish:

$$\mathfrak{C} \text{ is non-degenerate} \Leftrightarrow \prod_{(ij) \in \mathfrak{C}} J_{(ij)} \neq 0. \quad (\text{II.4})$$

Define the vortex loop $\tilde{\mathfrak{B}}(\mathfrak{C})$ as

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = - \prod_{(ij) \in \mathfrak{C}} u_{ij}. \quad (\text{II.5})$$

In case $\tilde{\mathfrak{B}}(\mathfrak{C}) = 1$ we say that the loop \mathfrak{C} is *vortex-free*. In case $\tilde{\mathfrak{B}}(\mathfrak{C}) = -1$ we say that \mathfrak{C} is *vortex-full*. We say that a state is vortex-free or vortex-full, in case all loops \mathfrak{C} are vortex-free or vortex-full. In case \mathfrak{C} bounds a surface, one can interpret the vortex configuration $\tilde{\mathfrak{B}}(\mathfrak{C})$ in terms of flux through the surface.

II.4. Fermionic Fock Representation

We represent the Hilbert space $\tilde{\mathcal{H}}_c$ as a fermionic Fock space generated by $|\Lambda|/2$ real creation operators a_μ^* and their adjoints a_μ are the corresponding annihilation operators. Here $\mu = 1, \dots, |\Lambda|/2$. Each creation-annihilation pair gives rise to two Majoranas

$$m_{\mu 1} = a_\mu + a_\mu^*, \quad \text{and} \quad m_{\mu 2} = i(a_\mu - a_\mu^*). \quad (\text{II.6})$$

II.5. The \mathbb{Z}_2 Gauge Group on $\tilde{\mathcal{H}}_c$

It is convenient to introduce the gauge group \mathfrak{G}^c that acts on $\tilde{\mathcal{H}}_c$. The generators of this group are the operators

$$U_j^c = c_j \mathcal{U}^c, \quad \text{where } \mathcal{U}^c = \prod_{j=1}^{4N} c_j. \quad (\text{II.7})$$

We later choose an order for the product \mathcal{U}^c , but conjugation by \mathcal{U}^c does not depend on the choice. The group \mathfrak{G}^c has dimension $2^{|\Lambda|+1}$, since $(U_j^c)^2 = -I$.

A general gauge transformation $W \in \mathfrak{G}^c$ on $\tilde{\mathcal{H}}_c$ depends upon $|\Lambda| + 1$ two-valued parameters $\mathbf{n} = \{n_0, n_1, \dots, n_{|\Lambda|}\}$. It has the form

$$W(\mathbf{n}) = (-1)^{n_0} (U_1^c)^{n_1} (U_2^c)^{n_2} \dots (U_{|\Lambda|}^c)^{n_{|\Lambda|}}, \quad (\text{II.8})$$

where $n_k = 0, 1$. Conjugation by the unitary $W(\mathbf{n})$ acts on the c_k 's as an automorphism that we also denote by $W(\mathbf{n})$. We write,

$$W(\mathbf{n})(c_k) = W(\mathbf{n}) c_k W(\mathbf{n})^* = (-1)^{n_k} c_k. \quad (\text{II.9})$$

As Osterwalder and Seiler pointed out in their original study of the Wilson action on a lattice [33], when one studies reflection positivity in gauge theory, a useful technique is to perform a unitary gauge transformation which removes the interaction terms across the reflection plane. Our Hamiltonians have a gauge symmetry as well, which allows one to fix the sign of the interactions across the reflection plane in our proof of Theorem 2, as it also was in [28].

II.6. Reflection-Symmetry

In certain sections we consider lattices that are symmetric under a reflection ϑ in a hyperplane Π , that intersects no lattice sites. The reflection defines two disjoint subsets of the lattice Λ_{\pm} of $\Lambda = \Lambda_- \cup \Lambda_+$ that map into each other,

$$\vartheta \Lambda_{\pm} = \Lambda_{\mp}, \quad \vartheta^2 = \text{Id}, \quad \text{with } \vartheta : i \mapsto \vartheta i. \quad (\text{II.10})$$

The reflection ϑ acts on loops as

$$\vartheta(\mathfrak{C}) = \vartheta(\{i_1, i_2, \dots, i_{\ell}, i_1\}) = \{\vartheta i_1, \vartheta i_2, \dots, \vartheta i_{\ell}, \vartheta i_1\}. \quad (\text{II.11})$$

We say that a loop \mathfrak{C} is reflection-symmetric under the action of ϑ , if $\vartheta(\mathfrak{C}) = -\mathfrak{C}$.

We represent ϑ on $\tilde{\mathcal{H}}_c$ as an *anti-unitary* transformation with

$$\vartheta(c_j) = \vartheta c_j \vartheta^{-1} = c_{\vartheta j}. \quad (\text{II.12})$$

The transformation ϑ defines an anti-linear automorphism of the algebra generated by the c_j 's, which we also denote by ϑ .

Definition 1. *The Hamiltonian \tilde{H}_u is reflection-symmetric if $\vartheta(\tilde{H}_u) = \tilde{H}_u$.*

II.7. The Fermionic Algebra on $\tilde{\mathcal{H}}_c$

Define the fermionic algebra \mathfrak{A}_c as the algebra generated by the c_j 's for $j \in \Lambda$. Let $\mathfrak{A}_c^{\text{even}}$ denote the even subalgebra of \mathfrak{A}_c , generated by even monomials in the fermionic operators. Similarly let $\mathfrak{A}_{c,\pm} \subset \mathfrak{A}_c$ denote the subalgebras generated by the c_j 's with $j \in \Lambda_{\pm}$. Also let $\mathfrak{A}_{c,\pm}^{\text{even}}$ denote the even subalgebras of $\mathfrak{A}_{c,\pm}$.

II.8. Reflection Positivity

Reflection positivity (RP) for Majoranas is a condition on a Hilbert space, an algebra of operators on the Hilbert space, a reflection ϑ through a plane Π , and a Hamiltonian. Here we study the Hilbert space $\tilde{\mathcal{H}}_c$, the algebras $\mathfrak{A}_{c,\pm}^{\text{even}}$, an implementation of the reflection ϑ on $\tilde{\mathcal{H}}_c$, and a reflection-symmetric Hamiltonian \tilde{H} . The RP condition states that

$$\text{Tr}_{\tilde{\mathcal{H}}_c} \left(B \vartheta(B) e^{-\tilde{H}} \right) \geq 0, \quad \text{for all } B \in \mathfrak{A}_{c,\pm}^{\text{even}}. \quad (\text{II.13})$$

Time-reflection positivity was originally discovered in quantum field theory by Osterwalder and Schrader in the context of relating classical fields with quantum fields [32]. In particular they introduced the method of ‘‘multiple reflection bounds,’’ involving iterated applications of a reflection-positivity bound. Such bounds have been key for the first mathematical proof of the existence of phase transitions (ground-state degeneracy) in quantum field theory [16], and in proving that certain field theories have infinite volume limits [14].

RP has also had many applications in the study of phase transitions for classical and quantum spin systems on a lattice; see Fröhlich, Simon, and Spencer [11], Dyson, Lieb, and Simon [7], and Fröhlich, Israel, Lieb, and Simon [12] for more details. In the context of nearest-neighbor hopping interactions, the vortex configuration of the ground state has been analyzed by Lieb [28] and Macris and Nachtergaele [31]. Recently one has shown that RP is also valid for a class of many-body Majorana interactions [19]; this family of interactions includes the two-body \tilde{H}_u in (II.2) with certain restrictions on the coupling constants J_{ij} .

II.9. Vortex Loops and Reflection Positivity

We study vortex loops $\tilde{\mathfrak{B}}(\mathfrak{C})$ for ground states of the family of Hamiltonians $\{\tilde{H}_u\}$ with ground state energies $\{\tilde{E}_0(u)\}$.

Theorem 2. *Let \tilde{H}_u denote a Hamiltonian of the form (II.2). Let \mathfrak{C} denote a non-degenerate, reflection-symmetric loop with respect to a reflection ϑ in the plane Π . Assume that the magnitudes of the couplings are reflection-symmetric, $|J_{(ij)}| = |J_{(\vartheta i \vartheta j)}|$. Then $\min_u \tilde{E}_0(u)$ is achieved for a ‘‘vortex-free’’ configuration of the u_{ij} ’s, namely*

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = 1. \quad (\text{II.14})$$

Proof. Consider a loop \mathfrak{C} of length $2L$ symmetrically crossed by the hyper-plane Π . This means that $\Lambda_{\pm} \cap \mathfrak{C}$ each contain L sites. Relabel the sites of \mathfrak{C} as $1, \dots, 2L$ so that the bonds in order on $\mathfrak{C} \cap \Lambda_-$ are $(i i + 1)$ with $i = 1, \dots, L - 1$. Similarly on $\mathfrak{C} \cap \Lambda_+$ the bonds are $(i i + 1)$ with $i = L + 1, \dots, 2L - 1$. Choose the starting point of \mathfrak{C} so that the bonds cutting Π are $(2L, 1)$ and $(L, L + 1)$.

Define $\Lambda_{\Pi\pm} \subset \Lambda_{\pm}$ as those sites in Λ_{\pm} that border Π . Decompose $\tilde{H}_u = \tilde{H}_{u,-} + \tilde{H}_{u,0} + \tilde{H}_{u,+}$ where $\tilde{H}_{u,\pm} \in \mathfrak{A}_{\pm}$ and

$$\tilde{H}_{u,0} = \sum_i J_{(i \vartheta i)} u_{i \vartheta i} i c_i \vartheta(c_i), \quad \text{with } i \in \Lambda_{\Pi-}. \quad (\text{II.15})$$

Perform a gauge transformation $W(\mathbf{n}) \in \mathfrak{G}^c$ of the form (II.8), with $n_i = 0$ except for $i \in \Lambda_{\Pi-}$. Choose n_i to ensure that the interactions in $\tilde{H}_{u,0} = W(\mathbf{n}) \tilde{H}_{u,0} W(\mathbf{n})^*$ across Π are positive, namely

$$J_{(i \vartheta i)} u_{i \vartheta i} (-1)^{n_i} > 0, \quad \text{for } i \in \Lambda_{\Pi-}. \quad (\text{II.16})$$

Also define the Hamiltonians $\tilde{H}_{u,1}$ and $\tilde{H}_{u,2}$ as

$$\tilde{H}_{u,1} = \tilde{H}_{u,-} + \tilde{H}_{u,0} + \vartheta(\tilde{H}_{u,-}), \quad \text{and} \quad \tilde{H}_{u,2} = \vartheta(\tilde{H}_{u,+}) + \tilde{H}_{u,0} + \tilde{H}_{u,+}, \quad (\text{II.17})$$

where

$$\tilde{H}_{u,-} = W(\mathbf{n}) \tilde{H}_{u,-} W(\mathbf{n})^*, \quad \text{and} \quad \tilde{H}_{u,+} = W(\mathbf{n}) \tilde{H}_{u,+} W(\mathbf{n})^*. \quad (\text{II.18})$$

Since $\vartheta(\tilde{H}_{u,0}) = \tilde{H}_{u,0}$, the Hamiltonians $\tilde{H}_{u,1}$ and $\tilde{H}_{u,2}$ are reflection-symmetric,

$$\vartheta(\tilde{H}_{u,1}) = \tilde{H}_{u,1}, \quad \text{and} \quad \vartheta(\tilde{H}_{u,2}) = \tilde{H}_{u,2}. \quad (\text{II.19})$$

Furthermore the coupling constants in $\tilde{H}_{u,0}$ that cross the reflection plane Π are positive.

The Hamiltonians $\tilde{H}_{u,1}$ and $\tilde{H}_{u,2}$ satisfy the hypothesis of Theorem 3 in [19]. In that paper one studies reflection-positivity for a class of interacting Majorana systems including the present one satisfying (II.16) and (II.19). From this result one concludes the reflection-positivity conditions. For $B \in \mathfrak{A}_{\pm}^{\text{even}}$,

$$\text{Tr}_{\tilde{\mathcal{H}}_c} \left(B \vartheta(B) e^{-\tilde{H}_{u,1}} \right) \geq 0, \quad \text{and} \quad \text{Tr}_{\tilde{\mathcal{H}}_c} \left(B \vartheta(B) e^{-\tilde{H}_{u,2}} \right) \geq 0. \quad (\text{II.20})$$

A direct consequence of the reflection-positivity conditions (II.20) is the reflection-positivity bound for any $\beta \geq 0$,

$$\text{Tr}_{\tilde{\mathcal{H}}_c} e^{-\beta \tilde{H}_u} \leq \left(\text{Tr}_{\tilde{\mathcal{H}}_c} e^{-\beta \tilde{H}_{u,1}} \right)^{1/2} \left(\text{Tr}_{\tilde{\mathcal{H}}_c} e^{-\beta \tilde{H}_{u,2}} \right)^{1/2}. \quad (\text{II.21})$$

This bound is a special case of the reflection-positivity inequality for interacting Majorana systems proved in Proposition 8 of [19]. The reflection-positivity bound (II.21) allows one to establish an inequality on the ground state energy $\tilde{E}_0(u)$ of the Hamiltonian \tilde{H}_u in terms of the ground-state energies $\tilde{E}_0(u,1)$ and $\tilde{E}_0(u,2)$ of the Hamiltonians $\tilde{H}_{u,1}$ and $\tilde{H}_{u,2}$, namely

$$0 \geq \tilde{E}_0(u) \geq \frac{\tilde{E}_0(u,1) + \tilde{E}_0(u,2)}{2}. \quad (\text{II.22})$$

Taking β large in (II.21) proves (II.22).

Conjugation by the gauge transformation $W(\mathbf{n})$ does not change the ground state energy $\tilde{E}_0(u)$ of \tilde{H}_u , so $\tilde{E}_0(u) = \tilde{E}_0(u)$. Nor does conjugation by the gauge transformation $W(\mathbf{n})$ change the value of any vortex loop $\tilde{B}(\mathfrak{C})$. Thus $\min_u \tilde{E}_0(u)$ is obtained from some configuration $u = u_0$ that is both reflection-symmetric and has positive interactions across Π . Call this Hamiltonian \tilde{H}_{u_0} .

Let $\tilde{H}_{u_0}(\mathfrak{C})$ denote the Hamiltonian that is the restriction of \tilde{H}_{u_0} to bonds $(ij) \in \mathfrak{C}$. Decompose $\tilde{H}_{u_0}(\mathfrak{C})$ as

$$\tilde{H}_{u_0}(\mathfrak{C}) = \tilde{H}_{u_0,-}(\mathfrak{C}) + \tilde{H}_{u_0,0}(\mathfrak{C}) + \tilde{H}_{u_0,+}(\mathfrak{C}), \quad (\text{II.23})$$

where positivity of the $J_{(ij)}$'s ensures

$$\begin{aligned} \tilde{H}_{u_0,-}(\mathfrak{C}) &= \sum_{i=1}^{L-1} J_{(i,i+1)} u_{i+1} i c_i c_{i+1}, \\ \tilde{H}_{u_0,+}(\mathfrak{C}) &= \sum_{i=L+1}^{2L-1} J_{(i,i+1)} u_{i+1} i c_i c_{i+1}, \\ \tilde{H}_{u_0,0}(\mathfrak{C}) &= J_{(1,2L)} u_{1,2L} i c_1 \vartheta(c_1) + J_{(L,L+1)} u_{L,L+1} i c_L \vartheta(c_L). \end{aligned} \quad (\text{II.24})$$

With our chosen representation

$$J_{(1,2L)} u_{1,2L} > 0, \quad \text{and} \quad J_{(L,L+1)} u_{L,L+1} > 0, \quad (\text{II.25})$$

and also reflection-symmetry $\vartheta(\tilde{H}_{u_0,-}(\mathfrak{C})) = \tilde{H}_{u_0,+}(\mathfrak{C})$ yields for $i = 1, \dots, L-1$,

$$\begin{aligned} J_{(i,i+1)} u_{i+1}, i c_{2L-i} c_{2L-i+1} \\ = J_{(2L-i, 2L-i+1)} u_{2L-i} c_{2L-i+1} i c_{2L-i} c_{2L-i+1}. \end{aligned} \quad (\text{II.26})$$

Consequently, since the loop \mathfrak{C} is non-degenerate, for $i = 1, \dots, L - 1$ one has

$$J_{(i, i+1)} J_{(2L-i, 2L-i+1)} u_{i i+1} u_{2L-i 2L-i+1} > 0. \quad (\text{II.27})$$

Multiply together conditions (II.25) with all the conditions (II.27), and identify site $2L + 1$ with site 1. One obtains

$$\mathfrak{B}(\mathfrak{C}) = - \prod_{(ij) \in \mathfrak{C}} u_{ij} = - \prod_{i=1}^{2L} u_{i i+1} = \text{sgn} \left(\prod_{i=1}^{2L} J_{(i, i+1)} \right) = 1. \quad (\text{II.28})$$

The first two equalities and the last equality in (II.28) are definitions, so one only needs to verify the third equality. There is one additional minus sign, which comes from $u_{1 2L} = -u_{2L 1}$, with the former appearing in (II.25) and the later in the product $\prod_{i=1}^{2L} u_{i i+1}$. This minus sign cancels the explicit minus sign in (II.28). \square

III. QUANTUM SPIN LADDERS

One way to realize the family of Hamiltonians \tilde{H}_u defined in (II.2) is to study nearest-neighbor spin interactions on a trivalent lattice. We consider the simplest example, the quantum spin ladder, corresponding to the case $d = 2$ in §II.

III.1. Even Spin Ladders

An open, even spin ladder is a $2 \times 2N$ square lattice array. The sites of the lattice are connected by bonds linking nearest-neighbor sites. We call one given plaquette the *unit cell* of the ladder. One obtains the lattice of the ladder as a union of N translates of the unit cell by integer multiples of twice the side-length of the unit cell, along one of its coordinate axes (which we choose horizontal). One completes the ladder with bonds (ij) that link site i with a nearest-neighbor site j .

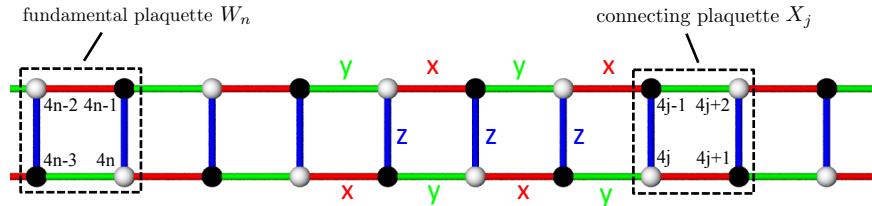


FIG. 1. Ladder

We illustrate such a ladder in Fig. 1, where we label the plaquettes, vertices, and bonds. Divide the $2N - 1$ plaquettes of the ladder into two sets: the first set comprises N *fundamental plaquettes* $p_1, p_3, \dots, p_{2k-1}, \dots, p_{2N-1}$ that are the translates of the unit cell that generates the ladder. The other set contains $(N - 1)$ *connecting plaquettes* $p_2, p_4, \dots, p_{2k}, \dots, p_{2N-2}$, each of which links two fundamental plaquettes, by sharing two of its bonds with two different fundamental plaquettes.

In order to discuss both “open” and “closed” ladders in a unified way, we introduce one additional connecting plaquette p_{2N} linking p_{2N-1} with p_1 , and two additional bonds to the open ladder, connecting the site $4N$ to the site 1, and connecting the site $4N - 1$ to the site 2. The closed ladder corresponds to periodic boundary conditions. Another way to characterize a closed ladder, is the property that one must remove at least four bonds to divide it into two disconnected pieces.

Label the sites in the fundamental plaquette p_{2n-1} clockwise, starting in the lower-left corner, by $4n - 3, 4n - 2, 4n - 1, 4n$. As a consequence, the sites in the connecting plaquette p_{2j} are labeled clockwise by $4j, 4j - 1, 4j + 2, 4j + 1$. The open ladders we consider have $(6N - 2)$ bonds, which we divide into three types. There are $(2N - 1)$ type- x bonds, $(2N - 1)$ type- y bonds, and $2N$ type- z bonds. All the vertical bonds will be type- z bonds. The horizontal bonds on top of each fundamental plaquette, and on the bottom of each connecting plaquette are type- x bonds. The remaining bonds are type- y bonds.

1. The Hamiltonian

The models we study here and other similar models arise frequently in the study of topological quantum information theory, see for instance [8, 34, 3, 39, 26, 6, 37, 18, 29]. The spins at each site $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ are Pauli matrices. Here i denotes the lattice site (using the labels above), and x, y, z denotes the three Pauli matrices. The Hamiltonian we study is a nearest-neighbor quadratic interaction of the form

$$H = - \sum_{(ij)} J_{(ij)} \sigma_i^{(ij)} \sigma_j^{(ij)}, \quad J_{(ij)} = J_{(ji)} \text{ real.} \quad (\text{III.1})$$

Here the sum over (ij) denotes a sum over unoriented bonds (ij) between nearest neighbor lattice sites in the ladder. Also $\sigma_i^{(ij)}$ equals σ_i^x , σ_i^y , or σ_i^z , according to whether the bond (ij) is type- x , type- y , or type- z , respectively, as defined above; thus the couplings labeled by a bond depends only on products of the same components of $\vec{\sigma}$ at different sites.

A simple case of this Hamiltonian which we call *homogeneous couplings* is the case for which every type- x bond has coupling J_x , every type- y bond has coupling J_y , and every type- z bond has coupling J_z . The open ladder Hamiltonian corresponds to taking the two coupling constants closing the ladder equal to zero, namely $J_{(4N1)} = J_{(4N-12)} = 0$.

III.2. Vortex Loops

For each loop \mathfrak{C} , we assign a *vortex-loop operator* (or simply a *vortex*) $\mathfrak{B}(\mathfrak{C})$. This is proportional to the *ordered* product along the loop of terms in the interaction. Recall that $\sigma_i^{(ij)} \sigma_j^{(ij)}$ is the term in the Hamiltonian (III.1) on the bond (ij) . Define

$$\mathfrak{B}(\mathfrak{C}) = i^{|\mathfrak{C}|+2} \prod_{(ij) \in \mathfrak{C}} \left(\sigma_i^{(ij)} \sigma_j^{(ij)} \right), \quad (\text{III.2})$$

similar to (5-6) in Kitaev [24]. As $\sigma_i^{(ij)}$ is self-adjoint with square I , we infer that $\mathfrak{B}(\mathfrak{C})$ is unitary. We devote the rest of this paper to the study of properties of the operators $\mathfrak{B}(\mathfrak{C})$.

IV. FERMIONIC LADDERS

IV.1. Mapping of Spins to Fermions

We use a representation of the Pauli matrices as quadratic expressions in Majoranas. Choose four Majoranas at each lattice site j and denote them b_j^x, b_j^y, b_j^z , and c_j . Define the algebra \mathfrak{A} as the algebra generated by the $b_j^{x,y,z}$ and c_j for $j \in \Lambda$. Let $\mathfrak{A}^{\text{even}}$ denote the even subalgebra of \mathfrak{A} , generated by even monomials in the fermionic operators. One defines for the single site j ,

$$\tilde{\sigma}_j^\alpha = i b_j^\alpha c_j, \quad (\text{IV.1})$$

as in the usual construction of boost operators for the Dirac equation, see for example §4c of [40]. For a single chiral component of the spin-1/2 Dirac wave function, the boost generator is isomorphic to the spin.

Denote the vector space of the four Majoranas as $\tilde{\mathcal{H}}_j$. In order to project onto a single chiral component, one restricts to the eigenspace $+1$ of the self-adjoint matrix $\gamma_j^5 = b_j^x b_j^y b_j^z c_j$ with square one. On the full Hilbert space \tilde{H} these γ_j^5 's mutually commute and commute with each $\tilde{\sigma}_j^{x,y,z}$. The corresponding orthogonal projection onto this eigenspace is $P = \prod_j P_j = \prod_j \frac{1}{2} (I + \gamma_j^5)$, and it yields $\mathcal{H} = P\tilde{\mathcal{H}}$ of dimension 2^{4N} . The $\sigma_j^\alpha = P_j \tilde{\sigma}_j^\alpha P_j$'s satisfy the correct multiplication laws for spin matrices on \mathcal{H} . We call \mathcal{H} the *spin-ladder Hilbert space*.

IV.2. Representation of the Hamiltonian

Introduce the three skew $4N \times 4N$ matrices u , A , and C with entries that are hermitian operators,

$$u_{ij} = -u_{ji} = u_{ij}^*, \quad A_{ij} = -A_{ji} = A_{ij}^*, \quad \text{and} \quad C_{ij} = -C_{ji} = C_{ij}^*. \quad (\text{IV.2})$$

We define these matrix elements to vanish unless i, j are nearest-neighbors. In this case

$$u_{ij} = i b_i^{(ij)} b_j^{(ij)}, \quad C_{ij} = i c_i c_j, \quad \text{and} \quad A_{ij} = J_{(ij)} u_{ij}, \quad (\text{IV.3})$$

with $J_{(ij)} = J_{(ji)}$ real. A representation of the spin-ladder Hamiltonian on the fermionic Hilbert space is

$$\tilde{H} = \sum_{(ij)} A_{ij} C_{ij} = \sum_{(ij)} J_{(ij)} u_{ij} i c_i c_j = \tilde{H}^*. \quad (\text{IV.4})$$

The u_{ij} operators mutually commute, and they also commute with the Hamiltonian \tilde{H} . They satisfy $u_{ij}^2 = +1$, so the eigenvalues of u_{ij} are ± 1 . Also all the γ_j^5 commute with \tilde{H} . Furthermore the Hamiltonian \tilde{H} commutes with P , so it maps the subspace \mathcal{H} into itself and on this subspace the Hamiltonian has the representation as a sum of self-adjoint operators,

$$H = P \tilde{H} P = \sum_{(ij)} P A_{ij} C_{ij} P. \quad (\text{IV.5})$$

The properties of \tilde{H} on $\tilde{\mathcal{H}}$ are different from those of H on \mathcal{H} , and in particular the eigenvalues might differ (aside from multiplicity). One should be careful not to jump to conclusions, as we give numerical evidence for the existence of both types of behavior in §VII, see also [36].

IV.3. Representation of the Vortices

A fermionic representation of $\tilde{\mathfrak{B}}(\mathfrak{C})$ of $\mathfrak{B}(\mathfrak{C})$ commutes with the projection P . Its projection $P \tilde{\mathfrak{B}}(\mathfrak{C}) P$, agrees with the original definition (III.2) of the vortex $\mathfrak{B}(\mathfrak{C})$. We give such a fermionic representation $\tilde{\mathfrak{B}}(\mathfrak{C})$, similar to [24] and observe that the spin vortices $\mathfrak{B}(\mathfrak{C})$ are mutually commuting, conserved quantities.

Proposition 3. *A fermionic representation of the vortex loop-operator is given in terms of the mutually-commuting operators u_{ij} as*

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = - \prod_{(ij) \in \mathfrak{C}} u_{ij}. \quad (\text{IV.6})$$

Each γ_k^5 commutes with $\tilde{\mathfrak{B}}(\mathfrak{C})$, namely

$$\left[\tilde{\mathfrak{B}}(\mathfrak{C}), \gamma_k^5 \right] = 0. \quad (\text{IV.7})$$

Proof. The contribution to the vortex loop-operator $\mathfrak{B}(\mathfrak{C})$ from the spins at site i_j , for $j \neq 1$, is $\sigma_{i_j}^{(i_{j-1} i_j)} \sigma_{i_j}^{(i_j i_{j+1})}$. (In case $j = \ell$, set $\ell + 1 = 1$.) This product has the fermionic representation $-b_{i_j}^{(i_{j-1} i_j)} c_j b_{i_j}^{(i_j i_{j+1})} c_j = b_{i_j}^{(i_{j-1} i_j)} b_{i_j}^{(i_j i_{j+1})}$. Taking the product of these representations and adding the contribution from the spins at site i_1 , one has a fermionic representation for $\mathfrak{B}(\mathfrak{C})$ defined in (III.2) equal to

$$\begin{aligned} \tilde{\mathfrak{B}}(\mathfrak{C}) &= -i^{|\mathfrak{C}|+2} b_{i_1}^{(i_1 i_2)} c_{i_1} b_{i_2}^{(i_1 i_2)} b_{i_2}^{(i_2 i_3)} b_{i_3}^{(i_2 i_3)} b_{i_3}^{(i_3 i_4)} \dots \\ &\quad b_{i_{\ell-1}}^{(i_{\ell-2} i_{\ell-1})} b_{i_{\ell-1}}^{(i_{\ell-1} i_{\ell})} b_{i_{\ell}}^{(i_{\ell-1} i_{\ell})} b_{i_{\ell}}^{(i_{\ell} i_1)} b_{i_1}^{(i_{\ell} i_1)} c_{i_1} \\ &= i^{|\mathfrak{C}|+2} c_{i_1} b_{i_1}^{(i_1 i_2)} b_{i_2}^{(i_1 i_2)} b_{i_2}^{(i_2 i_3)} b_{i_3}^{(i_2 i_3)} b_{i_3}^{(i_3 i_4)} \dots \\ &\quad b_{i_{\ell-1}}^{(i_{\ell-2} i_{\ell-1})} b_{i_{\ell-1}}^{(i_{\ell-1} i_{\ell})} b_{i_{\ell}}^{(i_{\ell-1} i_{\ell})} b_{i_{\ell}}^{(i_{\ell} i_1)} b_{i_1}^{(i_{\ell} i_1)} c_{i_1} \\ &= -c_{i_1} u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_{\ell-1} i_{\ell}} u_{i_{\ell} i_1} c_{i_1} \\ &= -u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_{\ell-1} i_{\ell}} u_{i_{\ell} i_1}. \end{aligned} \quad (\text{IV.8})$$

In the last equality we use the fact that c_{i_1} commutes with all the u_{ij} 's. This establishes the fermionic representation (IV.6). As each u_{ij} is hermitian and the u_{ij} mutually commute, we infer that $\tilde{\mathfrak{B}}(\mathfrak{C})$ is hermitian. Since $\tilde{\mathfrak{B}}(\mathfrak{C})$ is a product of b Majoranas, with an even number of b 's at each site $i_j \in \mathfrak{C}$, we infer that $\tilde{\mathfrak{B}}(\mathfrak{C})$ commutes with each γ_j^5 . Therefore $\tilde{\mathfrak{B}}(\mathfrak{C})$ commutes with P . \square

From the representation (IV.6) for $\tilde{\mathfrak{B}}(\mathfrak{C})$ and the representation (IV.4) for \tilde{H} in terms of the mutually-commuting, self-adjoint operators u_{ij} with square one, one infers the following two corollaries:

Corollary 4. *The fermionic vortex representatives $\tilde{\mathfrak{B}}(\mathfrak{C})$ are all self-adjoint and have eigenvalues ± 1 . Different $\tilde{\mathfrak{B}}(\mathfrak{C})$ mutually commute,*

$$\left[\tilde{\mathfrak{B}}(\mathfrak{C}), \tilde{\mathfrak{B}}(\mathfrak{C}') \right] = 0. \quad (\text{IV.9})$$

All the $\tilde{\mathfrak{B}}(\mathfrak{C})$ are conserved by \tilde{H} , namely

$$\left[\tilde{\mathfrak{B}}(\mathfrak{C}), \tilde{H} \right] = 0. \quad (\text{IV.10})$$

Corollary 5. *The vortex loop operators $\mathfrak{B}(\mathfrak{C})$ are self-adjoint on \mathcal{H} , and have eigenvalues ± 1 . Different $\mathfrak{B}(\mathfrak{C})$ mutually commute,*

$$[\mathfrak{B}(\mathfrak{C}), \mathfrak{B}(\mathfrak{C}')] = 0. \quad (\text{IV.11})$$

The vortex loop-operators are all conserved, namely

$$[\mathfrak{B}(\mathfrak{C}), H] = 0. \quad (\text{IV.12})$$

IV.4. The Reduced Fermionic Hamiltonians

Define \tilde{H}_u as the Hamiltonian \tilde{H} restricted to an eigenspace of the u_{ij} 's. Therefore it is useful to represent the Hilbert space $\tilde{\mathcal{H}}$ in the form of a tensor product

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_u \otimes \tilde{\mathcal{H}}_c. \quad (\text{IV.13})$$

Here we consider the $6N$ mutually commuting variables u_{ij} corresponding to the products of $ib_i^{(ij)}b_j^{(ij)}$ on the $6N$ bonds of a closed ladder. In the case of an open ladder the couplings on the two extra bonds (1, $4N$) and (2, $4N - 1$) are zero. Each u_{ij} is self-adjoint and has square equal to one, so it can be represented on a two-dimensional Hilbert space. Therefore the Hilbert space \tilde{H}_u has dimension 2^{6N} , which is exactly $2^{\#_b/2}$, where $\#_b$ equals the total number of $b^{x,y,z}$ Majoranas. These Majoranas can be represented on a Hilbert space of the same dimension 2^{6N} .

Define the fermionic algebra \mathfrak{A}_c as the subalgebra of \mathfrak{A} generated by the c_j -Majoranas. Since this algebra commutes with all the u_{ij} 's, it acts as $I \otimes \mathfrak{A}_c$ on $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_u \otimes \tilde{\mathcal{H}}_c$.

V. EIGENVALUES OF \tilde{H} AND OF H

Let \tilde{E}_0 denote the ground-state energy of \tilde{H} given in (IV.4), and let E_0 denote the ground-state energy of H . We are interested to know when these two ground state energies coincide. By the variational principle, there is a normalized vector $\tilde{\Omega} \in \tilde{\mathcal{H}}$, such that

$$\tilde{E}_0 = \langle \tilde{\Omega}, \tilde{H}\tilde{\Omega} \rangle = \inf_{\|\tilde{\chi}\|=1} \langle \tilde{\chi}, \tilde{H}\tilde{\chi} \rangle \leq E_0.$$

One obtains E_0 by restricting $\tilde{\chi}$ to the range of P . So if $P\tilde{\Omega} = \tilde{\Omega}$, then $\tilde{E}_0 = E_0$. More generally, we investigate the eigenvalues of \tilde{H} , and determine in certain cases that they are the same as the eigenvalues of H . In other cases there is evidence that they are different.

For an open ladder, we prove in Theorem 6 that \tilde{H} and H have the same eigenvalues. We analyze the ground state of H using the fermionic representation and demonstrate that the ground state is vortex-free. In reflection symmetric cases we do this in §VI.1 using reflection positivity. In §VIII we analyze some non-reflection symmetric cases using perturbation theory.

In Proposition 7 we explain why the proof of Theorem 6 for the open ladder does not apply to the closed ladder. More to the point, numerical calculation shows that the spectra are really different, see the discussion in §VII and in particular in §VII.2.

Theorem 6. *Consider an open ladder. The eigenvalues of H defined in (III.1) are the same as those of \tilde{H} defined in (IV.4), aside from multiplicity.*

Proof. The operators γ_i^5 mutually commute and commute with \tilde{H} , so we can simultaneously diagonalize them. We find an operator Q_j with square ± 1 , which anti-commutes with γ_j^5 and commutes with \tilde{H} and γ_i^5 , for $i \neq j$. Let $\tilde{\Omega}$ be a simultaneous eigenstate of the γ_i^5 and \tilde{H} with eigenvalues (μ_i, \tilde{E}) , where $\mu_i = \pm 1$ are the eigenvalues of the γ_i^5 . Then the vector $Q_j \tilde{\Omega}$ is an eigenstate with the same eigenvalues except for the one μ_i with $i = j$, that has the opposite sign. (Note $Q_j \tilde{\Omega} \neq 0$, as $Q_j^2 = \pm 1$.) By applying Q_j for each negative μ_j , we obtain a simultaneous eigenstate with energy \tilde{E} , and with all the $\mu_i = +1$. Calling this vector $\tilde{\Omega}'$, the projected state $P\tilde{\Omega}' = \Omega'$ is an eigenstate of H with eigenvalue \tilde{E} . This also shows that to each eigenvalue E of H is associated 2^{4N} eigenvalues of \tilde{H} , of which all but one of the corresponding eigenvectors project to zero.

Define the operator Q_j by considering a non-self-intersecting path Γ through the ladder from site j to site $4N$. The operator Q_j equals the product of the $u_{i'j'}$ operators along the bonds $(i'j')$ on this path, followed by b_{4N}^x . This Q_j is a product of b operators, so its square is ± 1 . The operator b_{4N}^x does not enter the expression (IV.4) for \tilde{H} , and each term in \tilde{H} is a product of an even number of other fermion operators. Therefore Q_j commutes with \tilde{H} .

To complete the proof, we need to check the commutativity of Q_j with the operators γ_i^5 . Consider four cases: first suppose the path Γ does not pass through i . Then γ_i^5 commutes with each b belonging to Q_j , so it commutes with Q_j .

Second suppose that i is a site on the path Γ , but $i \neq j$ and $i \neq 4N$. In this case the site i contributes a product of two different b_i operators to Q_j ; this is the case, because in the ladders we consider, the three bonds ending at site i are of three different types, and the path Γ contains two of these bonds. Each of these two b_i 's anti-commutes with γ_i^5 , so their product commutes. Also γ_i^5 commutes with b_k 's at other sites, so it commutes with Q_j .

The third case is $i = 4N$. As before, γ_{4N}^5 commutes with the b 's at sites different from $4N$. Only one bond in Γ ends at site $4N$, so only one b_{4N} at site $4N$ arises from the path; for our ladders, this must be either b_{4N}^y or b_{4N}^z . But Q_j also includes the extra b_{4N}^x . So γ_{4N}^5 anti-commutes with this extra b_{4N}^x and therefore commutes with the product of the two distinct b_{4N} 's that occur in Q_j .

The fourth case is $i = j$. In this case only one bond in Γ enters site i , so only one b_i occurs in Q_j . Hence γ_j^5 anti-commutes with the b_j 's in Q_j . As γ_j^5 commutes with the b 's at other sites, γ_j^5 anti-commutes with Q_j . These cases cover all possibilities, so we have established all the desired properties of the operators Q_j . \square

We remark that an alternate proof could be based on the explicit form of the projection $P : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$ as a function of the variables u_{ij} derived in Appendix A of [36]. We now show that the proof of Theorem 6 does not extend in a straightforward way to the closed ladder. This is in line with the numerical calculations we perform in §VII suggesting that the spectrum of H is different from the spectrum of \tilde{H} for the closed ladder.

Proposition 7. *Consider a closed ladder Hamiltonian \tilde{H} of the form (IV.4) with all couplings $J_{(ij)}$ different from 0. There is no non-zero monomial Q_j in the b 's and c 's that anti-commutes with γ_j^5 and commutes with \tilde{H} and γ_k^5 for $k \neq j$.*

Proof. Each site k in the ladder gives rise to a 4-dimensional Hilbert space $\tilde{\mathcal{H}}_k$. There are 16 linearly-independent operators on $\tilde{\mathcal{H}}_k$, and this space is spanned by monomials M_k^α in the $b_k^{x,y,z}$ or c_k of degree 4 or less. Of these, four monomials that we denote $m_k^{1,\dots,4}$ are the Majoranas themselves and have degree 1, and four others $m_k^{1,\dots,4} \gamma_k^5$ have degree 3. We write these eight odd degree monomials as $M_k^{-,\alpha}$. Each $M_j^{-,\alpha}$ anti-commutes with γ_j^5 and commutes with γ_k^5 for $k \neq j$.

There are eight monomials $M_k^{+, \alpha}$ of degree 0, 2, or 4, and these commute with all the γ_j^5 . All 16 of the $M_k^{\pm, \alpha}$ commute with $\gamma_{k'}^5$ for $k' \neq k$. The monomials in the b 's and c 's are linearly independent and span the operators on $\tilde{\mathcal{H}}$, as shown in Proposition 1 of [19]. From these properties, we infer that

$$Q_j = \pm M_j^{-, \alpha} \prod_{k \neq j} M_k^{+, \alpha_k}. \quad (\text{V.1})$$

We now consider further restrictions on Q_j , imposed by the fact that one wants $[Q_j, \tilde{H}] = 0$. We show this is impossible for Q_j of form (V.1). These restrictions use the assumption that all $J_{(ij)} \neq 0$, so they do not apply in the case of an open ladder.

Let us denote the interaction on bond (ji) by $\langle ji \rangle$, so the Hamiltonian (IV.4) can be written

$$\tilde{H} = \sum_{(ji)} \langle ji \rangle, \quad \text{where} \quad \langle ji \rangle = J_{(ji)} u_{ji} i c_j c_i = -J_{(ji)} b_j^{(ji)} b_i^{(ji)} c_j c_i. \quad (\text{V.2})$$

We claim that

- I. $M_j^{-, \alpha}$ anti-commutes with either one or three terms in the sum (V.2).
- II. $\prod_{k \neq j}^3 M_k^{+, \alpha_k}$ anti-commutes with an even number of terms in (V.2).

These two properties show that Q_j of the form (V.1) cannot commute with \tilde{H} .

In order to establish property (I), notice that a single Majorana c_j anti-commutes with three terms $\langle ji \rangle$ in the sum (V.2), where i are the three nearest neighbors to j . Also the Majorana $b_j^{x,y,z}$ anti-commutes with one such term. As γ_j^5 commutes with $\langle ji \rangle$, the same anti-commutativity properties hold for $m_j^{1, \dots, 4} \gamma_j^5$ as for $m_j^{1, \dots, 4}$.

Property (II) also follows by considering the anti-commutation properties of the eight possible M_k^{+, α_k} . The identity and monomial of degree 4 commute with each $\langle ji \rangle$. The monomials M_k^{+, α_k} of degree 2 all anti-commute with two of the $\langle ji \rangle$'s. The statement then follows. \square

VI. LADDER HAMILTONIANS AND REFLECTIONS

In the following we consider ladder Hamiltonians \tilde{H} of the form (IV.4) with reflection-symmetric absolute value of the couplings $J_{(ij)}$, namely

$$|J_{(\vartheta i \vartheta j)}| = |J_{(ij)}|. \quad (\text{VI.3})$$

We determine the value of reflection-symmetric vortex loops in the ground states of \tilde{H} and H for such couplings.

The open or closed ladder in Fig. 1 satisfies (VI.3) in three cases:

Case I. Reflection through a horizontal plane, see Fig. 2. We make no restriction on the couplings $J_{(i+1)}$ on vertical bonds.

Case II. Vertical reflection plane bisecting an open ladder, see Fig. 3.

Case III. Reflection through any vertical plane bisecting a closed ladder, see Fig. 4. The dotted reflection plane intersects the ladder twice.

VI.1. Vortex Loops and Reflection-Symmetric Ladders

In this section we apply Theorem 2 to characterize the vortex configurations of the ground-state of \tilde{H} , for ladders satisfying conditions (VI.3).

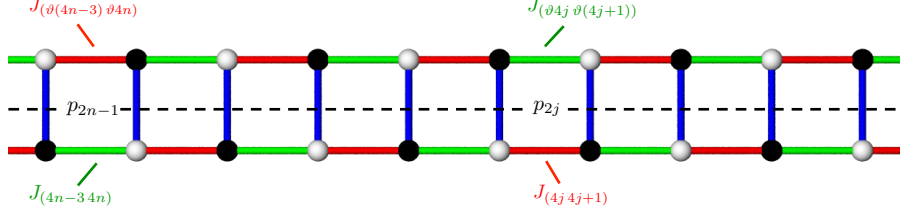


FIG. 2. Case I: Horizontal reflection plane for an open or closed ladder.

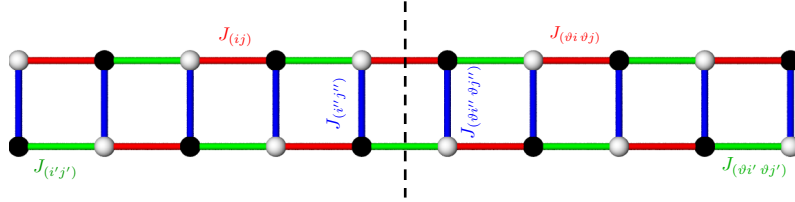
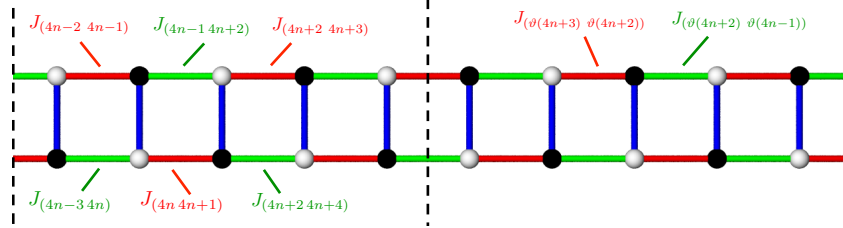
FIG. 3. Case II: Vertical reflection plane. Recall that $J_{(ij)} = J_{(ji)}$.

FIG. 4. Case III: Vertical reflection of a closed ladder.

Theorem 8. Let \tilde{H} denote a fermionic ladder Hamiltonian of the form (IV.4) satisfying condition (VI.3) with respect to a reflection ϑ and a reflection plane Π . Let \mathfrak{C} denote a non-degenerate, reflection-symmetric loop. Then the vortex configuration of \mathfrak{C} in the ground state of \tilde{H} is

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = \text{sgn} \left(\prod_{(ij) \in \mathfrak{C}} J_{(ij)} \right). \quad (\text{VI.4})$$

In case the couplings have all the same sign, then $\tilde{\mathfrak{B}}(\mathfrak{C}) = +1$ and the loop \mathfrak{C} is vortex-free.

Corollary 9 (Vortex-free ladders). For closed ladders with homogeneous couplings which all have the same sign, every loop \mathfrak{C} is vortex-free in the ground state of \tilde{H} .

Proof. Each Hamiltonian \tilde{H} acting on $\tilde{\mathcal{H}}$ corresponds to 2^{6N} Hamiltonians \tilde{H}_u acting on $\tilde{\mathcal{H}}_{c_2}$, some of which could be the same. Each \tilde{H}_u arises from a particular choice of $u_{ij} = \pm 1$. The eigenvalues of \tilde{H} are the union of the eigenvalues of these 2^{6N} Hamiltonians \tilde{H}_u . A gauge transformation of the variables $b_j^{x,y,z}$ transforms one \tilde{H}_u into another $\tilde{H}_{u'}$. This justifies our present study of the individual Hamiltonians \tilde{H}_u .

Each Hamiltonian \tilde{H}_u is of the form (II.2), although the couplings $J_{(ij)}$ may not be positive. In case all the $J_{(ij)} > 0$, we infer from Theorem 2 that the minimum energy of \tilde{H} is achieved for a \tilde{H}_u with a configuration of the u_{ij} 's such that

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = - \prod_{(ij) \in \mathfrak{C}} u_{ij} = 1, \quad (\text{VI.5})$$

for any loop \mathfrak{C} that is reflection-symmetric. Changing the sign of $J_{(ij)}$ with $(ij) \in \mathfrak{C}$ is equivalent to changing the sign of the corresponding u_{ij} , so one infers from (VI.5) that

$$\tilde{\mathfrak{B}}(\mathfrak{C}) = - \prod_{(ij) \in \mathfrak{C}} u_{ij} = \text{sgn} \left(\prod_{(ij) \in \mathfrak{C}} J_{(ij)} \right). \quad (\text{VI.6})$$

This completes the proof of the proposition. The corollary follows as every plaquette in the ladder is reflection-symmetric and hence vortex free, and the same then follows for the loop \mathfrak{C} . \square

VI.2. Implications for Reflection-Symmetric Spin Ladders

For open ladders, we know that the ground-state energies of \tilde{H} and H agree, as shown in Theorem 6. We also know that the projection P commutes with all the vortex operators, see Proposition 3. On the other hand, in the case of a closed ladder we do not know whether the spectra coincide, and in particular whether the ground-state energies are the same. We have shown the following:

Theorem 10. *The ground-state of the Hamiltonian H for an open spin ladder satisfying condition (VI.3) with respect to a reflection plane Π has the vortex configuration*

$$\mathfrak{B}(\mathfrak{C}) = \text{sgn} \left(\prod_{(ij) \in \mathfrak{C}} J_{(ij)} \right), \quad (\text{VI.7})$$

in each non-degenerate, reflection-symmetric loop \mathfrak{C} that crosses Π . In case the couplings have all the same sign, the ground-state is vortex-free in those loops.

VII. NUMERICAL EVIDENCE

In this section we give some numerical evidence for the spectral properties of H and \tilde{H} , both in the case of open and of closed ladders. We have shown in Theorem 6 that the spectra of H and \tilde{H} are identical for an open ladder. However this is not true for a closed ladder. Even a simple closed ladder with $N = 2$ (four plaquettes) shows by explicit numerical diagonalization that \tilde{H} has eigenvalues not present in the spectrum of H , see §VII.2. For this Hamiltonian, we plot the energies and show the vortex configurations for a number of eigenvalues.

We inspect the low-lying spectrum of the Hamiltonians H and \tilde{H} for a number of ladders of length N , in case that N is as large as 100, so with up to 400 spins and 1,600 Majoranas. We use Mathematica 8.0.4.0 and Matlab 7.10.0.499 (R2010a). In order to find which eigenvalues of \tilde{H} are eigenvalues of H , we use the method introduced in [36].

Our numerical analysis suggests that the ground state of \tilde{H} and also the ground state of H is vortex free, whether or not they have the symmetry (VI.3), leading to the conjecture at the end of the section.

VII.1. Open Ladders

We first analyze an open ladder with $N = 2$ (three plaquettes). In Fig. 5 we plot the low-lying eigenvalues of both H and \tilde{H} . We have chosen the couplings $J_z = 2$, and the x and y couplings to decrease from left to right on the top of the ladder, but not on the bottom. These couplings are neither reflection-symmetric nor homogeneous. The plaquettes are labeled p_n with $n = 1, 2, 3$ from left to right as illustrated in Fig. 5. The numerical eigenvalues of H and \tilde{H} agree, as we already have shown in Theorem 6. It is interesting that the one-vortex configurations yield the first excited states (aside from multiplicity) and the placement on the ladder of the vortex that creates the minimal-energy excitation corresponds to the configuration of coupling constants that one intuitively expects.

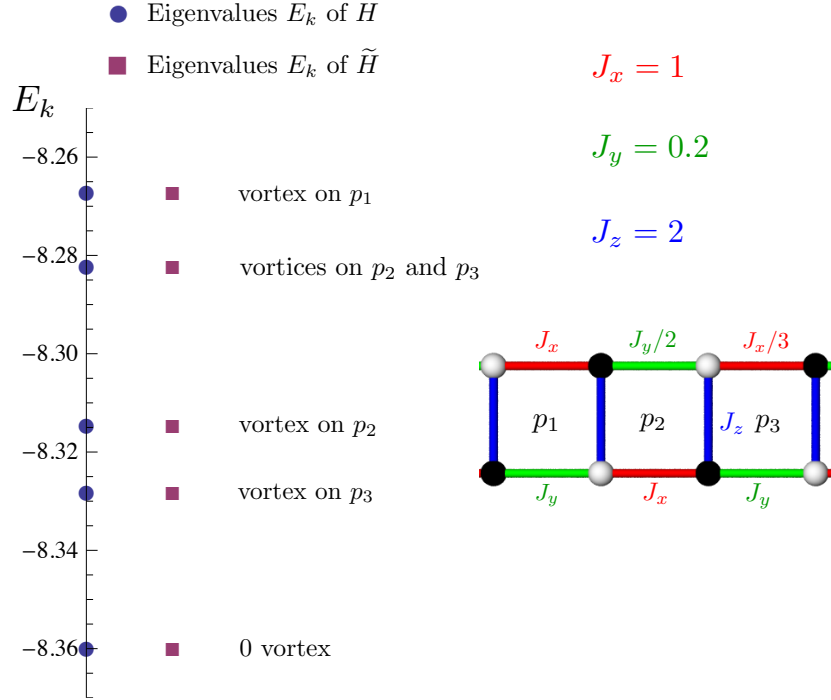


FIG. 5. Low-lying eigenvalues of H and \tilde{H} for an open ladder with the illustrated couplings. We plot eigenvalues of H with circles and those of \tilde{H} with squares, and we ignore multiplicities. Other couplings yield qualitatively similar plots.

1. Hamiltonians for Open Ladders of Length N

Next we consider a sequence of Hamiltonians \tilde{H} for open ladders with variable length N . We choose non-homogeneous couplings that decay on the upper rungs of the ladder from $2J_x$ and $3J_y/2$ on the left, to $J_x + J_x/(2N - 1)$ and $J_y + J_y/(2N - 2)$ on the right. On the bottom rungs we take homogeneous couplings. We plot the case $J_x = 1$, $J_y = 0.2$, and $J_z = 2$, as illustrated in Fig. 6. We find that the ground state energy corresponds to a vortex-free configuration. We then consider the minimal energy excitation above the ground state (neglecting multiplicity).

Among the configuration we have tested, the minimal energy excitation above the vortex-free configuration appears to occur with a single vortex on a plaquette p_{2N-j} for small j . The effect of the boundary of the ladder at plaquette p_{2N-1} seems to raise slightly the energy of the single vortex in that plaquette, as illustrated in two curves labeled by p_{2N-2} and p_{2N-1} . We have computed other single-vortex excitations that confirm this picture.

We also plot the excitation energy of a configuration with two vortices on plaquettes p_{2N-2} and p_{2N-1} . This is approximately twice the energy of a single vortex.

VII.2. Closed Ladders

We present numerical evidence for several closed ladders, and contrast the results with the case of the open ladders. In spite of the fact that we observe numerically that \tilde{H} and H have different spectra, the ground-state energy of \tilde{H} coincides with the ground-state energy of H and the ground-state vortex-loop configuration is vortex-free.

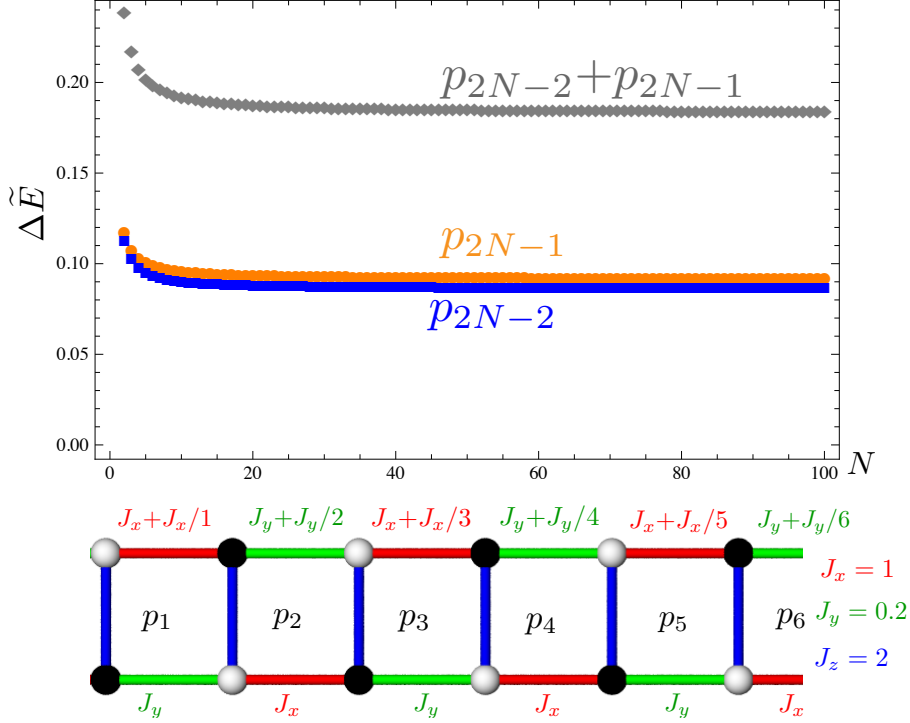


FIG. 6. We plot the lowest energies of excitation for an open ladder of length N , with a vortex on plaquette p_{2N-2} , on plaquette p_{2N-1} , or on both. The choice of the couplings is shown in the figure, and the vortex configurations are explained in the text.

1. Hamiltonians \tilde{H} and H for Closed Ladders of Length $N = 2$

We first analyze the $N = 2$ ladder with couplings of the same sort as in Fig. 5, but with non-zero couplings on the bonds closing the ladder, as illustrated in Fig. 7. We plot the low-lying eigenvalues of H and \tilde{H} , aside from multiplicity. We label the eigenvalues we plot by their vortex loop configuration.

2. Hamiltonians \tilde{H} and H for Closed Ladders of Length N

Here we consider the two smallest excitations above the ground state of the Hamiltonians \tilde{H} and H for ladders of variable length $2 \leq N \leq 100$. We choose non-homogeneous couplings that decay on the upper rungs of the ladder from $2J_x$ and $3J_y/2$ on the left, to $J_x + J_x/(2N - 1)$ and $J_y + J_y/(2N)$ on the right. On the bottom rungs we take homogeneous couplings. We plot the case $J_x = 1$, $J_y = 0.2$, and $J_z = 2$. See Fig. 8.

We find that the lowest energy of the configurations we tested is a zero-vortex state. We redefine this energy to be zero. However, we also find that the energy for the state with lowest energy and having a vortex in the big loop, decays rapidly with N . We plot the energy $\Delta\tilde{E}$ (relative to the vortex-free state) for one vortex in the big loop (BL), two vortices in the big loop and on plaquette p_{2N} (BL+ p_{2N}), and finally three vortices in the big loop, on p_{2N-1} and p_{2N} (BL+ p_{2N-1} + p_{2N}). The configurations BL and BL+ p_{2N} appear to be the lowest-energy excitations of \tilde{H} . By computing the eigenvalues of H , we find that the minimal-energy configuration is vortex-free, and the eigenvalue equals the ground state energy of \tilde{H} . However the lowest-energy excitations of H appear to arise from the vortex-loop configurations BL and BL+ p_{2N-1} + p_{2N} .

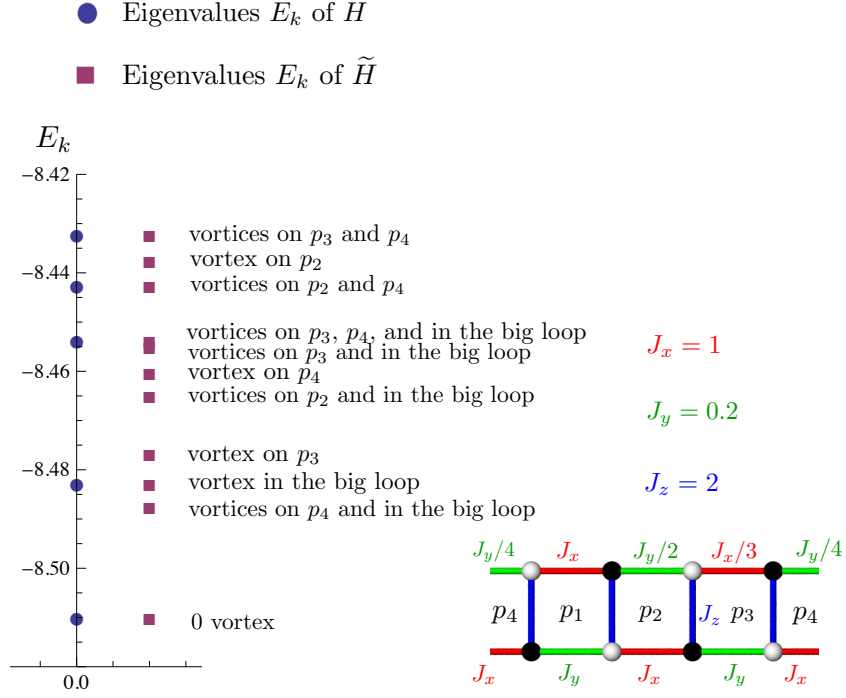


FIG. 7. Low-lying eigenvalues of H and \tilde{H} for an $N = 2$ closed ladder without symmetry. Here \tilde{H} has eigenvalues that do not occur in H . We refer to the “big loop” as a loop with four horizontal bonds (around either the top or the bottom of the ladder). As in Fig. 5 we ignore multiplicities.

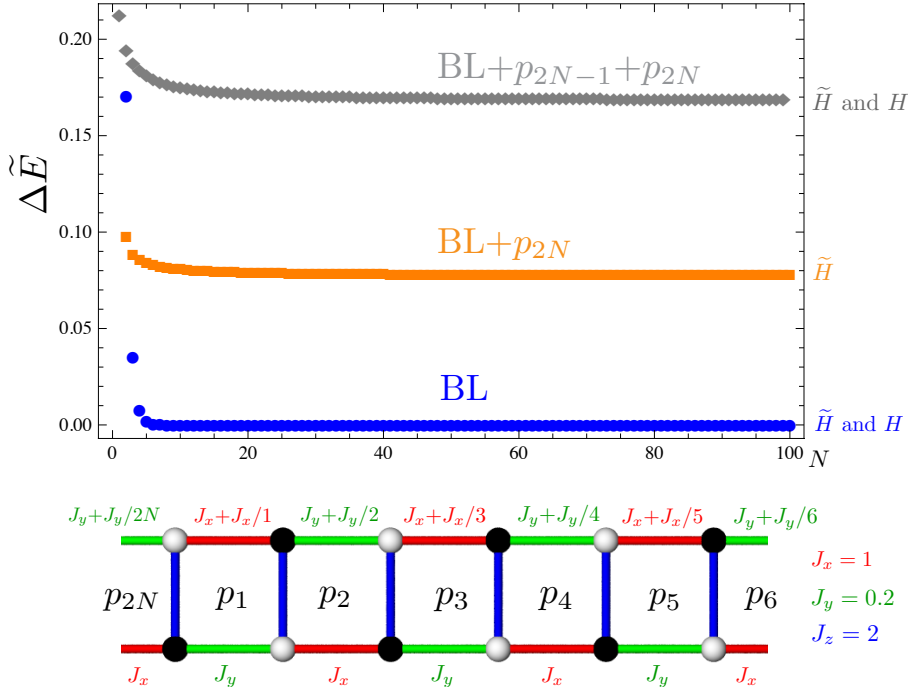


FIG. 8. Excitation energies for closed ladders of length N compared with the vortex-free configuration. The choice of the couplings is shown in the figure, and the vortex configurations are explained in the text.

3. Remark

We have performed numerical calculations for different ladder lengths and coupling configurations that we do not show here, but they all result in similar behavior.

VIII. PERTURBATIVE RESULTS WITHOUT REFLECTION SYMMETRY

In the previous sections we found that ladders satisfying (VI.3) and all positive couplings (or all negative couplings) have ground states with no vortex in any reflection-symmetric loop \mathfrak{C} . It is of interest to understand whether the vortex-free property extends to open and closed ladders that do not satisfy (VI.3). Here we investigate this question by perturbation theory, and find evidence that certain ladders have vortex-free ground-state configurations.

We study ladders for which the x -couplings are all equal to J_x , but for which

$$J_x \gg J_{(i, i+3)}, J_{(i, i+1)} \geq 0. \quad (\text{VIII.1})$$

For homogeneous couplings with $J_z \gg |J_x|, |J_y|$, the ground-state of the open ladder in lowest-order perturbation theory (depending upon N) has been shown to be vortex-free when $J_x J_y > 0$ or vortex-full when $J_x J_y < 0$ [6]. Qualitatively this situation is different from the one we study here, as our perturbation satisfying (VIII.1) gives a vortex contribution to the energy only in third order perturbation theory, rather than in second order. On the other hand the perturbation theory evidence in [6] that the ground state is vortex-free or vortex-full agrees with Conjecture 13 in §IX.

Write the Hamiltonian as

$$H = H_0 + V, \quad (\text{VIII.2})$$

where

$$H_0 = -J_x \sum_{(ij)_x} \sigma_i^x \sigma_j^x, \quad \text{and} \quad V = - \sum_{(ij)_y} J_{(ij)} \sigma_i^y \sigma_j^y - \sum_{(ij)_z} J_{(ij)} \sigma_i^z \sigma_j^z, \quad (\text{VIII.3})$$

where $(ij)_{x,y,z}$ denotes type- x, y, z bonds. We consider perturbations of H_0 by V . In the case of the open ladder, σ_1 and σ_{4N} do not occur in H_0 .

Proposition 11 (Open Ladder). *Assume that $0 < J_{(ij)}$ for all bonds (ij) . Also assume that there are constants $0 < M_1, 0 < M_2$ such that $J_{(ij)} < M_1$ for y and z bonds (ij) and $M_2 < J_x$. Then for M_1/M_2 sufficiently small, the ground state of Hamiltonian (III.1) is vortex-free.*

Remark. We believe that in Proposition 11 one can choose M_1/M_2 sufficiently small, uniformly in N . Establishing such a result about the boundedness of the magnitude of differences of eigenvalues of H requires detailed analysis of the local nature of the perturbation. One needs to estimate non-perturbatively the error in the low-energy perturbation analysis, within a small region of couplings bounded by M_1/M_2 , uniformly in N . Cluster expansions have been used to do this, both in field theory [15] and in lattice systems. For the latter a framework is given in [4, 5] and several related papers. Working out the details to bound the energy differences for the ladder Hamiltonian H remains an interesting project.

Proof. First we establish the notation we use. The ground-state eigenspace \mathcal{P}_0 of the Hamiltonian H_0 has 2^{2N+1} ground states, which we label by the eigenvalues of σ_j^x , for $j = 1, \dots, 4N$, with the constraint $\sigma_i^x \sigma_j^x = +1$ for all bonds $(ij)_x$. We use m to denote the set of eigenvalues of σ_j^x for $j = 1, \dots, 4N$ that satisfy the constraint. Let $\mathcal{P}_0^\perp = 1 - \mathcal{P}_0$. Note that $\mathfrak{B}(\partial p_n)$ commutes with H_0 and thus with \mathcal{P}_0 . Decompose the perturbation V in two parts, $V = V_z + V_y$ with

$$V_z = \sum_{j=1}^{2N} V_z^j = - \sum_{j=1}^{2N} J_{(2j-1, 2j)} \sigma_{2j-1}^z \sigma_{2j}^z, \quad (\text{VIII.4})$$

and

$$V_y = \sum_{j=1}^{2N-1} V_y^j = - \sum_{j=1}^{2N-1} J_{(2j-1, 2j+2)} \sigma_{2j-1}^y \sigma_{2j+2}^y. \quad (\text{VIII.5})$$

The First-Order Effective Hamiltonian. The first-order effective Hamiltonian is

$$\mathcal{P}_0 H_{\text{eff}}^{(1)} \mathcal{P}_0 = \mathcal{P}_0 V \mathcal{P}_0 = 0. \quad (\text{VIII.6})$$

The Second-Order Effective Hamiltonian. The second-order effective Hamiltonian has matrix elements

$$(\mathcal{P}_0 H_{\text{eff}}^{(2)} \mathcal{P}_0)_{m,m'} = \frac{1}{2} \sum_l V_{m,l} V_{l,m'} \left[\frac{1}{E_m - E_l} + \frac{1}{E_{m'} - E_l} \right], \quad (\text{VIII.7})$$

where l labels eigenstates in \mathcal{P}_0^\perp . Here $V_{m,l}$, and E_l are the corresponding matrix elements of V and H_0 . As $\mathcal{P}_0 V_j^z V_{j'}^z \mathcal{P}_0 = 0$, $\mathcal{P}_0 V_j^y V_{j'}^y \mathcal{P}_0 = 0$ for $j \neq j'$, and $\mathcal{P}_0 V_j^z V_{j'}^y \mathcal{P}_0 = 0$ for all j and j' , so one obtains

$$\begin{aligned} & \mathcal{P}_0 H_{\text{eff}}^{(2)} \mathcal{P}_0 \\ &= -\frac{1}{4J_x} \left(\sum_{j=1}^{2N-1} J_{(2j-1, 2j+2)}^2 + \sum_{j=1}^{2N} J_{(2j-1, 2j)}^2 \right. \\ & \quad \left. + J_{(1,4)}^2 + J_{(4N-3, 4N)}^2 + J_{(1,2)}^2 + J_{(4N-1, 4N)}^2 \right) \mathcal{P}_0. \end{aligned}$$

This Hamiltonian does not involve the σ 's, so it does not introduce any splitting of the different vortex configurations.

The Third-Order Effective Hamiltonian. The third-order effective Hamiltonian has matrix elements

$$\begin{aligned} & (\mathcal{P}_0 H_{\text{eff}}^{(3)} \mathcal{P}_0)_{m,m'} \\ &= -\frac{1}{2} \sum_{l,m''} \left[\frac{V_{m,l} V_{l,m''} V_{m'',m'}}{(E_{m'} - E_l)(E_{m''} - E_l)} + \frac{V_{m,m''} V_{m'',l} V_{l,m'}}{(E_m - E_l)(E_{m''} - E_l)} \right] \\ &+ \frac{1}{2} \sum_{l,l'} V_{m,l} V_{l,l'} V_{l',m'} \left[\frac{1}{(E_m - E_l)(E_m - E_{l'})} + \frac{1}{(E_{m'} - E_l)(E_{m'} - E_{l'})} \right]. \end{aligned} \quad (\text{VIII.8})$$

We claim this simplifies to

$$\begin{aligned} \mathcal{P}_0 H_{\text{eff}}^{(3)} \mathcal{P}_0 &= - \sum_{k=2}^{2N-2} \frac{J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)}}{8J_x^2} \mathfrak{B}(\partial p_k) \mathcal{P}_0 \\ & \quad - \frac{J_{(1,2)} J_{(1,4)} J_{(3,4)}}{2J_x^2} \mathfrak{B}(\partial p_1) \mathcal{P}_0 \\ & \quad - \frac{J_{(4N-3, 4N-2)} J_{(4N-3, 4N)} J_{(4N-1, 4N)}}{2J_x^2} \mathfrak{B}(\partial p_{2N-1}) \mathcal{P}_0. \end{aligned} \quad (\text{VIII.9})$$

The minimal energy configuration for the Hamiltonian (VIII.9) therefore occurs in the case that all $\mathfrak{B}(\partial p_k) = +1$. The single sum over k reflects the extensive nature of the eigenvalues in perturbation theory, see for example [2]. The splitting of the degenerate ground states occurs in case a single vortex $\mathfrak{B}(\partial p_k) = -1$. This raises the energy of such a state by the quantity

$$\delta E = \begin{cases} \frac{J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)}}{4J_x^2}, & \text{for } k = 2, \dots, 2N-2 \\ \frac{J_{(1,2)} J_{(1,4)} J_{(3,4)}}{J_x^2}, & \text{for } k = 1 \\ \frac{J_{(4N-3, 4N-2)} J_{(4N-3, 4N)} J_{(4N-1, 4N)}}{J_x^2}, & \text{for } k = 2N-1 \end{cases}. \quad (\text{VIII.10})$$

Which plaquette p_k gives rise to the minimal energy shift depends upon the choice of the coupling constants $J_{(ij)}$. In every case, the energy shift is positive as long as $J_{(ij)} > 0$. For given M_1 and M_2 , the energy shifts δE due to a single vortex on one plaquette—as given by third-order perturbation theory—are bounded away from zero, and also from above, uniformly in N .

We justify the expression (VIII.9) as follows. The first sum in (VIII.8) vanishes because $\mathcal{P}_0 V \mathcal{P}_0 = 0$. The perturbation V_z^j contains the product $\sigma_{2j-1}^z \sigma_{2j}^z$, and V_y^j contains the product $\sigma_{2j-1}^y \sigma_{2j+2}^y$, so the only possible third-order terms have the form $V_z V_y V_z$, $V_z V_z V_y$, or $V_y V_z V_z$, where

$$V_z V_y V_z = - \sum_{j,k,l} J_{(2j-1, 2j)} J_{(2k-1, 2k+2)} J_{(2l-1, 2l)} \sigma_{2j-1}^z \sigma_{2j}^z \sigma_{2k-1}^y \sigma_{2k+2}^y \sigma_{2l-1}^z \sigma_{2l}^z,$$

etc. There are only two possible choices of indices such that $\mathcal{P}_0 V_z V_y V_z \mathcal{P}_0$ does not vanish, namely $j = k$, $l = k + 1$, and $l = k$ and $j = k + 1$. One thus obtains

$$\begin{aligned} & \mathcal{P}_0 V_z V_y V_z \mathcal{P}_0 \\ &= -2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathcal{P}_0 \sigma_{2k-1}^x \sigma_{2k}^z \sigma_{2k+1}^z \sigma_{2k+2}^x \mathcal{P}_0 \\ &= 2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathfrak{B}(\partial p_k) \mathcal{P}_0. \end{aligned} \quad (\text{VIII.11})$$

Here we use $\mathcal{P}_0 \sigma_{2k}^z \sigma_{2k+1}^z \mathcal{P}_0 = -\mathcal{P}_0 \sigma_{2k}^y \sigma_{2k+1}^y \mathcal{P}_0$ and the definition of $\mathfrak{B}(\partial p_k)$ in (III.2). Similarly

$$\begin{aligned} & \mathcal{P}_0 V_z V_z V_y \mathcal{P}_0 \\ &= 2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathcal{P}_0 \sigma_{2k-1}^x \sigma_{2k}^z \sigma_{2k+1}^z \sigma_{2k+2}^x \mathcal{P}_0 \\ &= -2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathfrak{B}(\partial p_k) \mathcal{P}_0. \end{aligned} \quad (\text{VIII.12})$$

The terms in (VIII.11)–(VIII.12) that do not contain the boundary plaquettes $\mathfrak{B}(\partial p_1)$ and $\mathfrak{B}(\partial p_{2N-1})$ cancel identically; they have the same energy denominators and opposite signs. Finally

$$\begin{aligned} & \mathcal{P}_0 V_y V_z V_z \mathcal{P}_0 \\ &= 2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathcal{P}_0 \sigma_{2k-1}^x \sigma_{2k}^z \sigma_{2k+1}^z \sigma_{2k+2}^x \mathcal{P}_0 \\ &= -2 \sum_k J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)} \mathfrak{B}(\partial p_k) \mathcal{P}_0. \end{aligned} \quad (\text{VIII.13})$$

Therefore the contribution to $\mathcal{P}_0 H_{\text{eff}}^{(3)} \mathcal{P}_0$ that does not involve the boundary plaquettes p_1 and p_{2N-1} is

$$- \sum_{k=2}^{2N-2} \frac{J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)}}{8J_x^2} \mathfrak{B}(\partial p_k) \mathcal{P}_0.$$

The situation is different for terms entering in the perturbations $V_z V_y V_z$, $V_z V_z V_y$, and $V_y V_z V_z$ and involving plaquettes p_1 and p_{2N-1} . The reason is that σ_1 and σ_{4N} do not enter into H_0 . Taking this into account, the coefficients of the boundary terms differ. However, they are still negative and the third-order effective Hamiltonian is (VIII.9).

The fact that the perturbation theory result applies in a region of couplings for small M_1/M_2 is a consequence of the analyticity of the eigenvalues, see [20] and §II.1.3–II.1.4 of [21]. \square

Proposition 12 (Closed Ladder with $N > 2$). *Under the hypothesis of Proposition 11, the ground state of the Hamiltonian H in (III.1) with closed boundaries is vortex-free on each plaquette p_1, \dots, p_{2N} . The effective Hamiltonian to third order is*

$$\begin{aligned} \mathcal{P}_0 H_{\text{eff}}^{\leq 3} \mathcal{P}_0 &= H_0 \mathcal{P}_0 - \frac{1}{4J_x} \left(\sum_{j=1}^{2N-1} J_{(2j-1, 2j+2)}^2 + \sum_{j=1}^{2N} J_{(2j-1, 2j)}^2 + J_{(2, 4N-1)}^2 \right) \mathcal{P}_0 \\ &\quad - \sum_{k=1}^{2N-1} \frac{J_{(2k-1, 2k)} J_{(2k-1, 2k+2)} J_{(2k+1, 2k+2)}}{8J_x^2} \mathfrak{B}(\partial p_k) \mathcal{P}_0. \end{aligned} \quad (\text{VIII.14})$$

Remark. The perturbative expansion up to third order will not give a splitting in energy due to a vortex on the big loop (the shortest loop around the top or bottom of the closed ladder). This will occur only in perturbation theory of order $O(N)$; but a single vortex in this loop gives an energy shift that is exponentially small in N .

Proof. The Hamiltonian H in (III.1) possesses two additional bonds $(2, 4N - 1)$ and $(1, 4N)$ that do not occur in the open ladder. They yield interaction terms

$$H_x^{\text{closed}} = -J_x \sigma_1^x \sigma_{4N}^x \quad \text{and} \quad V_y^{\text{closed}} = -J_{(2, 4N-1)} \sigma_2^y \sigma_{4N-1}^y. \quad (\text{VIII.15})$$

We incorporate H_x^{closed} in the unperturbed Hamiltonian H_0 and V_y^{closed} in the perturbation V_y . Define $V_y^{2N} = V_y^{\text{closed}}$, we have $V_y = \sum_{j=1}^{2N} V_y^j$. We now derive the first order, second order, and third order effective Hamiltonians.

The First-Order Effective Hamiltonian. As in the proof of Proposition 11, the first-order effective Hamiltonian vanishes.

The Second-Order Effective Hamiltonian. Also as in the proof of Proposition 11, the second-order is given in (VIII.7). For ladders with $N > 2$ the only second order terms that do not vanish are $\mathcal{P}_0 (V_z^j)^2 \mathcal{P}_0$ and $\mathcal{P}_0 (V_y^j)^2 \mathcal{P}_0$. The ladder being closed, all the energy denominators in (VIII.7) are the same. One thus obtains the second-order term.

The Third-Order Effective Hamiltonian. As for the open ladder, the third-order effective Hamiltonian is given in (VIII.8). For the same reason as in the case of the open ladder, the first sum in (VIII.8) vanishes. Again the relevant perturbations are $\mathcal{P}_0 V_z V_y V_z \mathcal{P}_0$, $\mathcal{P}_0 V_z V_z V_y \mathcal{P}_0$, $\mathcal{P}_0 V_y V_z V_z \mathcal{P}_0$. The cancelation of the terms (VIII.11) and (VIII.12) of the first two perturbations for the open ladder also takes place for the closed ladder. Furthermore, since the ladder is closed, the energy denominators appearing in $\mathcal{P}_0 V_z V_y V_z \mathcal{P}_0$, $\mathcal{P}_0 V_z V_z V_y \mathcal{P}_0$, $\mathcal{P}_0 V_y V_z V_z \mathcal{P}_0$ are all the same and no “boundary” terms appear in the third-order effective Hamiltonian. \square

IX. CONJECTURE

Based on the numerical calculations that we performed in §VII and the perturbation calculations we performed in §VIII, we formulate the following conjecture for ladder Hamiltonians:

Conjecture 13. *For a closed ladder, the ground state energies of H in (III.1) and \tilde{H} in (IV.4) coincide. For a closed or open ladder with the coupling constants $J_{(ij)}$ all positive or all negative, the ground states of H and \tilde{H} are vortex-free.*

It is known that one has qualitatively different behavior in two-dimensional spin systems with trivalent interactions [36].

X. ACKNOWLEDGEMENTS

Arthur Jaffe wishes to thank Daniel Loss for hospitality at the University of Basel where most of this work has been done, to Gian Michele Graf for hospitality at the ETH, and to Jürg Fröhlich for discussions. This work was supported by the Swiss NSF, NCCR QSIT, NCCR Nanoscience, NSERC, CIFAR, FRQNT, INTRIQ, and the Pauli Center of the ETH.

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