Topological Order and Reflection Positivity

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The interplay between the two fundamental concepts of topological order and reflection positivity allows one to characterize the ground states of certain many-body Hamiltonians. We define topological order in an appropriate fashion and show that certain operators have positive expectation value in all ground states. We apply our method to vortex loops in a model relevant to topological quantum memories.

I. INTRODUCTION

Topologically ordered systems have attracted much attention, since they represent promising candidates for the realization of fault-tolerant quantum computing architectures. Intuitively, topological order can be understood as the property that local perturbations cannot cause transitions between the different degenerate ground states of certain many-body Hamiltonians; such transitions require global perturbations. Therefore the subspace of topologically ordered ground states seems to be a good place to store and process quantum information[1–3].

Another fundamental concept that we use is reflection positivity. This notion originally arose in the theory of random fields, as the property that justifies *inverse* Wick rotation from random fields to quantum fields [4], both at zero and at positive temperature [5]. This property has also played an important role in the analysis of phase transitions and ground states in statistical mechanical systems [6–8].

In this work, we show that these two fundamental concepts allow one to characterize the ground states of certain reflection-symmetric Hamiltonians. In particular, we show that the expectation value of certain local operators are positive in all ground states. In certain examples this means the the ground states are vortex-free.

Understanding the ground-state properties of topologically ordered systems is not only interesting from a fundamental point of view, but is also relevant to topological quantum computation: the ground states encode the logical-qubit states.

We apply our method to a model for Majorana interactions on a planar lattice introduced in [9]. We focus on this model because of its equivalence to the toric code model, the archetypical model for a topologically-ordered quantum memory [1], in lowest-order perturbation theory. We show in this example that with our choice of the signs of the coupling constants, all the ground states are free of vortices.

The paper is organized as follows: In Section II, we review the concept of reflection positivity. In Section III, we introduce the concept of W-topological order that is a special case of the more general topological order defined in [2]. In Section IV we show how topological order and reflection positivity allow one to characterize the properties of the degenerate ground states. In particular, we show that the expectation of certain operators is non-negative. Finally, we apply our method in Section V to a model of interacting Majoranas on a planar lattice presented in [9]. We show that when the model is reflection-symmetric, then no vortices are present in the ground states.

II. REFLECTION-POSITIVITY

Consider a lattice Λ which is divided in two parts $\Lambda_\pm$ mapped into each other by reflection in a plane Π. We represent this reflection as an anti-unitary ϑ on the Hilbert space $\mathcal{H}$ of our model. To each vertex of the lattice, we associate one or more operators O_i, and the reflection maps them into

$$\vartheta(O_i) = \vartheta\, O_i\, \vartheta^{-1} = O^\dagger_{\vartheta i}\,, \tag{1}$$

where site ϑi is the reflection of site i.

Let $A \in \mathfrak{A}$ denote a sum of products of operators O_i, where i is in Λ_-. We consider a self-adjoint Hamiltonian H with the reflection-positivity property,

$$0 \leqslant \mathrm{Tr}(A\,\vartheta(A)\,e^{-\beta H})\,, \tag{2}$$

for all $0 < \beta$. In the rest of this work let $A\,\vartheta(A) = W_A$.

III. W-TOPOLOGICAL ORDER

Let us denote the ground-state subspace of H and the orthogonal projection onto this subspace by $\mathcal{P}$. One says that $\mathcal{P}$ has W-topological ordered if $\mathcal{P}W\mathcal{P}$ is a scalar multiple of $\mathcal{P}$. This definition is a specialization of general topological order defined in [2]. In other words, the operator W cannot cause transitions between different ground states.

IV. TOPOLOGICAL ORDER ENSURES POSITIVITY

Consider a Hamiltonian H and operators $A \in \mathfrak{A}$ with the reflection positivity property (2). Assume that the ground-state subspace has W_A-topological order. Then we infer positivity,

$$0 \leqslant \langle \Omega, W_A \Omega \rangle \,, \tag{3}$$

for any ground state Ω of H.

A. Explanation

Suppose H has N orthonormal ground states denoted by Ω^u, with $\mu = 1, \ldots, N$. Normalize the ground-state energy of H to be zero. Assuming (2) and taking the $\beta \to \infty$ limit leads to,

$$0 \leqslant \sum_{\mu=1}^{N} \langle \Omega^\mu, W_A \, \Omega^\mu \rangle. \tag{4}$$

We infer from (4) that the expectation value of W_A is non-negative in at least one of the ground states. Since the ground states are W_A-topologically ordered,

$$\langle \Omega^\mu, W_A \, \Omega^\mu \rangle = \alpha \langle \Omega^\mu, \Omega^\mu \rangle = \alpha \,, \tag{5}$$

with α a constant, independent of μ. As the expectation must be non-negative for some μ, we conclude that $0 \leqslant \alpha$, and

$$0 \leqslant \langle \Omega^\mu, W_A \, \Omega^\mu \rangle \qquad \text{for} \quad \mu = 1, \ldots, N \,. \tag{6}$$

V. EXAMPLE

A. Vortex Loops

In certain situations, the operator W_A equals a vortex loop. In this case, the positivity (6) says that any ground state of H is free of vortices of type W_A. We will explain this result in more detail in the context of a particular example.

A loop C of length $2l$ is an ordered sequence $\{i_1, i_2, \ldots, i_{2l}, i_1\}$ of nearest-neighbor sites in Λ. Associated to each loop C, we define a vortex loop $W(C)$ as a product of O_i's,

$$W(C) = O_{i_1} O_{i_2} \cdots O_{i_{2l}} \,. \tag{7}$$

Since we want $W(C)$ to have the form $A\vartheta(A)$, we choose $A = O_{i_1} \cdots O_{i_l}$, so by (1),

$$W(C) = O_{i_1} O_{i_2} \cdots O_{i_l} O^\dagger_{\vartheta i_1} O^\dagger_{\vartheta i_2} \cdots O^\dagger_{\vartheta i_l} \,. \tag{8}$$

In the following example, we consider the case where each of the operators O_i is a Majorana. Then $c_i = c_i^\dagger = c_i^{-1}$ and $\{c_i, c_j\} = 2\delta_{ij}$. In this case, we introduce a phase i^l and redefine $W(C)$ in place of (7) as

$$W(C) = i^l c_{i_1} c_{i_2} \cdots c_{i_l} c_{\vartheta i_1} c_{\vartheta i_2} \cdots c_{\vartheta i_l} \,. \tag{9}$$

Thus, we have $W(C) = W(C)^\dagger = W(C)^{-1}$, so the vortex loop has eigenvalues ± 1, and its expectation in a unit vector lies between -1 and $+1$.

We say that the loop C is vortex-free when the expectation of $W(C)$ is $+1$ and vortex-full when the expectation is -1. In the intermediate cases, we say that the loop is partially free and partially full, according to the sign of the expectation of $W(C)$.

In the special case that $W(C)$ commute with H, and therefore $W(C)$ is conserved, it is possible to choose an orthonormal basis of ground states of H for which $W(C) = \pm 1$. Then the loop C is either vortex-free or vortex-full in each of these ground states.

B. Topological quantum memory

As an example, we consider the Hamiltonian proposed in [9], describing interactions between Majoranas localized on the vertices of a planar lattice. Those authors show that for small values of the parameter in the Hamiltonian, the model possesses W-topological order for local operators W. In fact, in lowest-order perturbation theory, the low-energy effective Hamiltonian is identical to the toric code.

1. *Hamiltonian*

The ideas above lead to a characterization of the configuration of vortex loops in the ground states. The model studied in [9] has a Hamiltonian of the form

$$H = \sum_j H_{0,j} + \lambda \sum_{j<k} V_{jk} \,. \tag{10}$$

Here j labels square islands of the lattice, see Fig. 1, and the Hamiltonian $H_{0,j}$ is a product of four independent Majoranas c_{j_a}, c_{j_b}, c_{j_c}, and c_{j_d} of the form,

$$H_{0,j} = -c_{j_a} c_{j_b} c_{j_c} c_{j_d} \,. \tag{11}$$

The constant λ in (10) is dimensionless, and $V_{jk} = i c_j c_k$ are quadratic interactions between Majoranas. The superscript $^{(jk)}$ denotes a choice of the Majoranas determined by the directed bond (jk).

We illustrate the planar-lattice configuration in Fig. 1. Each pair of nearest-neighbor islands j and k define a directed bond (jk) that characterizes the coupling of the neighboring islands. Each island j consists of a square with independent Majoranas c^a_j, c^b_j, c^c_j, and c^d_j which we place on the four corners of the square island as specified in Fig. 1 (in particular a, b lie on the top of the square and c, d lie on the bottom of the square). For four nearest-neighbor squares labeled i, j, k, l on the lattice, define the loop $C_{ijkl} = \{i, j, k, l\}$ and the vortex operator

$$W(C_{ijkl}) = c_{i_c} c_{j_a} c_{j_b} c_{k_d} c_{k_a} c_{l_c} c_{l_d} c_{i_b} \,. \tag{12}$$

The vortex operators have the form (9) and thus eigenvalues ± 1.

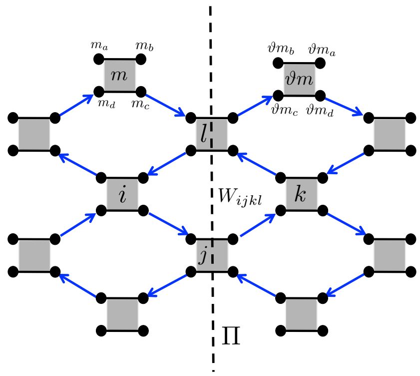

FIG. 1. In the lattice, we depict square islands by gray squares and Majoranas by black dots. Bonds (jk) which connect nearest-neighbor square islands are indicated by directed bonds between nearest-neighbor Majoranas. The orientation of the bonds define circuits around eight-site polygons that lie between the square islands.

2. *Majoranas and Reflection Positivity*

The vortex $W(C_{ijkl})$ that is bisected by the plane Π, as illustrated in Fig. 1, can be written in the form

$$W(C_{ijkl}) = W_A = A\,\vartheta(A)\,, \tag{13}$$

with $A = c_{l_d}c_{i_b}c_{i_c}c_{j_a}$. With our choice of V_{jk} the Hamiltonian H in (10) is reflection-symmetric,

$$\vartheta(H) = H\,. \tag{14}$$

In [10], we have studied a class of reflection-symmetric Majorana Hamitonians that includes the Hamiltonian (10), and we demonstrated that reflection positivity holds.

As a consequence, the expectation value of $W(C_{ijkl})$ in the thermal state at inverse temperature β is positive. Furthermore, in its topological phase, the loop $C_{ijkl} = \{i,j,k,l\}$ is vortex free in all ground states. When the ground state is non-degenerate, (2) implies that loop C_{ijkl} is vortex-free in the ground state. Finally, if the lattice is periodic, any loop C_{ijkl} can be used in this argument. Therefore, every ground state is free of each elementary vortex.

VI. CONCLUSIONS

In conclusion we remark that this work is based on general principles. Therefore we expect that the method described here could be useful in the study of other many-body, topologically-ordered Hamiltonians.

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