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A Modified Model of Euclidean Quantum Field Theory

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Kurt W. Symanzik

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Abstract

The analytic continuations to imaginary time of the Green's functions of local quantum field theory define Euclidean Green's functions. A neutral scalar field with quartic self-interaction is investigated. The coupled differential equations for the Euclidean Green's functions are solved, after regularization and introduction of a finite space-time volume, by a family of functional integrals. Hereby a number of properties of those functions is proven. Some of the Lagrangian-dependent properties translate immediately into properties of the real-time Green's functions if these exist.

Introduction

It is well known[1] that quantum field theory in Minkowski space (MQFT), if a Lagrangian is given, can be cast in the form of an infinite system of coupled integral equations* for the infinite set of Green's functions. These systems of equations have so far been of little help except for studying certain formal properties of Green's functions (e.g. properties under gange transformations in quantum electrodynamics[3] or how to define a Bethe-Salpeter kernel without recourse to perturbation theory[1)4)5)]). The main obstacle to a nonformal use of those systems is our inaptitude to formulate properly the boundary conditions on such systems to make them mathematically meaningful. Prescriptions on how to break such systems off have been given at times but seem so far lacking in convincing justification as well as success.

One feature of those equations that doubtless increases the task is already the poor definition of each single equation. E.g. in their momentum space form one encounters even under the most favorable of circumstances only conditionally convergent integrals, while in coordinate space they involve products of distributions. Dyson[6] has shown that this difficulty is overcome in perturbation theory by a rotation of the paths of integration. More generally, one may use simultaneous analytic

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* We shall not be concerned here with the systems of equations[2] that use no Lagrangian but intermediate state insertions and the asymptotic condition.

continuation[7] of all functions in the equations and define the original functions by the boundary values of their continuations. Continuing to imaginary times respectively energies yields the Euclidean Green's functions studied in their own right by Schwinger[8] and Nakano[9]. These functions can be defined even without reference to a Lagrangian and may be associated with a Euclidean quantum field theory (EQFT) whose characteristic symmetry group is not the Lorentz group but the orthogonal group in four dimensions.

Transition from MQFT to EQFT does not, however, abate the ultraviolet difficulties of MQFT but only makes them appear in terms of divergent, instead of meaningless, integrals. Renormalization of the coupled system of integral equations may be performed either with the help of limiting processes[10], or by renouncing manifest locality[11)5)]. Both these ways are not suitable for our purpose. We regularize the theory by the method of Pais and Uhlenbeck[12]. The indefinite metric then introduced in MQFT does not carry over to EQFT, where the metric remains positive definite. In certain of the properties of EQFT we derive the regularization does not seem to play a role; those properties may be expected to hold even for unregularized (and renormalized) EQFT if that limit of the regularized theory exists.

The infinite systems of equations so far described are equivalent to a functional differential equation for the generating functional of the Green's functions, the Schwinger functional[1]. In MQFT this equation is formally solved by the

Feynman history integral[13]. The corresponding solution in EQFT is a generalized Wiener integral, a "Wiener history integral". The relation between these two integrals is analogous to that between Feynman's path integral[14] and the elementary Wiener integral. Whereas the former is not an integral in the mathematical sense, but rather an approximation prescription, the latter is a true integral provided the time interval is finite. Infinite time intervals can be handled by limiting processes only, the existence and properties of the limit not being inferred from the Wiener integral itself but from the known solubility of the heat conduction equation (or of the Schrödinger equation in the case of the Feynman path integral). We find ourselves forced to a similar approach: the basic space- "time" interval for the Wiener history integral need be chosen finite. We prove existence of this integral by writing it as an integral over Hilbert space as developed by Segal[15] and Friedrichs and Shapiro[16]. The restriction to a finite space- "time" volume is the most drastic and detaches our model complete from physical reality. Nevertheless a few of the properties of EQFT Green's functions seem to be insensitive even to that, and certain inequalities we derive in our specific model must be valid also for the unmodified model and even for unmodified MQFT if its Green's functions exist at all.

We feel that the difficulty that requires us to choose a finite space- "time" volume is due to the special approach only and not one inherent in EQFT or MQFT. E.g. the Schrödinger equation approach does solve it. It is just our ignorance of appropriate boundary conditions for the functional Schrödinger equation that we hope to diminish in this paper.

In I.A we define EQFT axiomatically and in I.B in an alternative and more heuristic manner. In II we introduce a model for definiteness, although almost all of the later analysis applies to more general cases also. In III the modifications described before are made explicit and the existence of the modified model is shown. In IV general properties of modified EQFT, and in V properties specific to the model, are derived. In VI we give an operator formulation of modified EQFT. We then present our conclusions. For the convenience of the reader, we give in appendix A the definition of the Friedrichs-Shapiro integral and some of its properties. The remaining appendices complement the text on technical or peripheral points.

I. Formulation of EQFT and heuristic considerations

I.A. Axiomatic definition

For this subsection, we adopt the axiomatic approach to relativistic quantum field theory developed by Wightman[17]. We consider the theory of one hermitean scalar field $A(x)$ only.

Due to the stability of the vacuum (denoted by $\langle$ and $\rangle$), the spectrum condition, and their assumed temperedness as distributions, the vacuum expectation values

$$\langle A(x_o)A(x_1)\ldots A(x_n)\rangle \quad , \quad x_i = (x_i^o, x_i^1, x_i^2, x_i^3)$$

of products of field operators are, as functions of $\xi_i = x_{i-1}-x_i$ $(i=1\ldots n)$, boundary values of analytic functions $W_n((\zeta))$, $(\zeta) = (\zeta_1\ldots\zeta_n)$, with analyticity domain

the tube

$$\mathcal{R}_n = \left\{(\zeta) : \mathrm{Im}\,\zeta_i \in V^+, \forall_i \right\}$$

i.e. $\mathrm{Im}\,\zeta_i^o > 0$, $(\mathrm{Im}\,\zeta_i)^2 > 0$, with $g_{\mu\nu} = -\delta_{\mu\nu}(-1)^{\delta_{\mu o}}$. Due to relativistic invariance, $W_n((\zeta))$ is analytic and single-valued in the extended tube

$$\mathcal{R}'_n = \left\{(\zeta) : \exists\, \Lambda_+(C),\ (\zeta) = (\Lambda_+(C)\zeta'),\ (\zeta') \in \mathcal{R}_n \right\}$$

where $\Lambda_+(C)$ is a proper homogeneous complex Lorentz transformation:

$$\Lambda_+(C)^T g \Lambda_+(C) = g,\ \mathrm{Det}\ \Lambda_+(C) = 1 \ .$$

Due to local commutativity, $W_n((\zeta))$ is analytic and single valued in the permuted extended tube

$$\mathcal{R}''_n = \bigcap_{\text{all } p} P\mathcal{R}'_n$$

with

$$P\mathcal{R}'_n = \left\{(\zeta) : (\zeta) = (P\zeta'),\ (\zeta') \in \mathcal{R}'_n \right\}$$

where $P \in S_{n+1}$ is a permutation

$$P : (0,1\ldots n) \longrightarrow \big(P(0)P(1)\ldots P(n)\big)$$

and if $\zeta_i = z_{i-1}-z_i$, then $P\zeta_i = z_{p(i-1)}-z_{p(i)}$. In $\mathcal{R}''_n$, $W_n((P\zeta)) = W_n((\zeta))$ due to our use of one field only.

The Schwinger points

$$(\zeta_s) : \text{Re}\, \zeta_i^o = 0, \; \text{Im}\, \zeta_i^{1,2,3} = 0, \; \forall_i$$

lie in the interior of $\mathcal{R}_n''$ if $\zeta_{s_{i+1}} + \dots + \zeta_{s_k} \neq 0$ for all $1 \leq i+1 \leq k \leq n$. We may write

$$\zeta_{s_i} = x_{i-1} - x_i \;, \; x = (x^1, x^2, x^3, x^4)\text{real} \;, \; x^4 = ix^o \;,$$

and introduce the Schwinger functions[8]

$$W_n((\zeta_s)) \equiv S(x_o x_1 \dots x_n) \quad .$$

These Euclidean Green's functions are symmetric functions of n+1 4-vector arguments, invariant under the proper inhomogeneous orthogonal group in four dimensions (here called the Euclidean group), and real-analytic except at points of coincidence of some arguments. (Their analytic continuations are invariant under the complex Euclidean group and the original Wightman functions in different notation). They satisfy

$$\text{(I.1)} \qquad S(x_o \dots x_n) = S(x_o^T \dots x_n^T)^* = S(x_o^S \dots x_n^S)^* = S(-x_o \dots -x_n)$$

where $x^T = (x^1, x^2, x^3, -x^4)$, $x^S = (-x^1, -x^2, -x^3, x^4)$ and are, therefore, real if the theory is invariant under time reversal or space reflection.

The Green's functions

$$F(x_o \dots x_n) \equiv \langle TA(x_o) \dots A(x_n) \rangle \quad ,$$

where T is the symbol for operator ordering with increasing times from right to left, are for noncoinciding arguments symmetric tempered Lorentz invariant distributions. Assuming that these functions can be extended* to such distributions for all arguments, Ruelle[18] has shown that the Fourier transforms

$$\tilde{F}(p_1 \dots p_n) = \int dx_1 \dots dx_n \, e^{i \sum x_i p_i} F(0 x_1 \dots x_n)$$

are boundary values of analytic functions[1] which are invariant under the proper homogeneous Lorentz group. The Schwinger points $(p_s) \neq \text{Im}\, p_i^{1,2,3} = 0$, $\text{Re}\, p_i^o = 0$, $p_i^o = -ip_i^4$, $\forall_i$ lie inside the analyticity domain except for points where a nonempty partial sum of the vectors p_{s_i} vanishes. We shall write

$$i^n \tilde{F}((p_s)) \equiv \tilde{S}(p_o p_1 \dots p_n) \ , \ p_o = -p_1 - \dots - p_n \quad .$$

Then

$$(2\pi)^4 \, S(p_o \dots p_n) \delta(p_o + \dots + p_n) =$$

$$= \int dx_o \dots dx_n \, e^{-i \sum x_i p_i} S(x_o \dots x_n)$$

where $x_i p_i = x_i^1 p_i^1 + \dots + x_i^4 p_i^4$. If truncated Wightman functions[20]

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* See[18] for a precise statement of the assumption.

1 For descriptions of the domain of analyticity, see[18] and[19].

W^T and truncated Green's functions* F^T are introduced, the functions $\tilde{F}^T$ have no singularities at Schwinger points. Therefore, the functions $\tilde{S}^T(p_o \ldots p_n) = \tilde{F}^T((p_s))$ are symmetric real-analytic functions, invariant under the homogeneous proper orthogonal group, and satisfy

$$\text{(I.2)} \qquad \tilde{S}^T(p_o \ldots p_n) = \tilde{S}^T(p_o^T \ldots p_n^T)^* = \tilde{S}^T(p_o^s \ldots p_n^s)^* = \\ = \tilde{S}^T(-p_o \ldots -p_n)$$

with definitions analogous to those in (1). They possess analytic continuations into the tube

$$(\mathrm{Im}\ p) \; \varepsilon \; D^m \equiv \bigcap_{\text{all } I} D_I^m \;, \text{ with}$$

$$D_I^m = \left\{ (\mathrm{Im}\ p) = \sum_{k=1}^{4} \left(\sum_{i \varepsilon I} \mathrm{Im}\ p_i^k \right)^2 < m^2 \right\}$$

where I is a proper subset of $\{ 0,1 \ldots n \}$ and $m > 0$ is the lower bound of the mass spectrum (except for the vacuum) of the theory. It follows that, provided

$$\min_{i \neq j} (x_i - x_j)^2 > \varepsilon \quad ,$$

$$\text{(I.3)} \qquad \lim_{D \to \infty} S^T(x_o \ldots x_n) e^{\sum_{i=o}^{n} \alpha_i x_i} = 0 \;, \; (\alpha) \; \varepsilon \; D^m$$

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* For these we give a convenient definition in (11).

where $\alpha_i x_i = \alpha_i^1 x_i^1 + \ldots + \alpha_i^4 x_i^4$, $D = \{\max_{i,j}(x_i \ldots x_j)^2\}^{1/2}$. (3) shows the exponential decrease of $S^T(x_o \ldots x_n)$ for increasing distance between its arguments.

I.B. Heuristic considerations

Having established the existence of Euclidean Green's functions in every theory that satisfies Wightman's postulates, we will proceed in the remainder of this section and in II more heuristically, what seems justified as no physically nontrivial example of a Wightman theory is known. This will enable us to derive an interesting inequality for the generating functional of Euclidean Green's functions. The following considerations* apply to any theory merely invariant under time translation.

With a real function $J(\vec{x},t)$, $\vec{x} = (x^1,x^2,x^3)$, we define the two-parameter family of operators $U(t,t')$, $t \geq t'$, by

(I.4a) $$(\partial/\partial t)U(t,t') = i\int d\vec{x}\, A(\vec{x},t)J(\vec{x},t)U(t,t')$$

(I.4b) $$U(t',t') = 1$$

or

(I.5) $$U(t,t') = 1 + i\int_{t'}^{t} d\tau \int d\vec{x}\, A(\vec{x},\tau)J(\vec{x},\tau)U(\tau,t') \quad .$$

From (5) follows, if we assume uniqueness of its solution

(I.6) $$(\partial/\partial t')U(t,t') = -i\int d\vec{x}\, U(t,t')A(\vec{x},t')J(\vec{x},t') \quad .$$

From (4) and (6) we have, for $t \geq t'' \geq t'$,

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* Apart from the analytic continuation, they are thoroughly familiar, e.g.[21].

(I.7) $$U(t,t'')U(t'',t') = U(t,t') \quad ,$$

from (4)

$$U(t,t')^{+}U(t,t') = 1$$

and from (6)

$$U(t,t')U(t,t')^{+} = 1 \quad .$$

(4) or (6) and the consequence of (5) and (7)

(I.8) $$[\delta/\delta J(x)]U(t,t') = i\theta(t-x^{o})\theta(x^{o}-t')U(t,x^{o})A(x)U(x^{o},t')$$

substantiate the symbolic formula

(I.9) $$U(t,t') = T\ \exp\left[i\int_{t'}^{t} d\tau \int d\vec{x}\ A(\vec{x},\tau)J(\vec{x},\tau)\right] \quad .$$

The formal limit

(I.10a) $$F\left\{J\right\} = \lim_{\substack{t \to +\infty \\ t' \to -\infty}} \left\langle U(t,t') \right\rangle$$

is the generating functional of the Green's functions $F(x_1 \ldots x_n)$ considered in I.A:

(I.10b) $$F\left\{J\right\} = = 1 + \sum_{n=1}^{\infty} (n!)^{-1} i^{n} \int \ldots \int dx_1 \ldots dx_n J(x_1) \ldots J(x_n) F(x_1 \ldots x_n) .$$

Similarly,

(I.11) $$F^{T}\left\{J\right\} \equiv \ln F\left\{J\right\}$$

is the generating functional of the truncated Green's functions. (The limit $t \to +\infty$, $t' \to -\infty$, of $U(t,t')$ will actually exist as weak and, since $U(t,t')$ is unitary, also as strong operator limit provided $J(\vec{x},\tau)$ vanishes sufficiently strongly for $|\tau| \rightarrow \infty$).

We define the unitary operators

$$V(t,t') = e^{-iHt}U(t,t')e^{iHt'} \tag{I.12}$$

where H is the Hamiltonian, which satisfies with ground state $\rangle$

$$H \rangle = 0 \quad . \tag{I.13}$$

From

$$A(\vec{x},t) = e^{iHt}A(\vec{x},0)e^{-iHt}$$

and (4) we find

$$V(t,t') = 1 - i\int_{t'}^{t} d\tau \; H\left\{ J_\tau \right\} V(\tau,t') \tag{I.14a}$$

where

$$H\left\{ J_\tau \right\} \equiv H - \int d\vec{x} \; A(\vec{x},0)J(\vec{x},\tau) \tag{I.15}$$

is the energy operator perturbed by a ("time-independent") source term. In analogy to (7), (8), and (9) we have

$$V(t,t') = V(t,t'')V(t'',t') \quad , \quad t \geq t'' \leq t' \quad ,$$

$$[\delta/\delta J(x)]V(t,t') = i\theta(t-x^0)\theta(x^0-t')V(t,x^0)A(\vec{x},0)V(x^0,t') \quad ,$$

and

$$V(t,t') = T\,\exp\left[-i\int_{t'}^{t} d\tau \; H\left\{ J_\tau \right\}\right] \quad . \tag{I.14b}$$

We now define, similarly as Schwinger[8] did for the Green's functions separately, the analytic continuation of $V(t,t')$

$$(I.16) \qquad V_z(t,t') = 1 - iz \int_{t'}^{t} d\tau \; H\{J_\tau\} \; V_z \{\tau,t'\}$$

with the properties

$$(I.17) \qquad V_z(t,t') = V_z(t,t'')V_z(t'',t') \;, \quad t \geq t'' \geq t' \;,$$

$$(I.18) \qquad [\delta/\delta J(x)]V_z(t,t') =$$

$$= iz \; \theta(t-x^o)\theta(x^o-t')V_z(t,x^o)A(\vec{x},O(V_z(x^o,t') \;,$$

$$(\partial/\partial_z)V_z(t,t') = -i \int_{t'}^{t} d\tau \; V_z(t,\tau)H\{J_\tau\}V_z(t,t')$$

and the representation analogous to (9)

$$(I.19) \qquad V_z(t,t') = T \exp\left[-iz \int_{t'}^{t} d\tau \; H\{J_\tau\}\right] \;.$$

From (10) we have

$$(\partial/\partial t)V_z(t;t') = -iz \; H\{J_\tau\}V_z(t,t')$$

and thus

$$(I.20) \qquad (\partial/\partial t)[V_2(t,t')^+V_z(t,t')] =$$

$$= 2(\mathrm{Im}\; z)V_z(t,t')^+H\{J_t\}V_z\{t,t'\} \;.$$

(I.21) Let $E_o\{J_t\} = \underset{\Phi}{\text{g.}\ell\text{.b.}} \{\langle\phi|H\{J_t\}|\phi\rangle\langle\phi|\phi\rangle^{-1}\}$.

then from (20) follows

$$(\partial/\partial t)\ln\langle\phi|V_z(t,t')^+V_z(t,t')|\phi\rangle \leq 2(\text{Im } z)\, E_o\{J_t\}$$

for $\text{Im } z \leq 0$, and by integration

(I.22) $$\langle\phi|V_z(t,t')^+V_z(t,t')|\phi\rangle \leq$$
$$\leq \langle\phi|\phi\rangle \exp\left[2(\text{Im } z)\int_{t'}^{t} E_o\{J_\tau\}\, d\tau\right]$$

and therefore

(I.23) $$|\langle\phi|V_z(t,t')|\phi'\rangle| \leq$$
$$\leq ||\phi||\ ||\phi'||\ \exp\left[(\text{Im } z)\int_{t'}^{t} E_o\{J_\tau\}\, d\tau\right] .$$

from (13), (15), (21), and the usual convention $\langle A(x)\rangle = 0$ follows

(I.24) $$E_o\{J_t\} \leq 0 .$$

(23) suggests the existence of $V_z(t,t')$ in the lower z half plane and, because of (16), analyticity, if we can find a lower bound so $E_o\{J_t\}$ for some classes of J_t. Unfortunately, no such bound is known for relativistic theories of physical interest, except for the free field in a time-independent and not too deep external potential $V(\vec{x})$, for which [22]

(I.25) $$H = \frac{1}{2}\int d\vec{x}\left[\dot{A}(\vec{x})^2 + (\vec{\nabla}A(\vec{x}))^2 + m^2A(\vec{x})^2 + \right.$$

$$\left. + V(\vec{x})A(\vec{x})^2\right] \quad + \text{const}$$

and

$$E_o\{J_t\} = -\frac{1}{2}\iint J(\vec{x},t)G_V(\vec{x},\vec{x}')J(\vec{x}',t)d\vec{x}d\vec{x}' \ .$$

with

$$\left[-\Delta + m^2 + V(\vec{x})\right]G_V(\vec{x},\vec{x}') = \delta(\vec{x} - \vec{x}').$$

However, it follows from (21) and (15) that $E_o\ J$ is a concave functional of $J(\vec{x})$,

(I.26) $$E_o\left\{\frac{1}{2}(J_1 + J_2)\right\} \geq \frac{1}{2}E_o\{J_1\} + \frac{1}{2}E_o\{J_2\},$$

which generalises the well known concavity of the ground state energy with respect to the coefficient of any (hermitean) perturbation term.

The formal limit

(I.27a) $$\lim_{\substack{t\to+\infty \\ t'\to-\infty}} \langle V_{-i}(t,t')\rangle = S\{J\}$$

is the generating functional of the Euclidean Green's functions of I.A,

(I.27b) $$S\{J\} = 1 + \sum_{n=1}^{\infty}(n!)^{-1}\int\cdots\int dx_1\ldots dx_n J(x_1)\ldots J(x_n)S(x_1..x_n)$$

while

(I.28) $$S^T\{J\} = \ln S\{J\}$$

is the generating functional of the truncated functions $S^T(x_1 \ldots x_n)$. The limit $V_z(\infty, -\infty)$ will exist if $\mathrm{Im}\, z < 0$ and $J(\vec{x},\tau)$ vanishes sufficiently strongly for $|\tau| \to \infty$, and will be

$$V_z(\infty, -\infty) = \rangle \langle V_z(\infty, -\infty) \rangle \langle, \quad \mathrm{Im}\, z < 0$$

because the vacuum is the state of lowest energy. For z real, however, this limit does not exist already for the free field.

For $\mathrm{Im}\, z < 0$, the operators $V_z(t,t')$ form a semi-group with multiplication law (17). The formal inverse of (19),

$$V_z(t,t')^{-1} = \bar{T} \quad \exp\left[i\, z \int_{t'}^{t} d\tau H\{J_\tau\}\right],$$ does not

have a dense domain already for the free field if $\mathrm{Im}\, z < 0$.

The analytic function

(I.29) $$f(z) = \langle \Phi | V_z(t,t') | \Phi' \rangle \exp\left[i\, z \int_t^{t'} E_0\{J_\tau\} d\tau\right]$$

is, according to (24) absolutely bounded by $\|\Phi\| \cdot \|\Phi'\|$ in the closed lower half plane. If the adiabatic theorem applies with respect to switching on the source term in (15),

(I.30) $$\lim_{\substack{|z| \to \infty \\ 0 \geq \arg z \geq -\pi}} f(z) = \langle \Phi | J_t \rangle \langle J_{\tau_r + 0} | J_{\tau_r - 0} \rangle \cdots$$

$$\cdots \langle J_{\tau_1 + 0} | J_{\tau_1 - 0} \rangle \langle J_{t'} | \Phi' \rangle ,$$

if J_τ is continuous in τ except at times $\tau_1 \ldots \tau_r$, $t' < \tau_1 < \ldots < \tau_r < t$, where $|J_\tau\rangle$ is the normalised ground state to $H\{J_\tau\}$. The adiabatic theorem holds in the solvable case (25), where (30) can be verified without difficulty. More generally, it holds if $f(z)$ is expandable in powers of J and the limit (30) can be taken termwise, and if each term behaves either in the axiomatic fashion amended as described in 1.A, or like the terms in its renormalised* perturbation theoretical expansion.

From (23) we have[1]

$$(I.31) \qquad \left|\left\langle T \exp\left[-\int H\{J_\tau\}d\tau\right]\right\rangle\right| \le \exp\left[-\int E_0\{J_\tau\}d\tau\right].$$

If the adiabatic theorem holds sufficient, conditions for which we just gave,(27a), (29), and (30) permit to refine (31):

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*The field operator $A(x)$ should be the renormalised one whenever the renormalisation factor is not finite.

[1]This inequality is that one Feynman[23] used in order to obtain an upper bound to the ground state energy of the polaron system, in conjunction with the arithmetic-geometric mean inequality applied to a functional integral for the analog in the polaron case of the left bound side.

(I.32a) $$\lim_{T\to\infty} T^{-1} S^{T}\{J^{T}\} = -E_{o}\{J\}$$

and

(I.32b) $$\lim_{T\to\infty} e^{TE_{o}\{J\}} S\{J^{T}\} = |\langle J|\rangle|^{2}$$

where $J^{T}(\vec{x},t) = J(\vec{x})\theta(T-t)\theta(t)$.

II. A Model

The following considerations are entirely formal and intended only as introduction to the modified model defined and analyzed in III.

Let the Lagrangian density be

$$\text{(II.1)} \qquad L = \frac{1}{2}(\partial_o A)^2 - \frac{1}{2}\vec{\nabla} A \cdot \vec{\nabla} A - \frac{m^2}{2} A^2 - \frac{g}{4} A^4 .$$

The canonical formalism[24] gives the field equation

$$\text{(II.2)} \qquad (\partial_o^2 - \Delta + m^2)A(x) + g\, A(x)^3 = 0$$

and the commutation relations

$$\text{(II.3a)} \qquad [A(\vec{x},x^o),\ \partial_o\, A(\vec{x}',x^o)] = i\ \delta(\vec{x} - \vec{x}')$$

$$\text{(II.3b)} \qquad [A(\vec{x},x^o),\ A(\vec{x}',x^o)] = [\partial_o A(\vec{x},x^o), \partial_o\, A(\vec{x}',\ x^o)] = 0.$$

Schwinger's functional,[1] defined in (I.10), satisfies because of (I.8), (I.4), (I.6), (2), (3), and (I.7).

$$\text{(II.4)} \qquad (\partial_o^2 - \Delta + m^2)\ (-i)\ [\delta/\delta J(x)]\ F\{J\} +$$

$$+ g(-i)^3\ [\delta^3/\delta J(x)^3]\ F\{J\} = J(x)\ F\{J\}$$

with $F\{0\} = 1$. Because of (I.12) and (I.13),

$$\lim_{\substack{t \to +\infty \\ t' \to -\infty}} \langle V(t,t')\rangle$$ satisfies the same equation.

Similarly, we get for $F_z\{J\} \equiv \lim_{\substack{t \to +\infty \\ t' \to -\infty}} \langle V_z(t,t')\rangle$ from the corresponding equations (I.18) ff.

$$(z^{-2}\partial_o^2 - \Delta + m^2)\ (iz)^{-1}\ [\delta/\delta J(x)]\ F_z\{J\} +$$

$$+ (iz)^{-3}\ [\delta^3/\delta J(x)^3]\ F_z\{J\} = J(x)\ F_z\{J\}$$

and for $z = -i$

(II.5) $$(-\partial_o^2 - \Delta + m^2)\ [\delta/\delta J(x)]\ S\{J\} +$$

$$+ g[\delta^3/\delta J(x)^3]\ S\{J\} = J(x)\ S\{J\}.$$

In addition, $S\{0\} = 1$ and whatever else we may stipulate on the basis of the discussion in I.A. Especially, since the space - "time" domain is infinite, (5) integrates to

(II.6) $$[\delta/\delta J(x)]\ S\{J\} = \int G_o(x - y)\ dy \Big\{-g[\delta^3/\delta J(y)^3]\ \cdot$$

$$\cdot\ S\{J\} + J(y)\ S\{J\}\Big\}$$

where $G_o(x - y)$ is the Green's function (B.15) with $N = 1$ and $d = 4$.

Insertion of (I.27b) or (I.28) into (6) and comparing coefficients gives the infinite system of coupled integral equations for Euclidean Green's functions mentioned in the introduction. (5) is formally solved by

(II.7a) $$S\{J\} = N\{J\}/N\{0\}$$

where

(II.7b) $$N\{J\} = \int \exp\,[-\tfrac{1}{2}\,(\phi K \phi) - \tfrac{1}{4}\,g(\phi^4) + (J\phi)]\,\mathcal{D}_W(\phi)$$

provided a) it is defined, b) functional differentiation with respect to J can be carried out under the integral sign,

c) $\int [\delta/\delta\phi(x)] \exp[-\frac{1}{2}(\phi K\phi)-\frac{1}{4}g(\phi^4)+(J\phi)] \mathcal{D}_W(\phi) = 0.$

the abbreviations

$$(\text{II.8a}) \quad (\phi K\phi) \equiv (\phi(\overleftarrow{\partial}_o\overrightarrow{\partial}_o + \overleftarrow{\nabla}\cdot\overrightarrow{\nabla} + m^2)\phi) = $$

$$= \int dx[(\partial_o\phi(x))^2 + \vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x) + m^2\phi(x)^2],$$

$$(\text{II.8b}) \quad (\phi^4) \equiv \int \phi(x)^4 dx,$$

$$(\text{II.8c}) \quad (J\phi) \equiv \int dx J(x)\phi(x)$$

have been used, and $\mathcal{D}_W(\phi) \exp[-\frac{1}{2}(\phi K\phi)]$ is the measure differential (quasi-interval) of a suitably generalised Wiener integral with four-dimensional "time" The integral we shall actually use is the Hilbert space integral of Friedrichs and Shapiro.[16] Its definition and some of its properties are given in appendix A.

We shall find in V. that in this interpretation, and in any similar one[15] available in the literature, $N\{J\}$ in (7b) can only be assigned the value zero. Thus, in order to give (7) a meaning such that also properties b) and c) hold, we have to modify the integrals and, therefore, the model first, and can only investigate thereafter the possibility of returning to the unmodified model.

In order to assess the prospects for such a return, we briefly recapitulate the results of perturbation theoretical investigations[25] of the model (1). There, one has firstly to replace the A^4 term by a Wick product $:A^4:$, which amounts in (2) to replacing $A(x)^3$ by* $:A(x)^3: = A(x)^3 - 3G_o(0)A(x)$. (Actually, substituting $S(x,x)$ for $G_o(0)$ would appear more natural. We shall comment on this point in $\underline{V}$.) Since $G_o(0)$ is infinite, one should more precisely write

$$\text{(II.9)} \qquad :A(x)^3: = \lim_{\sup|\xi_i|\to 0} :A(x+\xi_1)A(x+\xi_2)A(x+\xi_3): \equiv$$

$$\equiv \lim_{\sup|\xi_i|\to 0} [A(x+\xi_1)A(x+\xi_2)A(x+\xi_3) -$$

$$- A(x+\xi_1)F_o(\xi_2-\xi_3) - A(x+\xi_2)F_o(\xi_1-\xi_3) -$$

$$- A(x+\xi_3)F_o(\xi_1-\xi_2)].$$

Here

$$F_o(\xi) = \frac{-i}{(2\pi)^4} \int dk^o d\vec{k} \exp[-ik^o x^o + i\vec{k}\vec{x}] \cdot$$

$$\cdot(-k_o^2 + \vec{k}^2 + m^2 - i0)^{-1} = \Delta_F(\xi)$$

and the three vectors ξ_i should have zero time component, $|\xi_i| \equiv |\vec{\xi}_i|$. With this modification, (2), (3) possess a perturbation theoretical solution for one, instead of three,

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*The model (1) is formally invariant under $A(x) \to -A(x)$. The modified model solved in III possesses the same invariance.

space dimensions. For two space dimensions, such a solution exists with redefined subtraction terms in (9), i.e., the mass renormalization is infinite. For three space dimensions, also the trilinear term in (9) must have a (ξ_1)-dependent factor, as given by Zimmermann,[10] and (3) must be reinterpreted, i.e., also amplitude - and coupling constant renormalization are infinite. In more space dimensions, the interaction term in (2) would have to be replaced by a formal polynomial of infinite order in the field operator.[26] These results, obtained in MQFT, hold of course also in EQFT, since the perturbation theoretical expressions, e.g., for Green's functions, allow explicit analytic continuation.

If the leading term in the interaction in (1) is of odd order, for instance of order three, similar results hold.[27] However, Baym[28] has shown that no space-translation invariant lowest energy state can then exist, what deprives also the corresponding EQFT of its basis. We shall find that we have to reject such models even in the modified Euclidean case discussed in III.

From (I.19) and (I.27a), we have

(II.10) $$S\{J\} = \langle T\, e^{-\int H\{J_\tau\}\, d\tau} \rangle .$$

(I.15) and the form of the Hamiltonian

(II.11) $$H = \int [\tfrac{1}{2}\dot{A}(\vec{x},0)^2 + \tfrac{1}{2}\vec{\nabla} A(\vec{x},0)\cdot\vec{\nabla} A(\vec{x},0) + \tfrac{g}{4} A(\vec{x},0)^4] + \text{const}$$

suggest to interpret (7) as the result of a diagonalization of the field operator in (10). This can indeed be justified.* In general, however, diagonalization of the field operator gives rise to dissimilar[1] formulas. In the regularized case discussed in III, diagonalization of the field operator only does not give a complete dynamical description, and there is no similarity between (10) and the function space integral that generalized (7b). In fact, since the classical Hamiltonian is then indefinite, it could anyway not be used to define a measure in function space.

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*The formal calculations that lead from (10), with (I.15), (11), and (13) to (7) are similar to those on the Feynman path integral [14], but involve integration over functions in three-space instead of ordinary integrals. This matter is implicit in vigorous form in IV.C.

[1] E.g., (I.14b) contains the Hamiltonian, the corresponding Feynman history integral [13], however, the Lagrangian.

III. Existence of the modified model

In this chapter we modify the model of II in such a manner that we can prove the existence of a solution. Two modifications are necessary: regularization, and introduction of a finite space-"time" volume: $\int dx \doteq V < \infty$.

III.A. Regularization

We use the method of Pais and Uhlenbeck[12] which consists of replacing the Klein-Gordon operator in (II.2) by a product of such operators:

$$(\partial_o^2 - \Delta + m^2) \longrightarrow \prod_{i=1}^{N} (\partial_o^2 - \Delta + m_i^2) \equiv \sum_{n=o}^{N} (\partial_o^2 - \Delta)^n s_{N-n}$$

with all m_i^2 positive. It was shown[12] that this replacement in (II.2), together with the corresponding commutation relations that replace (II.3), leads for $g = 0$ and $N \geq 2$ either to an indefinite energy operator, or necessitates the introduction of an indefinite metric, whereby the positivity of the eigenvalues of the energy operator can be maintained.

For $g > 0$ we show that if we suppose the theory to possess a vacuum $\rangle$ (i.e., a unique state of lowest energy eigenvalue) then the metric must be indefinite if $N \geq 2$.*

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*This could be shown by use of Baym's method [28], but the following one is even simpler here.

We introduce the function $\rho(\kappa^2)$, $\kappa \geq 0$, by[29]

(III.1) $$\langle A(x)A(y)\rangle = (2\pi)^{-3} \int d\kappa e^{-i\kappa(x-y)} \rho(\kappa^2)\theta(\kappa^0)\theta(\kappa^2) =$$

$$= \int_0^\infty d\kappa^2 \; \rho(\kappa^2) i\Delta_\kappa^{(+)}(x-y).$$

The canonical commutation relations* now entail for

(III.2a) $$\int_0^\infty d\kappa^2 (\kappa^2)^n \; \rho(\kappa^2) \equiv \sigma_n, \qquad n = 0 \ldots 2N - 1$$

The values

(III.2b) $$\sigma_n = 0, \qquad n = 0 \ldots N - 2,$$

$$\sigma_{n-1} = (-1)^{N-1},$$

$$\sigma_{N+L} = (-1)^{N-1} \text{ Det } \alpha_L, \; L = 0 \ldots N - 1,$$

where α_L is the $(L + 1)x(L + 1)$ matrix

$$(\alpha_L)_{ij} = s_{i+1-j}, \; s_o = 1, \; s_{-\kappa} = 0, \; \kappa = 1,2\ldots$$

Therefore, $\rho(\kappa^2)$ must for $N \geq 2$ change sign at least $N - 1$ times, as it does for $g = 0$ when $\rho(\kappa^2)$ is a sum of N δ - functions with alternating sign (if, for simplicity, we choose all masses different).[1] We shall see, however, in

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*They are simply related to those we derive in IV.D.

[1]The manifold of solutions of (2) is obtained by adding to the known $\rho(\kappa^2)$ for $g = 0$ an arbitrary function of κ^2 whose first 2N-1 moments in $(0,\infty)$ vanish.

VI that the metric we shall introduce in EQFT remains positive definite.

The ultimate intention is to let $m_1^2 \ldots m_N^2$ together with g vary in such a manner that in a certain limit positive definite metric is obtained also in MQFT. It is known that the renormalized perturbation theoretical expansion mentioned in II can so be obtained. There, particles of only one mass finally remain while the masses of all other particles, whose one-particle states have negative and positive norms, go to infinity as they do for $g = 0$. Gupta[30)] has pointed out that in perturbation theory the energy eigenvalues remain positive* under Pais-Uhlenbeck regularization, in the sense that the spectral representations of vacuum expectation values, generalizing the representation (1), are not affected as far as spectral support is concerned. Thus, the analytic continuation from MQFT to EQFT and vice versa might be possible before the regularization is removed, since it is possible in perturbation theory.

Other methods of regularization[31)] make use of nonlocal operator products for the interaction term.[1] For our purposes they seem less advantageous, however, since they do

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*The vacuum retains zero eigenvalue.

[1]At least in EQFT, Pais-Uhlenbeck regularization may by a linear transformation of variables be made to resemble such a nonlocal regularization.

not allow a Schrödinger equation to be derived unless one retains locality in at least one coordinate ("time"), which would require an unnatural breaking of Euclidean invariance. Finally, one may also, with a suppression of divergencies in mind, use in (II.1) a local nonpolynomial function of A as interaction term, as proposed by Fradkin[32)] and Efimov.[33)] Although our excessively strong regularization, to be specified later, would allow this, the question arises whether the regularization is removable such as to achieve a positive definite metric in MQFT without the theory becoming in the limit (if it exists) equivalent to the limit of one with polynomial interaction. However this may be, we have in mind mainly to investigate "orthodox" theories like quantum electrodynamics, which has a firm empirical and correspondence basis.

From now on we will understand $(\phi K \phi)$ in (II.7b) and (II.8a) in the sense of (B.1)* and (B.5), subsuming the "time" coordinate among the space coordinates.

III.B <u>Finite space "time" volume</u>

Essentially two possibilities offer themselves:

α) One may restrict the integration in (II.8b) to one over

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*The model we are actually treating admits "regularizing masses" also to coincide or to occur in pairs of conjugate complex masses. However, complex masses seem to rule out the existence of a corresponding MQFT, at least in perturbation theory.

a bounded domain Ω, with $\int_\Omega dx = V_\Omega < \infty$. More generally, one may replace in (II.8b)

$$\text{(III.3)} \qquad g(\phi^4) \longrightarrow \int g(x)\phi(x)^4 \, dx$$

where $g(x) = 1$ for $|x| < R$, $g(x) = 0$ for $|x| > R' \gg R$, and $g(x)$ infinitely differentiable, with ultimately $R \to \infty$. This "adiabatic switching-off" has the attractive feature of not causing "Stueckelberg divergencies,"[34] i.e., then to render $S\{J\}$ finite (at least in perturbation theory) no stronger regularization is needed than if Ω is the infinite space - "time." This method, however, precludes a Schroedinger equation to be derived. We therefore prefer the other possibility:

β) We integrate in all formulas (II.8) over a bounded domain $\hat{\Omega}$ only. We may then understand the "integral" in (II.7b) in several ways: (a) We "integrate" over all functions $\phi(x)$ for which $(\phi K \phi) < \infty$ and which satisfy inhomogeneous Dirichlet conditions

$$\underline{\vec{d}}\phi(x)|_{x \to s} = \underline{a}(s) \quad \text{on} \quad \partial\Omega$$

as defined in appendix B, esp. (B.6) (b) We "integrate" over all functions $\phi(x)$ for which $(\phi K \phi) < \infty$. (c) We "integrate" over all functions $\phi(x)$ for which $(\phi K \phi) < \infty$ and which satisfy incomplete Dirichlet conditions, i.e., Dirichlet data are prescribed on $(D) \subset (\partial\Omega)^N$ only. Clearly, (c) includes (b) and (a) as special cases.

If we extend the "partial integrability" postulated after (II.7b) as (c) to the property

$$\text{(III.4)} \quad \int \mathcal{D}_W(\phi)F(\phi) = \int \mathcal{D}_W(\phi + \phi_o)F(\phi + \phi_o) = = \int \mathcal{D}_W(\phi)F(\phi + \phi_o)$$

provided ϕ_o is chosen such that $\phi + \phi_o$ satisfies the same boundary conditions, if any, ϕ originally is supposed to satisfy, then we can for $g = 0$ "carry out" the integration (II.7b) at once, using (B.37) to obtain in case (a)

$$\text{(III.5a)} \quad N\{\underline{a},J\}_{g=o} = \int_{\underline{\vec{d}}\phi=\underline{a} \text{ on } \partial\Omega} \exp\left[-\tfrac{1}{2}(\phi K\phi) + (J\phi)\right] \mathcal{D}_W(\phi) =$$

$$= \exp\left[-\tfrac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a} - JG_D\overleftarrow{\underline{D}}\cdot\underline{a} + \tfrac{1}{2}(JG_DJ)\right]\cdot$$

$$\cdot \int_{\underline{\vec{d}}\rho=0 \text{ on } \partial\Omega} \exp\left[-\tfrac{1}{2}(\rho K\rho)\right] \mathcal{D}_W(\rho)$$

and (B.38) to obtain in case (b)

$$\text{(III.5b)} \quad N\{J\}_{g=o} = \int_{\phi \text{ unrestr.on } \partial\Omega} \exp\left[-\tfrac{1}{2}(\phi K\phi)+(J\phi)\right] \mathcal{D}_W(\phi) =$$

$$= \exp\left[\tfrac{1}{2}(JG_NJ)\right]\cdot \int_{\rho \text{ unrestr.on } \partial\Omega} \exp\left[-\tfrac{1}{2}(\rho K\rho)\right] \mathcal{D}_W(\rho),$$

where the remaining "integrals" are independent of the data J and $\underline{a}$ and eliminated in (II.7a). Case (c) gives rise to the generalization of (5a) with $\partial\Omega \to (D)$, $G_D \to G$, $\overrightarrow{\underline{D}} \to (D)\cdot\overrightarrow{\underline{D}}$, $\underline{G} \to (D)\cdot\overrightarrow{\underline{D}}\, G\, \overleftarrow{\underline{D}}\cdot(D)$. An obvious generalization of (4) now leads to

$$\text{(III.6a)} \qquad \int \mathcal{D}(\underline{a}) N\{\underline{a}, J\}_{g=0} \;\alpha\; N\{J\}_{g=0}$$

and

$$\delta\{\underline{a} - \underline{d}[\delta/\delta J]\}\; N\{J\}_{g=0} \;\alpha\; N\{\underline{a}, J\}_{g=0}$$

where for (6a) the formula (B.31) is needed while (6b) may be evaluated by using for the δ - functional either its Fourier representation

$$\text{(III.7a)} \qquad \delta\{\rho\} \;\alpha\; \int \mathcal{D}(\mu) \exp[i(\rho\mu)]$$

or the formula, with κ positive definite,

$$\text{(III.7b)} \qquad \delta\{\rho\} \;\alpha \lim_{\|\kappa^{-1}\| \to 0} \exp[-\tfrac{1}{2}(\rho\kappa\rho) + \tfrac{1}{2}\,\mathrm{Tr}\ln\kappa]$$

and (B.29) - (B.31). Extending in (6a) the integration, or in (6b) the δ - functional, only over (N) resp. (D) produces the formula for case (c), while in the most general case transitions between various sets (D) may be effected.

Clearly, (5) and the generalization to incomplete data are the (upon normalization) unique solutions of (II.5) for $g = 0$, with the differential operator replaced by K and the boundary condition

$$\underline{\vec{d}}\,[\delta/\delta J(x)]\ S\{J\}|_{x\to s} = \underline{a}(s)\ S\{J\}$$

on (D), whereas on $(\partial\Omega)^N - (D)$ homogeneous Neumann conditions apply. To introduce inhomogeneous Neumann conditions, the integrand in (5b) and in case (c) would have to be modified.

For $g > 0$, we have to give an explicit definition of the integral (II.7b) and to verify that it has the properties (b) and (c) required in II. To this end we transform (II.7b) formally into a Friedrichs-Shapiro (FS) integral by the substitutions

(a) $\phi = - G_D\underline{\overleftarrow{D}} \cdot \underline{a} + G_D J + H_D \Psi$

(b) $\phi = G_N J + H_N \Psi$

and the obvious generalization of (a) to incomplete data, where we use the integral operators defined in (B.47).

We shall for brevity only consider (a) further, since the treatment of the more general, incomplete case adds only notational complications. Therefore, we drop from now on the subscript $_D$.

If, in carrying out the substitution (a), we proceed as if $\vec{\underline{d}}\, H = 0$ on $\partial\Omega$ and $H^T K H = 1$, we arrive* at

$$N\{\underline{a}, J\} \equiv \int_{\vec{\underline{d}}\phi = \underline{a}} \exp[-\tfrac{1}{2}(\phi K \phi) - \tfrac{1}{4}\, g(\phi^4) +$$

$$+ \tfrac{3g}{2}\, G_0(0)(\phi^2) + (J\phi)]\,\mathcal{D}_W(\phi) =$$

$$\text{(III.8a)} \quad = \exp[-\tfrac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a} - JG\overleftarrow{\underline{D}}\cdot\underline{a} + \tfrac{1}{2}\,(JGJ)]\cdot$$

$$\cdot \int \exp[-\tfrac{1}{2}\,(\Psi\Psi) - \tfrac{1}{4} g((H\Psi + f)^4) +$$

$$+ \tfrac{3g}{2}\, G_0(0)((H\Psi + f)^2)]\,\mathcal{D}_{FS}(\Psi)$$

with the abbreviation

$$\text{(III.8b)} \quad GJ - G\overleftarrow{\underline{D}}\cdot\underline{a} = f,$$

where we use for the FS integral the notation introduced at the end of appendix A. In (8a) we have inserted the counterterm[1] commented upon in connection with (II.9). From

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*The "Wiener integral" in (8a) has only symbolic meaning. The question of its direct definition will be briefly taken up in IV.D.

[1]This term has no bearing on all our discussions except that it does influence the inequalities derived in V in a characteristic manner.

(B.15) we have $G_o(0) < \infty$ if $N \geq [\frac{1}{2}d] + 1$. We shall assume in the following this and the cone condition[35] on $\partial\Omega$ to be satisfied, such that also $G_N(x,x) < \infty$ for all $x \in \bar{\Omega}$.

As discussed in appendix B, the choice of $(\phi K \phi)$ in (8a), and therefore of $\underline{G}$ and $\underline{D}$, is not unique. (B9) and (B.45) show, however, that in (8a) only the factor $\exp[-\frac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a}]$ is affected, which corresponds to a particularly simple* canonical transformation in more elementary situations.

III.C. Existence and uniform convergence of the FS integral (8)

We now prove that for certain classes of J and $\underline{a}$ the FS integral (8) exists, and is even uniformly convergent in the sense of appendix A, if

$$\text{(III.9)} \qquad 0 \leq g < A < \infty, \qquad \underset{x}{\text{g.l.b.}}\; G_N(x,x) = B < \infty,$$

$$(JGJ) < C < \infty, \qquad \underline{a}\cdot\underline{G}\cdot\underline{a} < D < \infty.$$

We first note that with

$$\text{(III.10)} \quad N_o\{\underline{a},J\} = \exp[-\tfrac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a} - JG\overleftarrow{\underline{D}}\cdot\underline{a} + \tfrac{1}{2}(JGJ)]$$

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*The change under incomplete Dirichlet conditions is slightly less trivial since then also the kernels G and H are changed.

the integral $N\{\underline{a},J\}/N_o\{\underline{a},J\} \equiv Z\{\underline{a},J\} = Z\{f\}$ depends on f only. Due to the last three conditions in (9) f is uniformly bounded throughout Ω and, in fact, uniformly continuous throughout $\Omega' \subset \Omega$ where Ω' lies strictly inside Ω and $\int_{\Omega-\Omega'} dx$ is arbitrarily small. This follows from

(III.11a) $|\,{}_x|GJ|^2 \leq G(x,x)(JGJ) < G_o(0)C < BC,$

(III.11b) $|\,{}_x|GJ- \;{}_y|G|^2 < [G(x,x)+G(y,y)-2G(x,y)](JGJ),$

(III.12a) $|\,{}_x|G\overleftarrow{\underline{D}}\cdot\underline{a}|^2 \leq {}_x|G\overleftarrow{\underline{D}}\cdot\underline{G}^{-1}\cdot\overrightarrow{\underline{D}}\,G|_x \;\; \underline{a}\cdot\underline{G}\cdot\underline{a} =$

$$= [G_N(x,x) - G(x,x)]\;\underline{a}\cdot\underline{G}\cdot\underline{a} < BD,$$

and

(III.12b) $|\,{}_x|G\overleftarrow{\underline{D}}\cdot\,\underline{a} - \;{}_y|G\overleftarrow{\underline{D}}\cdot\,\underline{a}|^2 \leq [G_N(x,x) +$

$$+ G_N(y,y) - 2G_N(x,y) - G(x,x) - G(y,y) +$$

$$+ 2G(x,y)]\;\underline{a}\cdot\underline{G}\cdot\underline{a}$$

The brackets in the last expressions in (11b) and (12b) differ only by analytic functions, with uniformly bounded derivatives in Ω', from the same expressions with G_N

and G replaced by G_o from (B.15), which either vanish identically or vanish if $x - y \longrightarrow 0$ since $G_o(0) - G_o(x-y)$ then vanishes.

Next we remark that the argument used in the proof of lemma A.4 also gives uniform β-convergence of the exponential integral if we show uniform α-convergence of the exponent and give an apriori upper bound for the exponential integral with g replaced by $\alpha\beta/(\alpha-\beta)$. Such bound is immediate from

$$\text{(III.13)}\quad \exp[-\tfrac{1}{4}\, g((H\Psi+f)^4)+\tfrac{1}{2}\, 3g\, G_o(0)((H\Psi+f)^2)] \leq$$

$$\leq \exp\ [\tfrac{9g}{4}\, G_o(0)^2\, V_\Omega].$$

We now show the uniform convergence for the exponent (without the measure term, of course), in (8a) with $\alpha = 2$. We have with (B.46)

$$\text{(III.14)}\quad |\ {}_x|HP_nH^T|_y\ |^2 \leq (\ {}_x|HP_nH^T|_x\ )(\ {}_y|HP_nH^T|_y\ ) <$$

$$< G(x,x)\ G(y,y) < G_o(0)^2 < B^2$$

and

$$\text{(III.15)}\quad |\ {}_x|HP_nH^T|_y\ -\ {}_x|HQ_mH^T|_y\ |\ =$$

$$= |\ {}_x|H(1-P_n)H^T|_y\ -\ {}_x|H(1-Q_m)H^T|_y\ |\ \leq$$

$$\leq [\ {}_x|H(1-P_n)H^T|_x\ ]^{1/2}\ [\ {}_y|H(1-P_n)H^T|_y\ ]^{1/2} +$$

$$+ [\ {}_x|H(1-Q_m)H|_x\ ]^{1/2}\ [\ {}_y|H(1-Q_m)H|_y\ ]^{1/2}.$$

The expression

$$\text{(III.16)} \qquad {}_x|H(1-P_n)H^T|_x$$

is uniformly continuous in Ω' since

$$|\ {}_x|H(1-P_n)H^T|_x - {}_y|H(1-P_n)H^T|_y\ |^2 =$$

$$= |\ (\ {}_x|H + {}_y|H)(1-P_n)(H^T|_x - H^T|_y\ )\ |^2 \leq$$

$$\leq [\ (\ {}_x|H + {}_y|H)(1-P_n)(H^T|_x + H^T|_y\ )\ ]\ \cdot$$

$$\cdot\ [\ (\ {}_x|H - {}_y|H)(H^T|_x - H^T|_y\ )\ ] <$$

$$< 4\ G_o(0)\ [G(x,x) + G(y,y) - 2\ G(x,y)],$$

and we showed that the last bracket is continuous in Ω and thus uniformly continuous in Ω'. Therefore, it suffices to make (16), and the same expression with $P_n \to Q_m$, smaller than ε at a finite number of points in Ω' to have it smaller than 4ε throughout Ω'. This can always be achieved by choosing n and m large enough independently of A, C, or D in (9). It now easily follows from earlier remarks and (A.3) that (11a), (12a), (14), and our result about (15) suffice for uniform convergence of the integral $Z\,f$ in (8), if (9) holds, since contributions from $\Omega - \Omega'$ can be made arbitrarily small due to the boundedness of all integrands.

The same argument we just used may also be employed to prove the uniform convergence of derivatives of $Z\,\underline{a},J$ with respect to $\underline{a}$, J, and g. It need first be made clear in which sense these derivatives are to be taken. (9) shows that the derivative with respect to g must be a right-derivative at $g = 0$. The other derivatives, being functional ones, must conform with the conditions mentioned in connection with (A.1). It is not difficult to see that the limit (A.1) exists if we construct

$$\text{(III.17a)} \int dx\, J'(x)[\delta/\delta J(x)]\, Z\,\underline{a},J\ ,\ (J'GJ') < \infty$$

and

$$\text{(III.17b)} \int dO(s)\, \underline{a}'(s)[\delta/\delta\underline{a}(s)]\, Z\,\underline{a},J\ ,\ |\underline{a}'|_{N-1/2} < \infty$$

with the functional derivatives carried out under the integral sign. Since the FS integrals obtained in this way are easily seen to converge uniformly due to lemma A.8, the derivatives in (17) may be interpreted as genuine derivatives* of $Z\ \underline{a},J$. One also proves with the Schwartz inequality, (9), (11a), (12a), (13), and (14) that (17a) and (17b) are bounded linear forms on the Hilbert spaces indicated. Thus $[\delta/\delta\underline{a}(s)]\ Z\ \underline{a},J$ is itself an element of the Hilbert space with norm $|\ |_{-N+1/2}$. To obtain a similar result for (17a), we recall that $Z\ \underline{a},J$ depends only on f, such that

$$\text{(III.18)} \quad \overrightarrow{\underline{D}}\,[\delta/\delta J(x)]\ Z\ \underline{a},J\ |_{x\to s} + [\delta/\delta\underline{a}(s)]\ Z\ \underline{a},J = 0.$$

From (17b) we have $|\overrightarrow{\underline{D}}[\delta/\delta J]\ Z|_{-N+1/2} < \infty$. This, (17a), and theorem A of appendix B give

$$\text{(III.19)} \quad \|\ [\delta/\delta J]Z\ \underline{a},J\ \|_N < \infty, \quad |\overrightarrow{\underline{d}}\ [\delta/\delta J]|_{N-1/2} < \infty,$$

whereby we have used that the J' with $(J'G_N J') < \infty$ are

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*We shall see later that $Z\ \underline{a},J$ is entire analytic in J, and in our model incidentally also in $\underline{a}$, such that for the definition of the functional derivative also the method of Donsker and Lions [36] could be employed.

a subclass of those with $(J'GJ') < \infty$. Multiple derivatives are easily seen to belong to the product spaces corresponding to (17b) and (19).

Actually, (18) is true only if we consider $Z\ \underline{a}, J$ as a functional of f. Also $N_0\ \underline{a}\ , J\ = \exp\ [fKf) - f\overleftarrow{\underline{d}}\cdot\overrightarrow{\underline{D}f}]$ can so be written. However, we shall later break $N\ \underline{a}, J$ up and will have to consider the $\underline{a}$- and J- dependence separately. Then, since with (B.31)

$$|JG\overleftarrow{\underline{D}}\cdot\underline{a}| \leq [(JG_N J) - (JGJ)]^{1/2}\ [\underline{a}\cdot\underline{G}\cdot\underline{a}]^{1/2},$$

for the existence of $N_0\ \underline{a}, J$ the requirement $(JGJ) < C$ must be strengthened to

(III.20) $$(JG_N J) < \infty,$$

and we may only state, as follows from (10), (17b), and (18),

(III.21) $$|\overrightarrow{\underline{D}}\ [\delta/\delta J]\ N + [\delta/\delta\underline{a}]\ N\,|_{-N+1/2} = 0.$$

If incomplete Dirichlet data are given, $[\delta/\delta\underline{a}(s)]N\ \underline{a}, J\ \equiv 0$ on $(\partial\Omega)^N - (D)$. (21) shows that there homogeneous Neumann conditions are satisfied, as was the case for $g = 0$ discussed earlier.

We finally remark that the uniform convergence proofs given in this section allow to evaluate the FS integrals in question to any prescribed accuracy.

III.D. Verification of functional differential equations

We shall now prove that $N\{\underline{a},J\}$, defined by (8), satisfies

(III.22a) $\| \overrightarrow{K} [\delta/\delta J]\, N + g[\delta^3/\delta J^3]\, N -$

$- 3g\, G_o(0)\, [\delta/\delta J]\, N - JN\|_{-N} = 0$

where

$\|J\|_{-N} = (JGJ),$

and

(III.22b) $|\overrightarrow{\underline{d}}\, [\delta/\delta J]\, N - \underline{a}\, N|_{N-1/2} = 0,$

or the integral relation

(III.23) $\| [\delta/\delta J]\, N + g\, G[\delta^3/\delta J^3]\, N -$

$- 3g\, G_o(0)\, G[\delta/\delta J]\, N - GJN + G\overleftarrow{\underline{D}} \cdot \underline{a}\, N\|_N = 0$

which is equivalent to (22) because of (B.43) and should be compared with (II.6).

First we note that the integral $Z\{f\}$ of (8) satisfies

(III.24) $\| G\, [\delta/\delta f]\, Z\, f \;- \int \exp[-\tfrac{1}{2}(\Psi\Psi)] H[\delta/\delta\Psi] \exp[-\tfrac{1}{4}\, g((H\Psi +$

$+ f)^4) + \tfrac{1}{2}\, 3g\, G_o(0)((H\Psi+f)^2)]\, \mathcal{D}_{FS}(\Psi)\|_N = 0$

due to (17), (18), and theorem A of appendix B. Also, from (17a) and its iterations, with $J_i^{'}(x) = \delta(x-x_i)$

(III.25) $$\prod_{i=1}^{n} ({}_{x_i}|G \frac{\delta}{\delta f}) Z\{f\} =$$

$$= \int \exp[-\tfrac{1}{2}(\Psi\Psi)] \prod_{i=1}^{n} ({}_{x_i}|H[\delta/\delta_\Psi])\exp[-\tfrac{1}{4}g((H_\Psi+f)^4) +$$

$$+ \tfrac{1}{2}\, 3g\ G_o(0)\ ((H_\Psi+f)^2)]\ \tilde{\mathcal{D}}_{FS}(\Psi)$$

which is a continuous function of the x_i. All approximations to the right-hand side of (25) by finite-dimensional integrals allow partial integration, and therefore that side is also equal to

(III.26) $$\int \exp [-\tfrac{1}{4}g((H\Psi+f)^4) + \tfrac{1}{2}\, 3g\ G_o(0)((H\Psi + f)^2)] \cdot$$

$$\cdot \prod_{i=1}^{n} (-\ {}_{x_i}|H[\delta/\delta\Psi])\ \exp[-\tfrac{1}{2}(\Psi\Psi)]\tilde{\mathcal{D}}_{FS}(\Psi).$$

Using this for $n = 1, 2,$ and 3, we may construct the integral in (24) in terms of derivatives with respect to f. The result* is $\| G[\delta/\delta f]Z\{f\} + g\,G[(G[\delta/\delta f] + f)^3 -$

---

*Whereby the formulae $-H[\delta/\delta\Psi]\exp[-\tfrac{1}{2}(\Psi\Psi) = \exp[-\tfrac{1}{2}(\Psi\Psi)](-H[\delta/\delta\Psi] + H\Psi)$ and $\exp[\eta(H\Psi+f)] = \exp[\eta(H\Psi - H[\delta/\delta\Psi] + f]\cdot \exp[\tfrac{1}{2}\eta G\eta] \sim \exp[\eta f]\exp[\eta G[\delta/\delta f]]\cdot \exp[\tfrac{1}{2}\eta G\eta] = \exp[\eta(f + G[\delta/\delta f])]$ are practical.

$$- 3\, G_o(0)\, (G[\delta/\delta f] + f)]\, Z\{f\} \|_N = 0.$$

Noting that from (10)

$$(G[\delta/\delta f] + f)\, N_o\{\underline{a}, J\}^{-1} =$$

$$= ([\delta/\delta J] + GJ - G\overleftarrow{\underline{D}} \cdot \underline{a})\, N_o\{\underline{a}, J\}^{-1} =$$

$$= N_o\{\underline{a}, J\}^{-1}\, [\delta/\delta f]$$

we obtain (23).

We finally derive another formula for $N\{\underline{a}, J\}$. We substitute in (8) $\Psi + H^T J \longrightarrow \Psi$. This substitution is admissable as it is for all finite-dimensional approximations to that integral, since $H^T J \varepsilon L_2$. The result is

$$\text{(III.27)}\quad N\{\underline{a}, J\} = \exp\, [- \tfrac{1}{2}\, \underline{a} \cdot \underline{G} \cdot \underline{a} - JG\overleftarrow{\underline{D}} \cdot \underline{a}] \cdot$$

$$\cdot \int \exp\, [- \tfrac{1}{2}\, (\Psi\Psi) - \tfrac{1}{4}\, g((H\Psi - G\overleftarrow{\underline{D}} \cdot \underline{a})^4) +$$

$$+ \tfrac{1}{2}\, 3g\, G_o(0)\, ((H\Psi - G\overleftarrow{\underline{D}} \cdot \underline{a})^2) + (JH\Psi)]\, \mathcal{D}_{FS}(\Psi).$$

With this representation* of $N\{\underline{a}, J\}$ it is formally quite simple to prove that the function whose norm is taken in (23)

---

*A substitution in (27) that would eliminate the $\underline{a}$ - dependence from the interaction terms is not possible, as the requisite shift term is not in general square integrable.

vanishes. However, the convergence of the integral in (24) and of its derivatives is not uniform with respect to J^*. (27) is useful nevertheless, as we shall see in IV.A.

IV. General properties of the solution

In this section we derive a number of properties of the integral (III.8) or (III.27), which we call "general" since they do not depend on the precise form of the interaction term. Although we give for definiteness the proofs, as we did in III, only for the special integral with polynomial interaction, they are easily seen to apply also to more general integrals such as for the models referred to earlier.[32)33)] We do not intend here to describe very general classes, but merely remark that if f is real, in (8) only real $H\Psi + f$ do matter, that for convergence of the integral the existence of even the first functional derivative of the interaction term is not necessary, but that existence of an apriori upper bound such as we obtained by use of (III.13) seems imperative in our method.

---

*The approximations to the integral in (27) are the same as those to the integral in (8) except for a factor $\exp [\frac{1}{2}(JHP_nH^TJ)]$. This factor does not converge uniformly if J is subject only to $(JGJ) < C$, which is the space natural to our problem.

We showed already that $N\{\underline{a},J\}$ possesses derivatives of all orders with respect to $\underline{a}$ and J, in the sense (III.17), (III.19), if (III.9) and (III.20) hold.

IV.A. Entire analyticity in J

We show that under conditions (III.9), with $(JGJ) < C$ replaced by $(\bar{J}GJ) < \infty$, $Z\{\underline{a},J\}$ has a permanently converging Volterra series expansion in J, admitting complex J also. With lemma A.4 it is easily seen that the integral in (III.27) converges for $(\bar{J}GJ) < \infty$, and that this is true also for all terms in its J-expansion. For the absolute value of the term of nth order we have with (III.13) the bound

$$|(\exp[\tfrac{1}{2}(JGJ)]\, Z\{\underline{a},J\})_n| \leq$$

$$\leq (n!)^{-1} \exp[\tfrac{1}{4}\, 9g\, G_o(0)^2 V_\Omega] \cdot$$

$$\cdot \int \exp[-\tfrac{1}{2}(\Psi\Psi)]\, |(JH\Psi)|^n \mathcal{D}_{FS}(\Psi)$$

and here, with (A.3) and the Minkowski inequality,

$$\int \exp[-\tfrac{1}{2}(\Psi\Psi)]\, |JH\Psi|^n \mathcal{D}_{FS}(\Psi) =$$

$$= \int \exp[-\tfrac{1}{2}(\Psi\Psi)[\mathrm{Re}JH\Psi)^2 + (\mathrm{Im}JH\Psi)^2]^{n/2} \mathcal{D}_{FS}(\Psi) \leq$$

$$\leq \left\{\int \exp[-\tfrac{1}{2}(\Psi\Psi)][(\mathrm{Re}J\ H\Psi)^2 + (\mathrm{Im}J\ H\Psi)^2]^n \mathcal{D}_{FS}(\Psi)\right\}^{1/2} \leq$$

$$\leq \left\{[\int \exp[-\tfrac{1}{2}(\Psi\Psi)](\mathrm{Re}J\ H\Psi)^{2n}\mathcal{D}_{FS}(\Psi)]^{1/n} + \right.$$

$$\left. + [\int \exp[-\tfrac{1}{2}(\Psi\Psi)](\mathrm{Im}J\ H\Psi)^{2n}\mathcal{D}_{FS}(\Psi)]^{1/n}\right\}^{n/2} =$$

$$= (JGJ)^{(1/2)n}[(n!)^{-1}2^{-n}(2n)!]^{1/2}.$$

The sum of these bounds behaves like exp $(\bar{J}GJ)$, which proves the statement. For the existence of the factor of the integral in (III.27) we have again the slightly stronger condition $(\bar{J}G_NJ) < \infty$ corresponding to (III.20).

A functional that admits a permanently converging Volterra series expansion for a class of functions, $(\bar{J}G_NJ) < \infty$ in our case, we call an entire analytic functional. We may call $N\{\underline{a},J\}$ entire analytic of order at most two, since $N\{\underline{a},Z_1J, + \ldots + Z_nJ_n\}$ is entire analytic of order at most two in each of the complex variables $Z_1 \ldots Z_n$ if $(J_iG_NJ_i) < \infty$, $i = 1 \ldots n$. We shall determine in V the order of $N\{\underline{a},J\}$ for our model more precisely.

Under the conditions (III.9), with $(\bar{J}G_NJ) < \infty$ instead of $(JGJ) < C$, we have

(IV.1a) $\quad |N\{\underline{a},J\}| \leq N\{\underline{a},\mathrm{Re}J\}$

and for real J

(IV.1b) $N\{\underline{a},J\} > 0$

with positive lower bound derived in V. (1) permits to define $\ln N\{\underline{a},J\}$ and, more suitably, $\ln(N\{\underline{a},J\}/N\{\underline{a},0\})$ or $\ln(N\{\underline{a},J\}/N\{0,0\})$. The latter are generating functionals of the truncated Euclidean Green's functions of the modified model, where in the second case Green's functions that have only $\underline{a}$-arguments occur in addition to those that have only J- or J-and $\underline{a}$-arguments.* The classical analoga of these functionals are discussed in appendix D.

IV.B. Positive definiteness and uniqueness

From (III.27), the definition of the FS integral, and the preceding section follows immediately that for k complex functions J_i, $i = 1 \dots k$, with $(J_i G_N J_i) < \infty$, all i,

$$\text{(IV.2)} \qquad \sum_{i,j}^{k} \overline{C}_i C_j N\{\underline{a}, \overline{J}_i + J_j\} \geq 0.$$

For all J_i imaginary we have positive definiteness of the functional in the usual restricted sense. Since continuity of $N\{\underline{a},J\}$ in J is implied by the differentiability (III.17a), $N\{\underline{a},J\}/N\{\underline{a},0\}$ for imaginary J is the functional Fourier transform of a nonnegative measure.[37)38)] To be on Hilbert space, we may instead of $N\{\underline{a},J\}$ consider

---

* The relation between the arguments is given by (III.18) and (III.21)

$$N'\{\underline{a}, J\} = N\{\underline{a}, J\} \exp[\tfrac{1}{2}\, \underline{a}\cdot\underline{G}\cdot\underline{a} + JG\overleftarrow{\underline{D}}\cdot\underline{a}]$$

and take as variable $H^T J$ instead of J. Using the Fourier representation for $N'\{\underline{a}, J\}$ in (III.23), carrying out the differentiations under the integral sign, and performing a partial integration gives a functional differential equation for the measure, whose solution is (III.27).

We will not attempt here to justify the admission of these operations. We know that they are permitted if we use the more complicated integral (III.8) and observe (III.9) and (III.20), and that they are permitted, under the same conditions, for (III.27) if we relax the FS invariance requirement somewhat*, since then uniform convergence with respect to J can again be achieved. In the latter case, however, uniqueness must rely on the aposteriori observation that the integrand obtained actually is FS invariant. In any case, the solution of (III.23) with J imaginary is unique if we supplement positive definiteness by the mentioned

---

*Note that for (III.8) or (III.27) to solve (III.23) it is not necessary that the integrands be invariant, or even semi-invariant [39], in the sense of Friedrichs and Shapiro. Lonvergence for a special sequence of projections would suffice. What is then lost in more general cases is uniqueness, invariance under symmetry operations the symmetry of Ω (e.g., spherical) may admit, and total additivity. The last two properties are not used in our context.

properties of the Fourier representation it implies.

It is likely that (2), with J complex instead of imaginary, results in the existence of a Fourier-Laplace representation (III.27). For uniqueness of the solution of (III.23) this more general setting is not required.

For J_i real (2) gives for $K = 2$ conveseity of $\ln N\{\underline{a}, J\}$ in J, wherefrom, by use of (I.32a), (I.26) follows. (2) for $K \geq 3$ seems not to lead to further restrictions for $E_0\{J\}$, however.

We cannot show that (III.23) has any solution other than (III.8) or (III.27) that the uniqueness problem might be less trivial for $g > 0$ than it is for $g = 0$, however, we can try to illustrate by the following example: reduce the number of dimensions from four to zero. Then (III.23) becomes the differential equation for a function of one variable.

$$N'(x) + g\,N'''(x) - 3g\,N'(x) = x\,N(x).$$

The solution is

$$N(x) = \int dz\,\exp[zx - \tfrac{1}{2}\,z^2 - \tfrac{1}{4}\,gz^4 + \tfrac{1}{2}\,3gz^2].$$

It is not unique since in the complex z-plane three linearly independent integration paths may be chosen, corresponding to the four allowed directions $z \rightarrow \pm\,\infty, \pm\,i\,\infty$. Only the path

$-\infty \ldots +\infty$ gives a function that satisfies the corresponding specialization of (2).

We are actually interested less in the solution of (III.23) than in the solution of the corresponding equation in infinite space, which resembles (II.6). It is not unlikely that then more than one solution exists even if (2) is imposed, while (III.23) with (2) has only one solution. A way to generate additional solutions of (III.22a) would be to let $\Omega \to \infty$ but to let at the same time the Dirichlet data $\underline{a}$ grow exponentially. In fact, (II.6) is not the only solution of (II.5), since on the right hand side a term $h(x)\,S\,J$ may be added, with $(-\partial_0^2 - \Delta + m^2)h(x) = 0$. For $g = 0$, (II.6) thus modified can be immediately solved and exhibits a loss of the nongeometric formal symmetry $A(x) \to -A(x)$ of (II.2), (II.3). However, with h nonzero also Euclidean invariance is lost. For $g \neq 0$, speculations[40)] suggest that for $\Omega = \infty$ nongeometric formal symmetries might be broken without loss of Euclidean invariance.

The discussion so far, based on (III.23), (2), and $N\,\underline{a},J\,/N\,\underline{a},0$, ignores the merely $\underline{a}$-dependent factor in (III.8a) or (III.27). It is specified by requiring (III.21) to hold in addition.*

---

*The compatibility of (21) with (23) follows from the observation that (23) can be written in terms of f of (III.8b), while (21) requires $N\,\underline{a},J$ to depend only on the combination f of $\underline{a}$ and J, and can e.g. be integrated with (A.2).

IV.C. The semigroup property

In this section we prove that $N\{\underline{a}, J\}$ has the property analogous to the semigroup property of fundamental solutions of the heat equation, or to the group property of fundamental solutions of the Schrodinger equation.

Let Ω be divided by a hypersurface into two subdomains Ω_1 and Ω_2 such that the cone condition remains satisfied throughout. We use the notations introduced in appendix B after (B.49b). Let $N_{1+2}\{\underline{a}_1, \underline{a}_2, J_1, J_2\}$ denote $N\{\underline{a}, J\}$, calculated for $\Omega_1 + \Omega_2$, with Dirichlet data $\underline{a}_{1,2}$ on $S_{1,2}$ and source function $J_{1,2}$ in $\Omega_{1,2}$. Let $N_1\{\underline{a}_1, \underline{a}_s, J_1\}$ and $N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\}$ denote the corresponding functionals for Ω_1 and Ω_2, with Dirichlet data $\underline{a}_s$ resp. $\underline{E}\cdot\underline{a}_s$ on S. We shall prove

$$(IV.3) \qquad N_{1+2}\{\underline{a}_1, \underline{a}_2, J_1, J_2\} =$$

$$= C_{12} \int N_1\{\underline{a}_1, \underline{a}_s, J_1\} \; N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\} \mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s)$$

with $\underline{F}$ from (B.55), (B.49), and c_{12} a finite normalization factor.

We first show that the integral on the right hand side of (3) exists, and converges uniformly if

$$\text{(IV.4)} \qquad 0 \leq g < A, \ \underset{x \varepsilon \Omega_{1,2}}{\text{l.u.b.}} \ G_{1,2N}(x,x) = B_{1,2} < \infty,$$

$$(J_{1,2}G_{1,2}J_{1,2}) < C_{1,2} < \infty,$$

$$\underline{a}_{1,2}\cdot(s_{1,2})\cdot\underline{G}_{1,2}\cdot(s_{1,2})\cdot\underline{a}_{1,2} < D_{1,2} < \infty.$$

With (III.8) and (B.55) we have to consider*

$$\text{(IV.5)} \qquad \int \exp\Big\{- \underline{a}_s\cdot(s)\cdot\underline{G}_1\cdot(s_1)\cdot\underline{a}_1 -$$

$$- \underline{a}_s\cdot\underline{E}\cdot(s)\cdot\underline{G}_2\cdot(s_2)\cdot\underline{a}_2 +$$

$$+ \frac{1}{2}\sum_{i=1,2} [- \underline{a}_i\cdot(s_i)\cdot\underline{G}_i\cdot(s_i)\cdot\underline{a}_i -$$

$$- 2J_iG_i^{(i)}\overleftarrow{\underline{D}}\cdot(s_i)\cdot\underline{a}_i + J_iG_iJ_i] - J_1G_1^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot\underline{a}_s -$$

$$- J_2G_2^{(2)}\overleftarrow{\underline{D}}\cdot(s)\cdot\underline{E}\cdot\underline{a}_s\Big\} \ Z_1\Big\{\underline{a}_1,\underline{a}_s,J_1\Big\}\cdot$$

$$\cdot \ Z_2\Big\{\underline{a}_2,E\cdot\underline{a}_s,J_2\Big\} \ \exp[- \tfrac{1}{2}\underline{a}_s\cdot\underline{H}^{-1}\underline{a}_s]\mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s).$$

---

*It is easily verified that for $g = 0$, i.e. $Z_1 = Z_2 = 1$, the evaluation of (5) as Gaussian integral leads, with (B.52) - (B.55) and formulas derived from these, to (3) with $C_{12} = 1$.

We know from III.C that $Z_{1,2}$ exist and are bounded by $\exp[\frac{1}{4} 9g\, G_o(0)^2 V_{\Omega_{1,2}}]$, if $(J_{1,2} G_{1,2} J_{1,2}) < \infty$ and

$$[\underline{a}_{1,2}\cdot(s_{1,2}) + \underline{a}_s \cdot \{(s), \underline{E}\cdot(s)\}]\cdot \underline{G}_{1,2}\cdot$$

$$\cdot\ [(s_{1,2})\cdot \underline{a}_{1,2} + \{(s),(s)\cdot\underline{E}\}\cdot\underline{a}_s] < \infty.$$

Since the other factors in the integrand in (5) are trivial, by lemma A.3 for the existence of (5) only the FS invariance, with $\alpha > 1$, of Z_1 and Z_2 need be shown. Since these are bounded, by lemma A.2 the choice $\alpha = 1$ suffices. With the abbreviations

$$\underline{F}^{-1}\cdot P_n \cdot \underline{F}\cdot \underline{a}_s \equiv \underline{a}_n, \quad \underline{F}^{-1}\cdot Q_m \cdot \underline{F}\cdot \underline{a}_s \equiv \underline{a}_m,$$

$$-\ G_1 \overleftarrow{\underline{D}} \cdot [(s)\cdot \underline{a}_{n,m} + (s_1)\cdot \underline{a}_1] \equiv a_n$$

$$a_{n,m} + H_1 \psi \equiv f_{n,m}$$

we have to show

$$\text{(IV.6)} \qquad \int |Z_1\{\underline{a}, \underline{a}_n, J_1\} - Z_1\{\underline{a}_1, \underline{a}_m, J_1\}|\cdot$$

$$\cdot\ \exp[-\tfrac{1}{2}\, \underline{a}_s \cdot \underline{H}^{-1}\cdot \underline{a}_s]\, \mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s)$$

$$\longrightarrow 0 \quad \text{for} \quad n,m \longrightarrow \infty .$$

With (III.27), (III.13), and the Schwarz inequality we have

$$\text{(IV.7)}\qquad \exp\,[\tfrac{1}{2}\,J_1G_1J_1]\,|Z_1\{\cdots \underline{a}_n \cdots\} - Z_1\{\cdots \underline{a}_m \cdots\}| \leq$$

$$\leq \exp\,[\tfrac{1}{4}\,9g\;G_o(0)^2V_\Omega]\cdot\int \exp\,[-\tfrac{1}{2}(\Psi\Psi)+(J_1H_1\Psi)]\cdot$$

$$\cdot\;|\{-\tfrac{1}{4}g\;([f_n-f_m][f_n^3+f_n^2f_m+f_nf_m^2+f_m^3])\;+$$

$$+\;\tfrac{1}{2}\,3g\;G_o(0)\;([f_n-f_m][f_n+f_m])\}|\,\mathcal{D}_{FS}(\Psi)\;\leq$$

$$\leq [(a_n-a_m)^2]^{1/2}\;\exp[\tfrac{1}{4}\;9g\;G_o(0)^2V_\Omega + (J_1G_1J_1)]\cdot$$

$$\cdot\;\{\int \exp\,[-\tfrac{1}{2}(\Psi\Psi)]((-\tfrac{g}{4}\;[f_n^3+\ldots+f_m^3]\;+$$

$$+\;\tfrac{1}{2}\,3g\;G_o(0)\;[f_n+f_m])^2)\mathcal{D}_{FS}(\Psi)\}^{1/2}.$$

With (A.3) the last integral can be obtained exactly, and is, after repeated use of the Hölder inequality, found to be bounded by

$$A + B(a_n^6 + a_m^6)$$

where A and B are finite constants. Using this in (7) and inserting into (6), we find, using the Schwarz inequality once more, that (6) requires

$$\text{(IV.8)} \qquad \int_{\Omega_1} {}_x|G_1\overleftarrow{\underline{D}}\cdot\underline{F}^{-1}\cdot(P_n-Q_m)^2\cdot\underline{F}^{T-1}\cdot\overrightarrow{\underline{D}}\,G_1|_x\,dx$$

$$\longrightarrow 0 \quad \text{for} \quad n,m \longrightarrow \infty,$$

since with (4) all other factors are uniformly bounded.

The integrand in (8) is an analytic function of x, and bounded by

$$4 \operatorname*{Max}_x \; {}_x|G_1\overleftarrow{\underline{D}} \cdot \underline{H}\cdot\overrightarrow{\underline{D}}\,G_1|_x =$$

$$= 4 \operatorname*{Max}_x \,(G(x,x) - G_1(x,x) < 4\; G_o(0)$$

with use of (B.52a). Introducing, as in III.C, a domain Ω_1' strictly inside Ω_1, with $V_{\Omega_1} - V_{\Omega_1'}$ sufficiently small, it suffices to show that the integrand of (8) can be made arbitrarily small at a finite number of points inside Ω_1'. Since

$$(P_n - Q_m)^2 = P_n(1 - Q_m) + (1 - P_n)Q_m$$

it suffices to have

$${}_x|G_1\overleftarrow{\underline{D}}\cdot\underline{F}^{-1}\cdot(1 - P_n)\cdot\underline{F}^{T-1}\cdot\overrightarrow{\underline{D}}G_1|_x$$

arbitrarily small for fixed x, which is possible with n large enough since P_n converges weakly (and strongly) to unity and

$$ {}_x|G_1\overleftarrow{\underline{D}} \cdot \underline{H}\cdot\overrightarrow{\underline{D}}\, G_1|_x < G_o(0) < \infty. $$

Having shown the existence and uniform convergence of the integral in (4) and (3), we prove that it satisfies (III.23) in $\Omega_1 + \Omega_2$, (III.21) on $S_1 + S_2$, and (2). Since these properties characterize, as discussed in IV.B, (III.8) up to a constant factor, (3) will then be proven.

(2) is immediate*, since the two factors in the integrand in (3) satisfy (2) for the restrictions J_1 and J_2 of

$$ J = \chi(\Omega_1)\ J_1 + \chi(\Omega_2)\ J_2 \quad \text{to} \quad \Omega_1 \quad \text{resp.}\ \Omega_2, $$

the definition of the FS integral permits to use the well-known result that the matrix obtained by multiplying the elements of two positive definite matrices is positive definite.

To verify (III.21), we differentiate under the integral sign and use the uniform convergence of also the integral so obtained, which can be shown. (B.52) - (B.55) permit to prove that the norm in (III.21) appropriate to N_{1+2} is

---

*The convergence proof for (3) with J_1, J_2 complex requires only a change of notation in the one given above for real J_1 and J_2.

correctly related to the norms for N_1 and N_2.

Finally, to verify (III.23) we first prove (III.22). (III.22b) is obtained similarly as (III.21) was. (III.22a) is valid when restricted to Ω_1 or Ω_2, as shown again by differentiating (3) under the integral sign and using uniform convergence. We now use theorem B of appendix B. The continuity of the Dirichlet data, condition α), in the sense of the theorem is an immediate consequence of the form of the integral and of (III.22b). On approach from Ω_1, the Neumann data take, with (III.21), the value

$$(IV.9a) \quad -\int([\delta/\delta \underline{a}_s(s)]\, N_1\{\underline{a}_1, \underline{a}_s, J_1\}) \cdot \cdot\, N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\}\mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s)$$

while the approach from Ω_2 gives Neumann data

$$(IV.9b) \quad -\, n\cdot\int N_1\{\underline{a}_1, \underline{a}_s, J_2\} \cdot \cdot([\delta/\delta \underline{a}_s(s)]\, N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\})\mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s)$$

partial integration* which is justified by reference to the finite - dimensional approximations to these integrals, shows

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*In passing, we remark that partial integration, together with (III.21), also establishes (III.6a) for $N\{\underline{a}, J\}$ instead of $N_0\{\underline{a}, J\}$, and for incomplete Dirichlet conditions.

that (9a) and (9b) are related as required in condition β) of the theorem. This establishes (III.23).

(3) shows in which sense $N\{\underline{a}, J\}$ possesses the Markovian property. It should be noted that the integral in (3) has no similarity to a Wiener integral, although the integral (III.8) can still be considered as related to a generalized Wiener integral by variable transformation, as we did show and also expressed in (III.8) by the notation. However, a definition of that Wiener integral by use of elementary intervals (i.e., with ranges specified at points instead of on higher-dimensional sets) gives the Markovian property only after the limit is performed, or in the limit, and not before as it does for the elementary Wiener integral (dimension $d = 1$ instead of 4).

For the purpose of illustration, we briefly compare the FS integral and the Wiener integral in the case just mentioned:

$$\int \exp\left[-\tfrac{1}{2}(\phi K \phi)\right] F\{\phi\} \mathcal{D}_W(\phi) =$$

$$= \int \exp\left[-\tfrac{1}{2}(\psi\psi)\right] F\{H_\psi\} \mathcal{D}_{FS}(\Psi).$$

Let $K = -D^2$ and $\Omega = [0,1]$. The choice of Friedrichs and Shapiro[39)] is

$$(H_\psi)(x) = \int_0^x \psi(y)dy = \int_0^1 \theta(x-y)\psi(y)dy.$$

Then

$${}_x|HH^T|_y = \min(x,y) = G(x,y)$$

is in fact the Green's function to K to Dirichlet condition (fixed endpoint) at $x = 0$ and Neumann condition (no restriction) at $x = 1$. Comparing H with

$$G^{1/2}(x,y) = \pi^{-1} \ln\left\{[\tan \tfrac{\pi}{4}(x+y)][\tan \tfrac{\pi}{4}|x-y|]^{-1}\right\}$$

we find that in (B.47a) we have

$$U(x,y) = [\partial/\partial y]G^{1/2}(x,y) =$$

$$= \tfrac{1}{2}[\sin \tfrac{\pi}{2}(x+y)]^{-1} + \tfrac{1}{2}[\sin \tfrac{\pi}{2}(x-y)]^{-1} =$$

$$= \sum_{n=1}^{\infty} (-1)^n U_n(x)\, U_n(1-y)$$

with

$$U_n(x) = 2^{1/2} \sin \frac{2n-1}{2} \pi x$$

the nth eigenfunction to $-D^2$ with the boundary condition stated above. Clearly, U is orthogonal. It was to accomodate such special choices of H that we kept it general rather than to restrict it to the form $G^{1/2}$.

Choosing in (3) Ω_1 infintesimal leads to the construction of the generator of the semigroup we are considering. This is the EQFT analog of the MQFT Schrődinger operator of Heisenberg and Pauli[24)], with the field operator, together with its first $N - 1$ normal derivatives due to regularization, being diagonal. Replacing (III.23), (III.21), and (2) by the EQFT Schrődinger equation, with boundary conditions to be inferred from (III.8) as long as Ω is bounded, gives an alternative construction of the theory. This will be followed up in a different paper. We only remark here that $N\{\underline{a}, 0\}/N\{0, 0\}$ is, in the formal limit when Ω becomes the half space, the ground state functional in the regularized MQFT that corresponds to our regularized EQFT (see also the following section).

IV.D. Canonical commutation relations

From (III.21) and (III.22b) follows

$$\text{(IV.10)} \quad \overrightarrow{\underline{D}_x \underline{d}_y} [\delta^2/\delta J(x)\delta J(y)] N\{\underline{a}, J\}(|_{x \to s, y=s'} - |_{x=s, y \to s'}) =$$

$$= 1\ \delta(s,s')\ N\{\underline{a}, J\}$$

in the sense of the appropriate distributions if generalization of (10) is obtained from (3), by differentiating under the integral sign and letting arguments go to S from different sides:

$$(\text{IV.11}) \quad \overrightarrow{\underline{D}}_x^{(1)} \overrightarrow{\underline{d}}_y^{(1)} [\delta^2/\delta J(x)\delta J(y)] N\{\underline{a}, J\} \Big(\Big|_{\substack{x \to s,\, y \to s' \\ x \varepsilon \Omega_1,\, y \varepsilon \Omega_2}} - \Big|_{\substack{x \to s,\, y \to s' \\ x \varepsilon \Omega_2,\, y \varepsilon \Omega_1}} \Big) =$$

$$= \int \Big\{ ([\delta/\delta \underline{a}_s(s)] N_1\{\underline{a}_1, \underline{a}_s, J_1\}) \underline{a}_s(s') N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\} -$$

$$- ([\delta/\delta \underline{a}_s(s)] N_2\{\underline{a}_2, \underline{E}\cdot\underline{a}_s, J_2\}) \underline{a}_s(s') N_1\{\underline{a}_1, \underline{a}_s, J_1\} \Big\} \cdot$$

$$\cdot \mathcal{D}_{FS}(\underline{F}\cdot\underline{a}_s) = 1 \; \delta(s_1 s') \; N\{\underline{a}_1 J\}$$

by partial integration. That (10) and (11) are the analog of the canonical commutation relations of MQFT is seen by choosing as Ω the whole space, a half space, or the region bounded by two parallel hyperplanes. In either case the separating surface S must be a plane parallel to the boundary planes. The transition from imaginary to real time is then formally possible, in the sense as it was done in I.B inversely, and the canonical commutation relations of regularized MQFT[12] result.

IV.E. Overregularization

If $N - [d/2] - 1 = n$ is positive, we have overregularized. In that case, one may first go over to incomplete Dirichlet data (e.g. by (III.6a), for $N\{\underline{a},J\}$, however), where the Dirichlet derivatives $d_{N-n} \ldots d_{N-1}$ remain unrestricted all over $\partial\Omega$, and then let n masses go to infinity as in (B.36), with the following changes of scale of functions and constants, writing $m_1 \ldots m_n \equiv M$:

$$\lim (M^{-1}J) = J', \quad \lim (M^{-4}g) = g',$$

$$\left.\begin{array}{l} \lim (M\, a_i) = a_i' \\ \lim (M^{-2}D_i) = D_i' \end{array}\right\} \quad \text{for } i = 0 \ldots N\text{-}n\text{-}1,$$

$$\lim (M^2 G_o(0)) = G_o(0)' = G_o(0)_{red}.$$

Hereby $N\{\underline{a},J\}$ goes to the solution with the primed data and $N' = [d/2] + 1$.

The problem is to show that regularization can be reduced further for the ratio $\lim_{\Omega\to\infty} (N\{\underline{a}',J'\}/N\{\underline{a}',0\})$.

The, at least partial, tying-up with the $\Omega \to \infty$ problem is due to the Stueckelberg divergences mentioned in III.A.

IV.F. Analytic properties in g

The same argument, based on lemma A.4, which resulted in existence of the integral (III.8) for $0 \leq g < \infty$ can be used to prove existence of $N\{\underline{a},J\}$ for g in the right half complex g-plane including the imaginary axis, with the bound

$$|N\{\underline{a},J\}_g| \leq N\{\underline{a},J\}_{\mathrm{Re}\ g}.$$

Moreover, $N\{\underline{a},J\}$ is there analytic in g. To prove this, we may either use the theorem of Vitali, or write

$$\text{(IV.12)} \quad N\{\underline{a},J\} = N_o\{\underline{a},J\} \exp[\tfrac{1}{4}\, 9g\ G_o(0)^2 V_\Omega] \cdot$$

$$\cdot \int \exp[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}\, g(([H\psi + f]^2 - 3G_o(0))^2)] \mathcal{D}_{FS}(\Psi).$$

Using

$$e^{-gA}(n!)^{-1}(|\Delta g|A)^n \leq e^{-n}n^n(n!)^{-1}(|\Delta g|/g)^n =$$

$$= (|\Delta g|/g)^n(2\pi n)^{-1/2}[1 + O(n^{-1})]$$

for $A \geq 0$ and $g > 0$, we find that the power series expansion of the integral in (12) around g_o, Re $g_o > 0$, converges for $|g - g_o| <$ Re g_o. Thus $N\{\underline{a},J\}/N\{\underline{a},0\}$ and $N\{\underline{a},J\}/N\{0,0\}$ are meromorphic functions of g in the right half g plane and, in fact, ratios of bounded holomorphic functions.

The analytic character of $N\{\underline{a}, J\}$ at $\text{Re } g \leq 0$ and especially at $g = 0$ we study in appendix C for the oversimplified situation $d = 0$ mentioned in IV.B already.

IV.G. Asymptoticity of the g-expansion

The formal g-expansion of $N\{\underline{a}, J\}$ is, from (III.27),

$$\text{(IV.13)} \quad N\{\underline{a}, J\} = \exp\left[-\frac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a} - J G \overleftarrow{\underline{D}}\cdot\underline{a}\right]\cdot$$

$$\cdot \sum_{n=0}^{\infty} (n!)^{-1}\left[-\frac{1}{4}\, g((-G\overleftarrow{\underline{D}}\cdot\,\underline{a} + [\delta/\delta J])^4) +\right.$$

$$\left. + \frac{1}{2}\, 3g\; G_o(0)\;((-G\overleftarrow{\underline{D}}\cdot\,\underline{a} + [\delta/\delta J])^2\right]^n \exp[\tfrac{1}{2}(JGJ)]$$

and is represented by graphs in the usual manner, except that here lines represent Dirichlet functions G instead of G_o functions (B.15), and these lines may also end on $\partial\Omega$. Dividing through $N\{0,0\}$ eliminates all "vacuum graphs" (i.e. with no J-end) except those with lines ending on $\partial\Omega$. Dividing through $N\{\underline{a}, J\}$ instead eliminates also these "surface graphs." The proof of these statements, and also of the connectedness of the graphs for logarithms, is simplest by use of (III.23), rewritten for the logarithms. These observations are the basis of the statements made in

IV.A on the interpretation of the ratios, and their logarithms, mentioned above. The ratio $N\{\underline{a},J\}/N\{\underline{a},0\}$ and its logarithm permit in their g-expansion the termwise limit $\Omega \rightarrow \infty$ to be taken if $\underline{a}$ does not increase exponentially, and the result is the usual regularized unrenormalized perturbation theoretical expansion of $S\{J\}$, containing regularized instead of unregularized G_o lines.

In appendix C this series is shown to diverge if e.g. J does not change sign, and in other cases. The divergence proof for $\Omega < \infty$ encounters the difficulty that in contrast to G_o, G is not positive throughout if $N > 1$. If, however, the finite domain Ω is introduced in the manner α) explained in III.B instead of β), which amounts to replacing in (III.8) G by G_o, H by e.g. $G_o^{1/2}$, and omitting all terms that contain $\underline{a}$, the g-expansion of the functional $N'\{J\}$ so obtained is shown to diverge, however small g and Ω, if J does not change sign and is not identically zero. The same holds for $N'\{J\}/N'\{0\}$ and its logarithm. The existence and analytic properties in g of $N'\{J\}$ and $N'\{J\}/N'\{0\}$ can be shown, however, with the same methods which we used for $N\{\underline{a},J\}$.

From these latter properties, it follows that the g-expansions of all the functionals mentioned are asymptotic ones for $g \rightarrow 0$ with $|\arg g| \leq \pi/2$, since the existence and boundedness of all g-derivatives in the closed right half plane, if (III.9) holds (with $0 \leq \mathrm{Re}\, g < A$ replacing the first condition) is straightforward to show. In fact, the asymptoticity for approach to zero through real positive g follows already from the results of III.C.

Finally, we remark that of the solutions of the zero-dimensional model mentioned in IV.A only the solution (C.1), with z-integration path along the real axis, admits $g \rightarrow +0$ where the g-expansion is asymptotic. This may be an alternative, but less general way to single out the "correct" solution than by positive - definiteness.

IV.H. The measure

The nonexistence of $N\{\underline{a}, J\}$ for $\Omega \rightarrow \infty$ and $g > 0$ does not preclude the ratios $N\{\underline{a}, J\}/N\{\underline{a}, 0\}$ to have such a limit $S\{J\}$. The connection with measure theoretical concepts may be expressed as follows: Define

$$(\mathrm{IV}.14) \quad h(g_1 \ldots g_n) = \Big\{ \int \exp[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}g((H\psi)^4) + \tfrac{1}{2}3g\, G_o(0)((H\psi)^2)] \prod_{i=1}^{n} \delta[g_i - (f_i\psi)] \mathcal{D}_{FS}(\psi) \Big\} \cdot \Big\{ \int \exp[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}g((H\psi)^4) + \tfrac{1}{2}3g G_o(0)((H\psi)^2)] \mathcal{D}_{FS}(\psi) \Big\}^{-1}$$

where $f_1 \dots f_n$ is a set of orthonormal functions, whose support we may assume to be in Ω for Ω sufficiently large. The observation that the numerator is an FS integral over the subspace of Hilbert space orthogonal to $\{f_i\}$ and that the integral without the δ-functions is invariant leads easily to the existence of $h(g_1 \dots g_n)$ for $\Omega < \infty$.

$$\int_{-\infty}^{g_1^0} \cdots \int_{-\infty}^{g_n^0} h(g_1 \dots g_n) dg_1 \dots dg_n$$

is the measure relevant for cylinder functions based in the space spanned by the f_i, and can be shown to be absolutely continuous relative to the measure for $g = 0$, when $h(g_1 \dots g_n) = (2\pi)^{-n/2} \exp[-\frac{1}{2} g_1^2 - \dots - \frac{1}{2} g_n^2]$.

For $g > 0$, however, $h(g_1 \dots g_n)$ apparently cannot be expressed any more in terms of elementary functions, although it is found to be an entire analytic function of $g_1 \dots g_n$ in our special case, with a method similar to the one we shall use in V.A. There is no reason against the conjecture that $h(g_1 \dots g_n)$ possesses a limit for $\Omega \to \infty$ that is a well-behaved function and possesses Euclidean invariance, whereas $\Omega < \infty$ admits only the invariance compatible with the symmetry (e.g., spherical) of Ω. In fact, we should have

(IV.15) $h(g_1 \ldots g_n) =$

$$= (2\pi)^{-n} \int \ldots \int \exp[-ig_1 w_1 - \ldots - ig_n w_n] \cdot$$

$$\cdot\, S\left\{i\, w_1 H_o^{T-1} f_1 + \ldots + i\, w_n H_o^{T-1} f_n\right\} dw_1 \ldots dw_n$$

We shall in V.E derive upper and lower bounds for $h(g_1 \ldots g_n)$ for finite Ω.

V. Special properties of the solution

In this section we derive properties of the functional integral (III.27) that depend on the circumstance that the interaction term is an analytical functional (V.A) and specifically a polynomial of order four (V.B - E).

V.A. Entire analyticity in $\underline{a}$

Let $\underline{a}$ in (III.27), and therefore f in (III.8), be complex. If in (III.9) $\underline{a} \cdot \underline{G} \cdot \underline{a} < \infty$ is replaced by $\overline{\underline{a}} \cdot \underline{G} \cdot \underline{a} < \infty$, the convergence proof given in III.C easily carries over to this case. Because of lemma A.4 only an upper bound need be given: With

(V.1) $$h = -G\overleftarrow{\underline{D}} \cdot \underline{a} = -G\overleftarrow{\underline{D}} \cdot \underline{a}_r - i\, G\overleftarrow{\underline{D}} \cdot \underline{a}_i = h_r + i\, h_i$$

we have for the integral in (III.27)

(V.2) $$\int \left| \exp[-\tfrac{1}{2}(\psi\psi) + (JH\psi) - \tfrac{1}{4}g((H\psi + h)^4) + \right.$$

$$\left. + \tfrac{1}{2}\, 3g\; G_o(0)((H\psi + h)^2)] \right| \mathcal{D}_{FS}(\psi) \leq$$

$$\leq \int \exp[-\tfrac{1}{2}(\psi\psi) + (JH\psi) - \tfrac{1}{4}g((H\psi + h_r)^4) +$$

$$+ \tfrac{1}{2}\, 3g((H\psi + h_r)^2 h_i^2) - \tfrac{1}{4}g(h_i^4) +$$

$$+ \tfrac{1}{2}\, 3g\; G_o(0)\; ((H\psi + h_r)^2) - \tfrac{1}{2}\, 3g G_o(0)(h_i^2)] \mathcal{D}_{FS}(\psi) \leq$$

$$\leq \exp\; [\tfrac{1}{2}\; (JGJ) + \tfrac{1}{4}\; 9g((h_i^2 + G_o(0))^2) -$$

$$- \tfrac{1}{4}g\; (h_i^4) - \tfrac{1}{2}\, 3g\; G_o(0)(h_i^2)].$$

Since also differentiability with respect to $\underline{a}$ in the sense of (III.17b) is easily shown, $N\{\underline{a}, J\}$ is an entire analytic functional of order at most four.

To see this more explicitly, we also prove directly the absolute convergence of the Volterra series for $\underline{\bar{a}} \cdot \underline{G} \cdot \underline{a} < \infty$, which implies that $|h|$ is bounded. The sum of the absolute values of the terms of the series is bounded by

$$B\{\underline{a},J\} = \int \exp[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}g((H\psi)^4) + g|((H\psi)^3h)| +$$

$$+ \tfrac{1}{2}3g|((H\psi)^2h^2)| + g|((H\psi)h^3)| + \tfrac{1}{4}g|(h^4)| +$$

$$+ \tfrac{1}{2}3g\; G_o(0)((H\psi)^2) + 3g\; G_o(0)|((H\psi)h)| +$$

$$+ \tfrac{1}{2}3g\; G_o(0)|(h^2)| + (JH\psi)]\mathcal{D}'_{FS}(\psi).$$

Writing $g = g_1 + g_2 + g_3$, $g_{1,2,3} > 0$, we have

$$B\{\underline{a},J\} \leq \exp[\tfrac{1}{4}\; g|(h^4)| + \tfrac{1}{2}3g\; G_o(0)|(h^2)|]\cdot$$

$$\cdot \int \exp\,[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}(g_2 + g_3)((H\psi)^4) +$$

$$+ (g_1^{-1}g^2 + \tfrac{1}{2}3g)((H\psi)^2|h|^2) + \tfrac{1}{2}3g\; G_o(0)((H\psi)^2) +$$

$$+ g(|(H\psi)|(|h^3| + 3G_o(0)|h|)) + (JH\psi)]\mathcal{D}_{FS}(\psi) \leq$$

$$\leq \exp\,[\tfrac{1}{2}(JGJ) + g_2^{-1}(((g_1^{-1}g^2 + \tfrac{1}{2}3g)|h^2| + \tfrac{1}{2}3g\; G_o(0))^2) +$$

$$+ \tfrac{1}{4}3g_3^{-1/3}((|h^3| + 3G_o(0)|h|)^{4/3})]$$

which shows that boundedness of $|h|$ is sufficient, but not necessary, for absolute convergence of the Volterra series.

V.B <u>Bounds based on convexity</u>

We define, with h again real

$$(V.3) \qquad f(u) \equiv \ln \int \exp[-\tfrac{1}{2}(\psi\psi)+(J\psi H)-\tfrac{1}{4}g((H\psi+h)^4) + $$

$$+ \tfrac{1}{2}u((H\psi+h)^2)]\mathcal{D}_{FS}(\psi).$$

Completing a square gives immediately

$$(V.4) \qquad f(u) \leq \tfrac{1}{2}(JGJ) + \tfrac{1}{4}g^{-1}u^2V_\Omega$$

On the other hand, from lemma A.6 and (A.3)

$$(V.5) \qquad f(u) \geq \tfrac{1}{2}(JGJ) + \left\{\int \exp[-\tfrac{1}{2}(\psi\psi)+(JH\psi)]\mathcal{D}_{FS}(\psi)\right\}^{-1} \cdot$$

$$\cdot \left\{\int \exp[-\tfrac{1}{2}(\psi\psi)+(JH\psi)]\cdot[-\tfrac{1}{4}g((H\psi+h)^4) + \right.$$

$$\left. + \tfrac{1}{2}u((H\psi+h)^2)]\mathcal{D}_{FS}(\psi)\right\} =$$

$$= \tfrac{1}{2}(JGJ) - \tfrac{1}{4}3g\int G(x,x)^2dx - \tfrac{1}{2}3g\int G(x,x)f(x)^2dx -$$

$$- \tfrac{1}{4}g(f^4) + \tfrac{1}{2}u\int G(x,x)dx + \tfrac{1}{2}u(f^2)$$

with f from (III.8b). Since, from A.5, f(u) is a convex function, we have

$$\underset{v < u}{\text{l.u.b.}} \left\{[v-u]^{-1}[f(v)-f(u)]\right\} \leq [d/du]f(u) \leq$$

$$\leq \underset{v > u}{\text{g.l.b.}} \left\{[v-u]^{-1}[f(v) - f(u)]\right\}$$

with coinciding extrema reached at $v \to u$ since $f(u)$ is continuously differentiable. However, if

$$f_L(u) \leq f(u) \leq f_U(u)$$

we have

$$\underset{v < u}{\text{l.u.b.}} \left\{[v-u]^{-1}[f_U(v) - f_L(u)]\right\} \leq [d/du]f(u) \leq$$

$$\leq \underset{v > u}{\text{g.l.b.}} \left\{[v-u]^{-1}[f_U(v) - f_L(u)]\right\} .$$

Inserting the bounds (4) and (5) gives

(V.6) $$[d/du]f(u) \lesseqgtr (2g)^{-1}u\ V_\Omega \pm (g^{-1}V_\Omega)^{1/2}\cdot$$

$$\cdot\ [(4g)^{-1}u^2V_\Omega - \frac{u}{2}\int G(x,x)dx - \frac{u}{2}(f^2) +$$

$$+ \frac{1}{4}\,3g\int G(x,x)^2dx + \frac{1}{2}\,3g\int G(x,x)f(x)^2dx+\frac{1}{4}g(f^4)]^{1/2}.$$

We use this result to obtain an upper bound for

$N\{\underline{a},J\}/N\{\underline{a},0\} \equiv S\{\underline{a},J\}$ which does not increase exponentially with V_Ω.

As found in IV.B, $\ln S\{\underline{a},J\} = S\{\underline{a},J\}^T$ is a convex functional of J. For such a functional (A.2) gives

$$S\{\underline{a},J\}^T \leq S\{\underline{a},0\}^T + \int dx\, J(x)[\delta/\delta J(x)]S\{\underline{a},J\}^T$$

Therefore, using the Hölder inequality

$$\begin{aligned}
\text{(V.7)} \quad & -(hJ) + S\{\underline{a},J\}^T \leq \Big\{\int \exp[(JH\psi) - \tfrac{1}{2}(\psi\psi) - \\
& - \tfrac{1}{4}g((H\psi+h)^4) + \tfrac{1}{2}3g\, G_o(0)((H\psi+h)^2)]\mathcal{D}_{FS}(\psi)\Big\}^{-1} \cdot \\
& \cdot \Big\{\int (JH\psi)\exp[(JH\psi) - \tfrac{1}{2}(\psi\psi) - \\
& - \tfrac{1}{4}g((H\psi+h)^4) + \tfrac{1}{2}3g\, G_o(0)((H\psi+h)^2)]\mathcal{D}_{FS}(\psi)\Big\} \leq \\
& \leq -(hJ) + \|J\|_{4/3}\Big[\Big\{\int \exp[(JH\psi) - \ldots]\mathcal{D}_{FS}(\psi)\Big\}^{-1} \cdot \\
& \cdot \Big\{\int ((H\psi+h)^4)\exp[(JH\psi) - \ldots]\mathcal{D}_{FS}(\psi)\Big\}\Big]^{1/4}
\end{aligned}$$

where $\|J\|_{4/3} = [\int |J(x)|^{4/3}dx]^{3/4}$. The radicand in (7) is

$$\text{(V.8)} \quad -4[\partial/\partial g]\ \ln\Big\{\int \exp[(JH\psi) - \ldots]\mathcal{D}_{FS}(\psi)\Big\} +$$

(V.8)Continued:

$$+ 6G_o(0)\left\{\int \exp[(JH\psi)- \dots]\mathcal{D}_{FS}(\psi)\right\}^{-1} \cdot$$

$$\cdot \left\{\int ((H\psi + h)^2) \exp[(JH\psi)- \dots]\mathcal{D}_{FS}(\psi)\right\}$$

An upper bound for the second term is obtained from (6) with $u = 3g\ G_o(0)$. The first term is a nonincreasing function of g since the logarithm is a convex function of g, and is therefore bounded by its value for $g = 0$, which can be obtained exactly. Using this in (8) and inserting in (7), we obtain the upper bound of $S\{\underline{a},J\}^T$ a lengthy expression, which is g-independent and of which it is easily seen that under the conditions (III.9) and $\| J \|_{4/3} < \infty$ it does increase only like $V_\Omega^{1/4}$ as $V_\Omega \longrightarrow \infty$. This shows that although in $N\{\underline{a},J\}/N\{\underline{a},0\}$ numerator and denominator increase both like $\exp[\text{const.}\ V_\Omega]$ as $V_\Omega \longrightarrow \infty$, not only the leading term in V_Ω, but also lower (fractional) powers must cancel at least until only $V_\Omega^{1/4}$ is left.

The perturbation theoretical interpretation of the Ω-dependence of $\ln N\{\underline{a},J\}$ was given in IV.G, e.g. in four dimensions, the "volume graphs" give rise to terms $\sim V_\Omega$, and $\sim V_\Omega^{3/4}$ and lower powers due to "distortion" near the surface. The "surface graphs" give terms $\sim V_\Omega^{3/4}$, $\sim V_\Omega^{1/2}$ etc. In perturbation theory, all terms that increase without limit as $V_\Omega \longrightarrow \infty$ cancel in $\ln N\{\underline{a},J\} - \ln N\{\underline{a},0\}$ for $\underline{a}$ not exponentially increasing with $V_\Omega^{1/4}$, and the formal limit is then $\underline{a}$-independent. We have so far been unable to show that this cancellation takes place to the same extent in the rigorous solution we are analyzing.

V.C. Variational bounds

For the ratio $N\{0,J\}/N\{0,0\} = S\{0,J\}$ a lower bound that is essentially Ω-independent is obtained by the following method due to Feynman[23)41)]. Using (III.27) for numerator and denominator, substituting $\Psi = \Psi_o + \Psi'$ in the numerator, with $\psi_o \varepsilon L_2$, we obtain from lemma A.6

(V.9)
$$S\{0,J\} \geq \underset{\Psi_o}{\text{l.u.b.}}[-\tfrac{1}{2}(\psi_o\psi_o) - \tfrac{1}{4}\, g((H\psi_o)^4) - \tfrac{1}{2}\, 3g \int (S(xx)-G_o(0))({}_x|H\psi_o)^2 dx + (JH\psi_o)] =$$

$$= \underset{\Phi_o,\vec{\underline{d}}\Phi_o = 0 \text{ on } \partial\Omega}{\text{l.u.b.}} [-\tfrac{1}{2}(\Phi_o K\Phi_o) - \tfrac{1}{4}\, g\,(\Phi_o^4) - \tfrac{1}{2}\, 3g \int (S(xx) - G_o(0))\Phi_o(x)^2\, dx + (J\Phi_o)]$$

where

$$S(xx) = [\delta^2/\delta J(x)^2]\; S\{0,J\}|_{J=o}$$

Apart from the term containing the difference $S(xx)-G_o(0)$, it is the classical Euclidean action integral that is to be maximized. This functional is briefly discussed in appendix D. If we let in (9) $J \longrightarrow 0$, we obtain in the sense of quadratic forms

$$\text{(V.10)} \qquad S \geq [K + 3g(S - G_o(0))\delta]^{-1}$$

and for the diagonal elements

$$\text{(V.11)} \qquad S(xx) \geq {}_x|[K + 3g(S - G_o(0))\delta]^{-1}|_x$$

We now let $\Omega \longrightarrow \infty$ (more precisely, $V_\Omega \gg m^{-4}$ where m is the smallest of the m_i in (B.1)). Then $S(xx) \longrightarrow \text{const} = c$. If $c < G_o(0)$, (11) is a contradiction. Therefore*

$$\text{(V.12a)} \qquad \lim_{\Omega\to\infty} S(xx) \geq G_o(0)$$

An upper bound for this limit from (6) with $u = 3gG_o(0)$, $J = h = 0$:

*The reader will be aware that we are temporarily dispensing of rigor in favor of simplicity.

(V.12b) $\lim_{\Omega \to \infty} S(xx) \leq [\frac{3}{2} + (\frac{3}{2})^{1/2}] G_o(0)$

That the bounds we obtain here for the limit are both finite we take as a further indication, beyond (7) and its interpretation and (9), that $\Omega \to \infty$ is actually possible and $S\{0,J\} \to S\,J$, the Euclidean invariant Schwinger functional.

To give rigorous bounds for the term involving $S(xx) - S_o(0)$ in (9) requires a modification of (3): the term $\frac{u}{2}((H\psi + h)^2)$ in the exponent need be replaced by $\frac{u}{2}(\phi_o^2(H\psi)^2)$ and h and J set zero. Then the method that lead to (6) may be taken over. Actually, in this method both bounds (4) and (5) may be improved by introducing variational parameters or integral kernels such that only Gaussian integrals result, e.g., for $f_U(u)$ of (4) a square may be completed differently, or for $f_L(u)$ of (5) one may[41] take a different, still Gaussian integrand as weight. We will not pursue this matter since it furnishes little additional insight here.

To conclude, we note that if in the functional integral (III.8a) we had replaced $\frac{1}{2} 3g\, G_o(0)((H\psi+f)^2)$ by $\frac{1}{2} 3g \int S(xx)[_x|H\psi+f(x)]^2 dx$, the term with $S(xx) - G_o(0)$ in (9) would have been missing and that bound correspondingly simpler. Although this replacement is suggested by perturbation theoretical considerations, see II, we were unable to prove existence of the functional integrals so modified. Although it is rather obvious that there should be one solution that is self-consistent, the dependence of the functional integral on itself introduces a stability problem: the solution just mentioned need not be stable. The technique we have available is too crude to decide this point and would certainly be unsuitable to deal with instability.

V.D. Miscellaneous bounds

We set $J = \underline{a} = 0$ in (III.8). Application of lemma A.6 gives

$$(V.13) \quad -\frac{1}{4}\, 3g \int G(x,x)^2 dx + \frac{1}{2}\, 3g\, G_o(0) \int G(x,x) dx \leq$$

$$\leq \ln N\{0,0\} \leq -\frac{1}{4}\, g \int S(xxxx) dx + \frac{1}{2}\, 3g G_o(0) \int S(xx) dx$$

or

$$(V.14a) \quad 3\int [S(xx) - G(x,x)][S(xx) + G(x,x) - 2G_o(0)] dx \leq$$

$$\leq -\int S(xxxx) dx + 3\int S(xx)^2 dx \leq 2\int S(xx)^2 dx$$

where the last inequality is Schwarz's. Since $\ln N\{0,0\}$ is a convex function of g,

$$(V.15a) \quad [\partial/\partial g]\left[\frac{1}{2}\, 3G_o(0) \int S(xx) dx - \frac{1}{4} \int S(xxxx) dx\right] \geq 0$$

If, to simplify, we let as before $\Omega \longrightarrow \infty$,

$$\text{(V.14b)} \quad 3[S(00) - G_0(0)]^2 \leq - S(0000) + 3\, S(00)^2 \leq 2\, S(00)^2$$

and

$$\text{(V.15b)} \quad [\partial/\partial g] \left\{ 3\, S(00)^2 - S(0000) - 3[S(00) - G_0(0)]^2 \right\} \geq 0$$

which conforms with the first inequality in (14b) since the curly bracket is zero for $g = 0$. On the other hand, the curly bracket cannot exceed $6\, G_0(0)^2$, using the right inequality in (14b). Thus, we have found a quantity that increases monotonically with g, and linearly for small g on the basis of IV.G, but which has an upper bound.

We note that the quantities appearing in (12), (14b), and (15b) have a direct meaning* also in regularized MQFT since imaginary time difference zero is equivalent to real time difference zero provided the approach to coinciding points takes place from space-like directions. The left and middle term in (14b), and thus (15), should even be finite in

---

*E.g., the quantity $\lim_{\Omega\to\infty} V_\Omega^{-1} \ln N\{0,0\}$, the curly bracket in (15b) apart from a constant, is the familiar $- i\, V^{-1} T^{-1} \ln_0 \langle S \rangle_0$ of MQFT perturbation theory.

two-dimensional unregularized MQFT, if the finiteness of the perturbation theoretical expansions of these expressions is assumed indicative.

The left inequality in (13) holds also for $\ln N\{0,J\}$ which therefore increases linearly as $V_\Omega \longrightarrow \infty$, or increases without bound if a removal of regularization (cp.IV.E) below $N = [d/2] + 1$ is attempted, unless $g = 0$. This proves the statements in II and IV.G, H.

We proved in IV.A that $N\{\underline{a},J\}$ is entire analytic in J of order at most two. This statement can easily be sharpened: with (1) we have from (III.27), similarly as in V.A,

$$N\{\underline{a},J\} \leq \exp[-\tfrac{1}{2}\,\underline{a}\cdot\underline{G}\cdot\underline{a} +$$

$$+ \tfrac{1}{4}\,9g_1^{-1}g^2G_o(0)^2V_\Omega + \tfrac{1}{4}\,3g_2^{-1/3}\|J\|_{4/3}^{4/3}]$$

with $g_1 + g_2 = g$, $g_{1,2} > 0$, and $\|J\|_{4/3}$ as in (7). Thus, with (IV.1b) the order as defined in IV.A is at most $4/3$. That it is precisely $4/3$* follows then from (9) (at least for $\underline{a} = 0$) and (D.6).

---

*A saddle point evaluation of $N\{\underline{a},J\}$ for J "large" gives the same result. However, the error estimation of this method for functional integrals is largely underdeveloped. In contrast, the bounds given here are rigorous.

From a bound on the analytic functional $N\{\underline{a}, J\}$, with (IV.1b) for J complex, also bounds on the derivatives with respect to J (or $\underline{a}$, from the analyticity in $\underline{a}$ proven in V.A) can be derived from

(V.16) $$|\int \dots \int J'(x_1)\dots J'(x_n)[\delta^n/\delta J(x_1)\dots\delta J(x_n)] \cdot$$

$$\cdot N\{\underline{a}, J\}\, dx_1 \dots dx_n| = |(2\pi i)^{-n} \oint \dots \oint z_1^{-2} \dots z_n^{-2} dz_1 \dots dz_n \cdot$$

$$\cdot N\{\underline{a}, J + z_1 J_1' + \dots + z_n J_n'\}| \leq$$

$$\leq \underset{R_1, \dots R_n}{\text{g.l.b.}} [(2\pi)^{-n} R_1^{-1} \dots R_n^{-1} \int_0^{2\pi} d\phi_1 \dots \int_0^{2\pi} d\phi_n \cdot$$

$$\cdot N\{\underline{a}, J + R_1 \cos\phi_1 J_1' + \dots + R_n \cos\phi_n J_n'\}$$

without going back to the functional integral. Especially, a bound for $-E_o\{J\}$ could be used because of (I.31).

We found already that for J' in (16) δ-functions are admissable in our model, but that derivatives with respect to $\underline{a}$ may not be taken locally in general.

V.E. Bounds on the measure

From (IV.14), we have with (A.3)

(V.17) $$h(g_1 \dots g_n) = N\{0,0\}^{-1} \cdot \lim_{V \to \infty} \{(2\pi)^{-n/2} V^{n/2} .$$

(V.17) Continued:

$$\cdot \int \exp[-\tfrac{1}{2}(\psi\psi)-\tfrac{1}{4}g((H\psi)^4)+\tfrac{1}{2}\,3g\;G_o(0)((H\psi)^2)\;-$$

$$-\tfrac{1}{2}\,v(g_1-(f_1\psi))^2-\ldots-\tfrac{1}{2}\,v(g_n-(f_n\psi)^2)]\mathcal{D}_{FS}(\psi)\;\leq$$

$$\leq N\{0,0\}^{-1}\lim_{v\to\infty}\{(2\pi)^{-n/2}v^{n/2}\;.$$

$$\cdot\;\exp[\tfrac{1}{4}\,9g\;G_o(0)^2V_\Omega]\int\exp[-\tfrac{1}{2}\,(\psi\psi)\;-$$

$$-\tfrac{1}{2}\,v(g_1-(f_1\psi))^2-\ldots]=N\{0,0\}\exp[\tfrac{1}{4}9gG_o(0)^2V_\Omega]\;\cdot$$

$$\cdot\;\lim_{v\to\infty}\{(2\pi)^{-n/2}[v/(1+v)]^{n/2}\exp[-\tfrac{1}{2}\,v(1+v)^{-1}(g_1^2+\ldots+g_n)^2]\}=$$

$$=(2\pi)^{-n/2}N\{0,0\}^{-1}\exp[\tfrac{1}{4}9g\;G_o(0)^2V_\Omega-\tfrac{1}{2}(g_1^2+\ldots+g_n^2)],$$

and with lemma A.6

$$\text{(V.18)}\qquad h(g_1\ldots g_n)\geq N\{0,0\}^{-1}\lim_{v\to\infty}\{(2\pi)^{-n/2}v^{n/2}\;.$$

$$\cdot\int\exp[-\tfrac{1}{2}(\psi\psi)-\tfrac{1}{2}\,v\sum(g_i-(f_i\psi))^2]\mathcal{D}_{FS}(\psi)\;\cdot$$

$$\cdot\;\exp[\{\int\exp[-\tfrac{1}{2}(\psi\psi)-\tfrac{1}{2}\,v\sum(g_i-(f_i\psi))^2]\mathcal{D}_{FS}(\psi)^{-1}\;.$$

$$\cdot\{\int\exp[-\tfrac{1}{2}(\psi\psi)-\tfrac{1}{2}\,v\sum(g_i-(f_i\psi))^2]\cdot[-\tfrac{1}{4}g((H\psi)^4)\;+$$

$$+\tfrac{1}{2}\,3g((H\psi)^2)]\mathcal{D}_{FS}(\psi)\}]$$

which can again be evaluated exactly. Using in (17) and (18) lower resp. upper bounds for N 0,0 , bounds for $h(g_1 \dots g_n)$ are obtained. The upper one goes to infinity and the lower one to zero as $V_\Omega \longrightarrow \infty$, however. The unsolved problem is to prove the existence of a limit of $h(g_1 \dots g_n)$ as $V_\Omega \longrightarrow \infty$, which would yield a measure presumably not equivalent to a Gaussian one and of physical interest. This subject is explored briefly from a different viewpoint in the next section.

VI. Operator formulation

In this section we show that the general properties proven in IV of the modified model of EQFT permit operator formulations of that model. We give one in detail for fixed Ω and $\underline{a}$.

The positive definiteness property (IV.2) permits to define $N\{\underline{a}, iJ\}$ as the expectation in a state $|\underline{a}\rangle$ of $\exp\left[i\int dx\, J(x)Q(x)\right]$ where $Q(x)$ is a commuting hermitean operator field, which acts on the Hilbert space obtained by completing the linear space of elements $\Pi\{Q\}|\underline{a}\rangle$, the norm of $\Pi\{Q\}|\underline{a}\rangle$ being $\langle\underline{a}|\Pi\{Q\}^{+}\Pi\{Q\}|\underline{a}\rangle^{1/2}$, and defining equivalence classes of vectors which differ only by vectors of norm zero in the usual way*.

Diagonalization of $Q(\cdot)$

(VI.1) $$\langle\phi|Q(x) = \phi(x)\langle\phi|$$

with normalization

(VI.2) $$\langle\phi|\phi'\rangle = \delta(\phi-\phi') \quad ,$$

(VI.3) $$\int |\phi\rangle N\{\underline{a},0\}^{-1} D_W(\phi)\langle\phi| = 1$$

such that

---

* We may also start from the algebra of unitary operators $\exp[i(JQ)]$ with $(JGJ) < \infty$. We will ignore, however, in this section the domain question since the availability of the solution of the model permits to interpret properly all formulas that we give.

$$F\{\phi\} = \int D_W(\phi')\delta(\phi'-\phi)F\{\phi'\}$$

where it is understood that $D_W(\phi)$ must appear only in a context that allows it to be combined with $\exp\left[-\frac{1}{2}(\phi K\phi)\right]$, gives with

(VI.4) $$\langle \phi|\underline{a}\rangle = \exp\left[-\frac{1}{4}(\phi K\phi) - \frac{1}{8}g(\phi^4) + \frac{1}{4}3g\, G_o(0)(\phi^2)\right]$$

correctly, with (III.8a),

$$\langle \underline{a}|\exp[i(QJ)]|\underline{a}\rangle = N\{\underline{a},0\}^{-1} N\{\underline{a},J\}$$

where we have in (4) the condition $\vec{\underline{d}}\phi = \underline{a}$ on $\partial\Omega$. For an operator characterization of $|\underline{a}\rangle$, we introduce the hermitean operator field $P(x)$ by

(VI.5a) $$[Q(x),P(y)] = i\delta(x-y)$$

(VI.5b) $$[P(x),P(y)] = 0$$

in addition to

(VI.5c) $$[Q(x),Q(y)] = 0$$

(1) entails

(VI.6) $$\langle \phi|P(x) = -i[\delta/\delta\phi(x)]\langle \phi|$$

which permits (4) to be written

(VI.7a) $$\left\{P(x) - \frac{1}{2}i\,\vec{K}Q(x) - \frac{1}{2}ig\, Q(x)^3 + \right.$$

$$\left. + \frac{1}{2}3ig\, G_o(0)Q(x)\right\}|\underline{a}\rangle \equiv$$

$$\equiv C(x)|\underline{a}\rangle = 0$$

together with the boundary condition

(VI.7b) $\quad \vec{\underline{d}}Q(x)|\underline{a}\rangle|_{x\to s} = \underline{a}(s)|\underline{a}\rangle$.

We have from (III.21)

(VI.8) $\quad \vec{D}Q(x)|\underline{a}\rangle|_{x\to s} = -[\delta/\delta\underline{a}(s)]|\underline{a}\rangle$

and from (IV.11)

(VI.9) $\quad \vec{\underline{D}}Q(x)\vec{\underline{d}}Q(y)\left(|_{x\uparrow s, y\downarrow s'} - |_{x\downarrow s, y\uparrow s'}\right) = i\delta(s,s')\delta(s,s')$

with normal direction $\uparrow$.

We may introduce

(VI.10) $\quad H = \frac{1}{2}\int dx\ C(x)^{+}C(x)$.

Then

(VI.11) $\quad H|\underline{a}\rangle = 0$

together with (7b) characterizes $|\underline{a}\rangle$ uniquely. One easily finds

(VI.12a) $\quad [H,Q(x)] = -i\ P(x)$

(VI.12b) $\quad [H,P(x)] = \frac{i}{4}\left[\vec{K} + 3g\ Q(x)^2 - 3g\ G_o(0)\right]$.

$$. \left[\vec{K}Q(x) + g\ Q(x)^3 - 3g\ Q(x)G_o(0)\right] -$$

$$- \frac{1}{2}3ig\ Q(x)\delta(0)$$

where the last term cancels the singularity in the expression $\frac{1}{4}i3g\ Q(x)^2\ \vec{K}Q(x)$ arising from the first brackets.

For $g = 0$, we have

$$[C(x),C(y)^{+}] = \overrightarrow{K}_x \delta(x-y)$$

which permits the formulation (1) - (12) to be solved elementarily in terms of the eigenfunctions to K with Dirichlet boundary conditions a. The spectrum of H is that of an infinite set of harmonic oscillators, one for each mode, and therefore discrete. The case $g = 0$, $\Omega = \infty$ may be solved similarly. The spectrum of H is now continuous (apart from the isolated eigenvalue zero) with threshold at s_N and higher ones at $2s_N$, $3s_N$ etc. We may also introduce the number operator.

$$N = \iint dxdy\ C(x)^{+}G(x,y)C(y)$$

which has the nonnegative integers as eigenvalues and commutes with H.

For $g > 0$ we know a solution to exist, which is obtained by giving to the Schrödinger representation (1), (6) and solving in terms of the functional integral. For $g > 0, \Omega = \infty$ we cannot show the existence of a solution.

For $\Omega = \infty$, the "Euclidean Hamiltonian" H may be supplemented by the "Euclidean momentum operators"

(VI.13) $$P_\mu = \int dx\ P(x)\partial_\mu Q(x) = -\int dx\ \partial_\mu P(x)\cdot Q(x),$$

(μ=1...4), which have the properties

(VI.14a) $$[P_\mu, Q(x)] = -i\partial_\mu Q(x) \quad ,$$

(VI.14b) $$[P_\mu, P(x)] = -i\partial_\mu P(x) \quad ,$$

(VI.15) $$[P_\mu, H] = [P_\mu, P_\nu] = 0$$

The theory defined in $\Omega = \infty$ by (5), (7a), (10) - (15) has all properties (including, at least in perturbation theory, the cluster property[42)*]) of a Hamiltonian quantum field theory[19], lacking, however, as do nonrelativistic theories, invariance under continuous kinematic symmetry groups other than space- and time translation.

The operator formulation just given is not unique, but seems to be the simplest. It differs from Schwinger's[8] by a similarity transformation and is for $g = 0$ equivalent to Nakano's[9].

Another operator formulation uses variable Ω and integration over Dirichlet data (which parametrize "intermediate states") whenever adjoining regions are joined as in IV.C. Being a rather direct operator transcription of the semigroup property, it seems of little use presently and will not be given here.

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* Our model allows us to introduce not only truncated but also irreducible functions[1)21)4)5)]. These decrease faster exponentially than merely truncated functions whenever certain arguments become far separated from others as in (I.3). As long as $\Omega < \infty$, this statement is not mathematically precise, however.

Conclusion

This paper derives from an attempt to analyse the content of the Lagrangian quantum field theory with emphasis on avoiding perturbation theory. Predictably, it raises more questions than it answers.

To each MQFT (with lowest energy state) there exists one and only one EQFT. We base our discussion on the coupled integral equations for EQFT- rather than MQFT Green's functions for reasons discussed in the introduction. Further modifications described there define our model. The functional differential equation for the generating functional is solved by a functional integral, and properties of this solution are studied. The more interesting ones are 1) positive definiteness, which renders this solution unique if others should exist, 2) entire analyticity in the source function, 3) the semigroup property, which permits to derive canonical commutation relations and the Schrödinger equation, 4) entire analyticity in the Dirichlet data, 5) asymptoticity of the perturbation expansion, 6) for real source function, the increase of the logarithm of the normalized functional with at most the fourth root (this being due to the quartic coupling used) of the space-"time" volume.

Of these properties, 2) and 3) are expected, while 4) simplifies the derivation of the Schrödinger equation. 5) gives support to the often expressed conjecture that the perturbation expansions in quantum field theory are asymptotic ones (in a slightly different model, we prove divergence of the perturbation theoretical expansions). 6) is reassuring as it makes more likely the existence of the limit of the normalized functional for infinite space-"time" volume. To obtain this limit itself, the functional integral method in the form used here is not suited. It is hoped that the Schrödinger equation approach, for which the basis is laid here and which must be returned to even in the entirely elementary problem of a non-linear oscillator, will be more successful in that respect.

The result 1) is of particular interest since it does not depend on the form of the Lagrangian at all, and one may surmise that its holds for the EQFT to any, even non-Lagrangian, MQFT. It would imply that there exists a functional integral representation of EQFT even though we could not prove its existence in our model for infinite space-"time" volume.

The methods of this paper may also be employed for models with Fermi fields, e.g. modified Euclidean quantum electrodynamics, provided the Fermi fields are eliminated first in closed form[43].

We have shown that problems defined by infinite sets of coupled integral equations are not necessarily unmanageable if adequate, although not entirely elementary, mathematical tools are used. The problem of appropriate representation of a non-trivial infinite set of functions has here been solved, however, only under the sacrifice of close connection with physics.

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Appendix A: The Friedrichs-Shapiro integral over Hilbert space.

In this appendix we give, for the convenience of the reader, the definition of the integral over Hilbert space by Friedrichs and Shapiro[16] (FS integral), and hereby partially reproduce verbally their own concise exposition. We then prove a few simple lemmata concerning that integral for later use.

Let $\mathcal{H}$ be a separable real Hilbert space, and let $\mathcal{L}$ be the collection of finite-dimensional subspaces E of $\mathcal{H}$. For each E, let the orthogonal projector onto E be P_E; it may be written, for $x \varepsilon \mathcal{H}$, in the form

$$P_E x = \sum_{\nu=1}^{n} e_\nu^E (e_\nu^E, x) \ , \ n = \dim E \ ,$$

with $(e_\mu^E, e_\nu^E) = \delta_{\mu\nu}$. For each E and a choice of the set (e_ν^E) define the Gaussian measure μ_E by

$$d\mu_E = \left(\frac{a}{\pi}\right)^{(1/2)n} e^{-a(x_1^2+\ldots+x_n^2)} dx_1\ldots dx_n$$

where $x_\nu = (e_\nu^E, x)$ and $a > 0$ (we shall by a suitable change of scale arrange $a = \frac{1}{2}$ in the following). Since the Gaussian measure is orthogonally invariant (even unitarily invariant if the preceding construction is generalized in the obvious manner), μ_E is independent of the choice of (e_ν^E).

The functional $f(x)$, defined over $\mathcal{H}$, is called a cylinder functional if there exists at least one $E \varepsilon \mathcal{L}$ such that $f(P_E x) = f(x)$ for all $x \varepsilon \mathcal{H}$. Then $f(x)$ is said to have a base in E.

Consider, for any $\alpha > 1$, the collection C_α^E of cylinder functionals $f(x)$ that have a base in E and for which $|f(x)|^\alpha$ is lebesgue-integrable over E with respect to μ_E. Then $C_\alpha = \bigcup_{E\varepsilon\mathcal{R}} C_\alpha^E$ is a linear space. For $f(x) \varepsilon C_\alpha^E$ the integral

$$I_E(|f|^\alpha) = \int_E |f(P_E(x)|^\alpha d\mu_E$$

is the same for all E in which $f(x)$ has a base, due to the consistency property[39] of Gaussian measures. For such functionals define the "Hilbert space integrals"

$$I_{\mathcal{H}}(|f|^\alpha) = I_E(|f|^\alpha)$$

and

$$I_{\mathcal{H}}(f) = I_E(f) \quad .$$

Consider "basic systems" of projections (P_n), corresponding to an orthonormal basis $g_1, g_2 \ldots$ of $\mathcal{H}$:

$$x \in \mathcal{H}, \; x = \sum_{i=1}^{\infty} x_i g_i \; , \; P_n x = \sum_{i=1}^{n} x_i g_i \quad .$$

For a given $\alpha \geq 1$ form the linear space $C_\alpha^{(P_n)} = \bigcup_n C_\alpha^{P_n}$ and introduce in $C_\alpha^{(P_n)}$ the norm

$$_\alpha||f||_{(P_n)} = \left[I_{\mathcal{H}}(|f|^\alpha)\right]^{1/\alpha} \quad .$$

In this norm $C_\alpha^{(P_n)}$ may be completed, and the norm extended, to a linear space $\mathcal{L}_\alpha^{(P)}$ with norm $_\alpha||f||_{(P)}$. The elements of $\mathcal{L}_\alpha^{(P)}$ are "ideal functionals" associated with sequences of cylinder functionals of $C_\alpha^{(P_n)}$ which are Gauchy sequences in the norm $_\alpha||\ ||_{(P_n)}$ or $_\alpha||\ ||_{(P)}$.

For any $\alpha \geq 1$, proper functionals over $\mathcal{H}$ are defined to be α-invariant, written $f \varepsilon \mathcal{L}_\alpha$, if and only if

(a) for all basic systems (P_n)

$$\left\{ f(P_n x) \right\} \varepsilon \mathcal{L}_\alpha^{(P_n)}$$

(b) for any two basic systems (P_n), (Q_m)

$$\lim_{m,n\to\infty} I_{\mathcal{H}}\left(|f(P_n\cdot) - f(Q_m\cdot)|^\alpha\right) = 0 \quad .$$

For any such $f(x)$ define

$$\int_{\mathcal{H}} |f(x)|^\alpha \, d\mu_{\mathcal{H}} = \lim_{n\to\infty} I_{\mathcal{H}}\left(|f(P_n\cdot)|^\alpha\right)$$

and

$$\int_{\mathcal{H}} f(x) d\mu_{\mathcal{H}} = \lim_{n\to\infty} I_{\mathcal{H}}\left(f(P_n\cdot)\right) \quad .$$

Integrable cylinder functionals are invariant. Functionals that are polynomials of finite order (i.e. the Volterra expansion exists and has only a finite number of terms) are invariant (with $\alpha = 2$) provided the coefficient functions satisfy certain conditions[39]. We will not give them here but shall later merely derive sufficient conditions in terms of quantities appearing in our context.

The integral defined above is not totally additive. Friedrichs and Shapiro have defined a "corona" integral that is totally additive, in general invariant under only a countable number of orthogonal (resp. unitary) transformations, and numerically equal to $I_{\mathcal{H}}$ if this exists for $\alpha = 1$. We will not describe this construction since we shall not use total additivity.

Rather, we shall use elementary inequalities, especially the Hölder and Minkowski inequalities, for $I_{\mathcal{H}}$. Since $I_{\mathcal{H}}$ is defined as the limit of Gauchy sequences of finite-dimensional integrals, for which those inequalities hold, they hold also for $I_{\mathcal{H}}$. We shall also use functional differentiation under the integral sign and partial integration and prove their validity in our cases, using again the approximating sequences for which it is easily established.

The following lemma is an immediate consequence of $I_E(1) = 1$ and the Hölder inequality:

Lemma A.1:

$$f(x) \varepsilon \mathcal{L}_\alpha \text{ implies } f(x) \varepsilon \mathcal{L}_\beta \text{ for } 1 \leq \beta < \alpha .$$

Lemma A.2:

If $f(x) \varepsilon \mathcal{L}_\alpha$, and if for any γ, with $1 \leq \gamma < \infty$, ${}_{\gamma+\varepsilon}||f(P_E\cdot)||_E$ has a uniform bound for all E and some $\varepsilon > 0$, then $f(x) \varepsilon \mathcal{L}_\gamma$.

Proof: Because of the preceding lemma, we need only consider $\gamma > \alpha$.

From the Hölder inequality and the Minkowski inequality, successively

$$_{\gamma}||f'-f''||_E \le {}_{\alpha}||f'-f''||_E^{\frac{\alpha\varepsilon_{\varepsilon}}{\gamma(\gamma-\alpha+\varepsilon)}} \; {}_{(\gamma+\varepsilon)}||f'-f''||^{\frac{(\gamma+\varepsilon)(\gamma-\alpha)}{\gamma(\gamma-\alpha+\varepsilon)}} \le$$

$$\le {}_{\alpha}||f'-f''||_E^{\frac{\alpha\varepsilon}{\gamma(\gamma-\alpha+\varepsilon)}} \cdot \left({}_{(\gamma+\varepsilon)}||f'||_E^{\frac{(\gamma+\varepsilon)(\gamma-\alpha)}{\gamma(\gamma-\alpha+\varepsilon)}} + \right.$$

$$\left. + {}_{(\gamma+\varepsilon)}||f''||_E^{\frac{(\gamma+\varepsilon)(\gamma-\alpha)}{\gamma(\gamma-\alpha+\varepsilon)}} \right) .$$

Substitution of $f'(x) = f(P_n x)$, $f'' = 0$ and $f'(x) = f(P_n x)$, $f''(x) = f(Q_m x)$ and comparison with (a) resp. (b) gives the statement. It suffices if for E all projections of the form $P_n \otimes Q_m$ are taken. ε may also be $+\infty$.

Lemma A.3:

If $f_i \in \mathcal{L}_\alpha$, $i=1\ldots n$, and if $1 \le \beta < \alpha$, and if for a set of nonnegative numbers c_i with $\sum_{i=1}^{n} c_i = 1$ the norms $\frac{\alpha\beta}{(\alpha-\beta)c_i} ||f_i(P_E\cdot)||_E$ have for all E uniform upper bounds, then $\prod_{i=1}^{n} f_i \in \mathcal{L}_\beta$.

Proof: For (a)

$$_{\beta}|| \prod_{i=1}^{n} f_i' ||E \le \prod_{i=1}^{n} {}_{\beta/c_i}||f_i'||_E \le \prod_{i=1}^{n} {}_{\frac{\alpha\beta}{(\alpha-\beta)c_i}}||f_i'||_E < \infty \quad .$$

For (b)

$$\prod_{i=1}^{n} f_i' - \prod_{i=1}^{n} f_i'' = (f_1'-f_1'')f_2''\ldots f_n'' + \ldots + f_1'\ldots f_{n-1}'(f_n'-f_n'')$$

and therefore

$$_{\beta}\left|\left|\prod_{i=1}^{n} f_i' - \prod_{i=1}^{n} f_i''\right|\right|_E \leq {}_{\beta}||(f_1'-f_1'')f_2''\cdots f_n''||_E + \cdots \leq$$

$$\leq {}_{\alpha}||f_1'-f_1''||_E \cdot {}_{\frac{\alpha\beta}{\alpha-\beta}}||f_2''\cdots f_n''||_E + \cdots \leq$$

$$\leq {}_{\alpha}||f_1'-f_1''||_E \cdot {}_{\frac{\alpha\beta}{(\alpha-\beta)c_2}}||f_2''||_E \cdots {}_{\frac{\alpha\beta}{(\alpha-\beta)c_n}}||f_n''|| + \cdots \quad .$$

Suitable substitution and comparison with (a) and (b) gives the statement. Remarks similar to those after the previous proof apply. The norm requirements in the lemma are obviously stronger than necessary.

Lemma A.4:

If $f \in \mathcal{L}_\alpha$, if $1 \leq \beta < \alpha$, and if ${}_{\frac{\alpha\beta}{\alpha-\beta}}||\exp f(P_E\cdot)||_E$ has for all E a uniform upper bound, then $e^f \in \mathcal{L}_\beta$.

Proof: (a) is obviously satisfied. From

$$|\mathrm{Re}(f'-f'')| + |\mathrm{Im}(f'-f'')| \leq \sqrt{2}\,|f'-f''|$$

and the conversity of the exponential function follows

$$|e^{f'}-e^{f''}| \leq \sqrt{2}\,|f'-f''|\ \mathrm{Max}(|e^{f'}|\ ,\ |e^{f''}|)$$

and therefrom, by use of the Holder and Minkowski inequalities

$$_{\beta}||e^{f'}-e^{f''}||_E \leq \sqrt{2}\ {}_{\alpha}||f'-f''||_E \cdot \left({}_{\frac{\alpha\beta}{\alpha-\beta}}||e^{f'}||_E + {}_{\frac{\alpha\beta}{\alpha-\beta}}||e^{f''}||_E\right) .$$

Substitution and comparison with (b) gives the statement. Remarks similar to those after the proof of lemma 2 apply. We shall in the following refer to such uniform bounds as required in theorems 1-4 for brevity as "apriori upper bounds".

In the analogous manner as it is shown for finite-dimensional integrals, we can derive

Lemma A.5:

If f is real, if g is nonnegative, and if $e^{\lambda_1 f} g \in \mathcal{L}_\alpha$ and $e^{\lambda_2 f} g \in \mathcal{L}_\alpha$ for some real $\lambda_1 < \lambda_2$, then in the interval $\lambda_1 < \lambda < \lambda_2$ $e^{\lambda f} g \in \mathcal{L}_\alpha$ and $\ln\left(\int_{\mathcal{H}} e^{\lambda f} g \, d\mu_{\mathcal{H}}\right)$ is a convex function of λ.

An immediate consequence is

Lemma A.6:

If in the hypothesis of lemma 5, $\lambda_1 \leq 0$, $1 \leq \lambda_2$, then

$$\frac{1}{2} \ln \frac{\int_{\mathcal{H}} e^{\lambda f} g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}}$$

is nondecreasing in λ. For $\lambda \to +0$ and $\lambda \to 1$ we obtain the inequality of the arithmetic and geometric means:

$$\exp\left(\frac{\int_{\mathcal{H}} f g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}}\right) \leq \frac{\int_{\mathcal{H}} e^{f} g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}} .$$

Substitution and comparison with (b) gives the statement. Remarks similar to those after the proof of lemma 2 apply. We shall in the following refer to such uniform bounds as required in theorems 1-4 for brevity as "apriori upper bounds".

In the analogous manner as it is shown for finite-dimensional integrals, we can derive

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An immediate consequence is

Lemma A.6:

If in the hypothesis of lemma 5, $\lambda_1 \leq 0$, $1 \leq \lambda_2$, then

$$\frac{1}{\lambda} \ln \frac{\int_{\mathcal{H}} e^{\lambda f} g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}}$$

is nondecreasing in λ. For $\lambda \to +0$ and $\lambda \to 1$ we obtain the inequality of the arithmetic and geometric means:

$$\exp\left(\frac{\int_{\mathcal{H}} f g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}}\right) \leq \frac{\int_{\mathcal{H}} e^{f} g \, d\mu_{\mathcal{H}}}{\int_{\mathcal{H}} g \, d\mu_{\mathcal{H}}} .$$

We now introduce the concept of uniform convergence of an FS integral and prove the analoga of elementary theorems.

Definition: Let u be a set of parameters in a parameter space U. Let $f(x,u) \in \mathcal{L}_\alpha$, $\alpha \geq 1$, be such that for all $u \in U$

(c) for all basic systems (P_n)

$$_\alpha\| f(P_n\cdot ,u)\|_{(P_n)} < M(U)$$

(d) for any two basic systems (P_n), (Q_m)

$$_\alpha\| f(P_n\cdot u) - f(Q_m\cdot ,u)\|_{(P_n \otimes Q_m)} < \varepsilon$$

if $n > n(\alpha,\varepsilon,P,U)$ and $m > m(\alpha,\varepsilon,Q,U)$.

Then we call the FS integral $I_{\mathcal{H}}\big(f(\cdot,u)\big)$ uniformly α-convergent with respect to u in U.

Lemma A.7:

A uniformly α-convergent FS integral $I_{\mathcal{H}}\big(f(\cdot,u)\big)$ may be integrated under the integral sign if $\int_u dm(u) < \infty$, and the resulting integrand is α-invariant:

$$\int dm(u)\ I_{\mathcal{H}}(f(\cdot,u)) = I_{\mathcal{H}}\Big(\int dm(u) f(\cdot,u)\Big) \ .$$

Proof: From (c) and

$$\Big|\int dm(u) f(P_n x,u)\Big|^\alpha \leq \Big(\int dm(u)\Big)^{\alpha-1} \int dm(u)\, |f(P_n x,u)|^\alpha$$

follows

$$_\alpha\Big\|\int dm(u) f(\cdot,u)\Big\|_{(P_n)} \leq M(U)\int dm(u) \ .$$

From

$$|\int dm(u)[f(P_n x,u) - f(Q_m x,u)]|^{\alpha} \leq$$

$$\leq \left(\int dm(u)\right)^{\alpha-1} \int dm(u)|P_n x,u) - f(Q_m x,u)|^{\alpha}$$

and (d) follows

(d) $\quad {}_{\alpha}||\int dm(u)f(P_n\cdot,u) - \int dm(u)f(Q_m\cdot,u)||_{(P_n \otimes Q_m)} < \varepsilon \int dm(u)$.

Therefore

$$I_{\mathcal{H}}\left(\int dm(u)f(\cdot,u)\right) \equiv \lim_{n\to\infty} I_{\mathcal{H}}\left(\int dm(u)f(P_n\cdot,u)\right) =$$

$$= \lim_{n\to\infty} \int dm(u)\ I_{\mathcal{H}}\left(f(P_n\cdot,u)\right) = \int dm(u)\ I_{\mathcal{H}}\left(f(\cdot,u)\right) \quad .$$

Remark: We shall apply lemma 7 so FS integration over Hilbert space by using it for the approximating finite-dimensional integrals.

Applying lemma 7 to an indefinite integral, we immediately obtain

Lemma A.8:

A FS integral may be differentiated under the integral sign if the resulting FS integral is uniformly convergent.

Remark: We shall use this theorem also for functional differentiation. Let $f(x)$ be defined for $x \varepsilon \mathcal{H}$, and let $y \varepsilon \mathcal{H}$. Then $[\delta/\delta x(s)]f(x)$ is defined by

(A.1) $$\int y(s)[\delta/\delta x(s)]f(x)dm(s) = \lim_{\varepsilon \to \infty} \varepsilon^{-1}[f(x+\varepsilon y) - f(x)]$$

whenever the limit exists, and is linear in y, for y in a linear space. $f(x+y)$ may be recovered from its functional derivative by

(A.2) $$f(x+y) = f(x) + \int_0^1 d\lambda \int y(s)[\delta/\delta x(s)]f(x+\lambda y)dm(s)$$

provided (formally)

$$[\delta/\delta x(s)][\delta/\delta x(s')]f(x) = [\delta/\delta x(s')][\delta/\delta x(s)]f(x) \quad .$$

Finally, let $P\{x\}$ be a polynomial of finite order as defined earlier. Then[39]

$$\int_{\mathcal{H}} \exp\left[\int J(s)x(s)dm(s) - \frac{1}{2}\iint x(s)x(s')A(s,s')dm(s)dm(s')\cdot\right.$$

$$\cdot P\{x\}\, d\mu_{\mathcal{H}} = P\{[\delta/\delta(Jdm)]\}\exp\left]\frac{1}{2}\iint J(s)J(s')([1+A]^{-1}((s,s')\cdot\right.$$

$$\left.\cdot dm(s)dm(s') - \frac{1}{2}\int dm(s)(\ln[1+A])(s,s)\right]$$

whenever the integrand is invariant. We rewrite this formula in the notation we shall actually employ:

(A.3) $$\int \exp\left[(J\phi) - \frac{1}{2}(\phi A\phi)\right]P\{\phi\}\exp\left[-\frac{1}{2}(\phi\phi)\right]\mathcal{D}_{FS}(\phi) =$$

$$= P\{\delta/\delta J\}\exp\left[\frac{1}{2}(J[1+A]^{-1}J) - \frac{1}{2}\mathrm{Tr}\ln(1+A)\right] \quad .$$

Appendix B: Green's functions to the regularized Yukawa operator.

Let

(B.1) $$K = \prod_{i=1}^{N} (-\Delta+m_i^2) = \sum_{k=o}^{N} (-\Delta)^k s_{N-k}$$

where Δ is the Laplacian in d-dimensional Cartesian coordinates. The m_i^2 are N positive constants (squared "masses") and s_n their nth elementary symmetric function. K is a formal elliptic operator.

The "half Green's formula" for a d-dimensional domain Ω with boundary $\partial\Omega$ is

(B.2) $$(\phi K\psi) = \int_{\Omega} dx\phi(x)\vec{K}\phi(x) + \oint_{\partial\Omega} dO(x) \cdot \left[\sum_{k=o}^{N-1} \phi(x)\overleftarrow{d}_k \cdot \vec{D}_k\phi(x)\right]\Big|_s = (\phi\vec{K}\psi) + \phi\underline{\overleftarrow{d}}\cdot\underline{\vec{D}}\psi \ .$$

Here $(\phi K\psi)$ is a bilinear symmetric positive definite form in Ω, involving derivatives in such a way that partial integration gives the right hand side. Arrows indicate the factor that is differentiated. $\overleftarrow{d}_k$ and $\overleftarrow{D}_k$ are two sets of N differential operators, each, and $\vec{d}_k$ and $\overleftarrow{D}_k$ shall denote the transposed operators. All integrals in this appendix will be understood as Lebesgue integrals. From (2) Green's formula follows:

(B.3) $$(\phi\vec{K}\psi) - (\phi\overleftarrow{K}\psi) = -\phi\underline{\overleftarrow{d}}\cdot\underline{\vec{D}}\psi + \phi\underline{\overleftarrow{D}}\cdot\underline{\vec{d}}\psi \ .$$

A natural choice for K in (2) would be

(B.4) $$K = \prod_{i=1}^{N} \left(\sum^{d} \overleftarrow{\partial}_k \overrightarrow{\partial}_k + m_i^2 \right) \quad .$$

This leads for $N > 1$ to complicated operators D_k, however. Simpler boundary terms are obtained with

(B.5) $$K = \sum_{u=o}^{[N/2]} s_{n-2u} (\overleftarrow{\Delta})^u (\overrightarrow{\Delta})^u + \sum_{u=o}^{[(N-1)/2]} s_{N-1-2u} \cdot$$
$$\cdot \sum_{k=1}^{d} (\overleftarrow{\Delta})^u \overleftarrow{\partial}_k \overrightarrow{\partial}_k (\overrightarrow{\Delta})^u \quad .$$

Then, with ∂_n the normal derivative pointing outward,

(B.6) $$\overleftarrow{d}_k = (\overleftarrow{\Delta})^{[k/2]} (\overleftarrow{\partial}_n)^{k-2[k/2]}$$

and

(B.7) $$\overrightarrow{D}_k = \sum_{u=o}^{N-k-1} (-1)^u s_{N-1-k-u} (\overrightarrow{\partial}_n)^{k+1-2[(k+1)]} (\overrightarrow{\Delta})^{u+[(k+1)/2]}$$

for $k = 0 \ldots N-1$.

It should be noted that since $\partial\Omega$ is closed, $\overleftarrow{\underline{d}}$ and $\overrightarrow{\underline{D}}$ are not unique. Our choice (6) of $\overleftarrow{d}_k$ is convenient since no explicit reference to the curvature of $\partial\Omega$ need be made. Comparing (2) for two different choices of K, e.g. (4) and (5), we may use the same $\overleftarrow{d}$ as given by (6) but obtain different $\overrightarrow{d}$:

(B.8) $$(\phi K_1 \psi) - (\phi K_2 \psi) = \phi \overleftarrow{\underline{d}} \cdot (\overrightarrow{\underline{D}}_1 - \overrightarrow{\underline{D}}_2) \psi = \phi (\overleftarrow{\underline{D}}_1 - \overleftarrow{\underline{D}}_2) \cdot \overrightarrow{\underline{d}} \psi \quad .$$

Therefore,

(B.9) $$\vec{\underline{D}}_1 - \vec{\underline{D}}_2 = \underline{K}_{12} \cdot \vec{d}$$

where $\underline{K}_{12}$ is a symmetric $N \times N$ form on $\partial\Omega$ for which we shall derive an explicit expression later.

Next we study extremal problems associated with $(\phi\underline{K}\phi) \equiv ||\phi||_N^2$ and the corresponding boundary value problems. If $d_k\phi$ has at a boundary point (or rather boundary element) a prescribed value, we say that a Dirichlet condition is imposed. If $D_k\phi$ has a prescribed value, a Neumann condition is imposed. Since $k = 0 \ldots N-1$, let (D) denote the manifold on an N-fold covering of $\partial\Omega$ on which Dirichlet conditions are imposed. Let $(\partial\Omega)^N - (D) = (N)$, where Neumann conditions could be imposed. If (D) covers part of $\partial\Omega$ N-fold, we shall say that on that part complete Dirichlet conditions are imposed. We postpone the question of in which sense Neumann conditions or inhomogeneous Dirichlet conditions may be imposed.

With a real continuous function $J(x)$ we define

(B.10) $$(JGJ) = \underset{\underline{\vec{d}}\phi = 0 \text{ on } (D)}{\text{l.u.b.}} \frac{(J\phi)^2}{||\phi||_N^2} \equiv ||J||_{-N}^2 \ .$$

The positive definite bounded kernel G is the Green's function to mixed boundary conditions:

(B.11a) $$\vec{K}G(x,y) = \delta(x-y)$$

(B.11b) $$\underline{\vec{d}}G(x,y)\Big|_{x \to s \,\varepsilon\, (D)} = 0 \ , \ \underline{\vec{D}}G(x,y)\Big|_{x \to s \,\varepsilon\, (N)} = 0$$

For $(N) = \emptyset$, $G = G_D$ the Dirichlet Green's function, for $(D) = \emptyset$, $G = G_N$ the Neumann Green's function. We have from (10)

$$(B.12) \qquad 0 < (JG_DJ) < (JG_oJ) < (JG_NJ) < s_N^{-1}(JJ)$$

where G_o is the Green's function to K in infinite space.

$$(B.13) \qquad G_o(x-y) = (2\pi)^{-d} \int e^{ik(x-y)} d^dk \prod_{i=1}^{N} (k^2+m_i^2)^{-1}$$

and (JG_oJ) the form restricted to Ω. G_N, in contrast to G_D, depends on the choice of K. From

$$(B.14a) \qquad \vec{K}[G(x,y) - G_o(x-y)] = 0 \quad .$$

$$(B.14b) \qquad \vec{\underline{d}}[G(x,y) - G_o(x-y)]\Big|_{x\to s\,\varepsilon\,(D)} \doteq \vec{\underline{d}}G_o(x-y)\Big|_{x\to s\,\varepsilon\,(D)} ,$$

$$(B.14c) \qquad \vec{\underline{D}}[G(x,y) - G_o(x-y)]\Big|_{x\to s\,\varepsilon\,(N)} = -\vec{\underline{D}}G_o(x-y)\Big|_{x\to s\,\varepsilon\,(N)}$$

follows[44] that $G(x,y) - G_o(x-y)$ is a real-analytic function of 2d arguments. The behavior of this difference function at $\partial\Omega$ depends on the smoothness of $\partial\Omega$. If $N \geq \left[\frac{1}{2}d\right] + 1$, $G_o(x-y)$ is finite, and from (12)

$$(B.15a) \qquad 0 \leq G_D(x,x) \leq G_o(0) < \infty$$

and

$$(B.15b) \qquad |G_D(x,y)| \leq G_D(x,x)^{1/2} G_D(y,y)^{1/2} < G_o(0) < \infty$$

with $G(x,x) = 0$ only on $\partial\Omega$. On a part of $\partial\Omega$ where the cone condition[35] is satisfied, also $G_N(x,y)$ is finite for one or both coordinates on $\partial\Omega$, as follows from (10), Sobolev's lemma[45], and (14). Then also $G(x,y)$ is finite because of

(B.16) $$(JG_DJ) \leq (JGJ) \leq (JG_NJ) \quad .$$

We now introduce a more concise notation. $\overleftarrow{\underline{\underline{D}}} = (\overleftarrow{\underline{d}}\ \overleftarrow{\underline{D}})$ is a 2N-component row vector and $\overrightarrow{D}$ the transposed column vector. Further, let

$$\eta_1 = \begin{pmatrix} 0 & -(N) \\ (D) & 0 \end{pmatrix} \quad , \quad \eta_2 = \begin{pmatrix} 0 & -(D) \\ (N) & 0 \end{pmatrix} = -\eta_1^T \quad ,$$

and $\eta = \eta_1 + \eta_2$. (D) and (N) stand for the N-dimensional restrictions on (D) and (N) , respectively. We have

$$\eta_1^2 = \eta_2^2 = 0 \quad , \quad \eta^2 = -1 \quad ,$$

and

$$-\eta_1\eta_2 = 1 + \eta_2\eta_1$$

is a projection operator. (3) may be written

(B.3') $$(\phi\overrightarrow{K}\psi) - (\phi\overleftarrow{K}\psi) = \phi\overleftarrow{\underline{\underline{D}}}\cdot\eta\cdot\overrightarrow{\underline{\underline{D}}}\psi$$

and (11b)

(B.11b') $$\eta_1\cdot\overrightarrow{\underline{\underline{D}}}G = G\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = 0 \quad .$$

The solution of the mixed boundary value problem

(B.17) $$\overrightarrow{K}\phi = J \quad , \quad \phi\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = ((N)\underline{b}\ (D)\underline{a}) \equiv \underline{A}$$

is, with (3') and (11'),

(B.18) $$\Phi = GJ - G\overleftarrow{\underline{\underline{D}}}\cdot\underline{A} \quad .$$

It follows that

(B.19a) $$\eta_1\cdot\overrightarrow{\underline{\underline{D}}}_{(2)}G_{(1)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_1 = -\eta_1$$

where the subscripts (1), (2) denote the order in which the arguments are put on the boundary.

By transposition,

(B.19b) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}_{(1)}G_{(2)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = \eta_2 \quad .$$

We also introduce the Green's function G' for which the roles of (D) and (N) are interchanged. It satisfies

(B.20a) $$\overrightarrow{K}\, G' = \delta$$

(B.20b) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}G' = G'\overleftarrow{\underline{\underline{D}}}\cdot\eta_1 = 0 \quad ,$$

(B.21) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}_{(2)}G'_{(1)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = -\eta_2,\ \eta_1\cdot\overrightarrow{\underline{\underline{D}}}_{(1)}G'_{(2)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_1 = -\eta_1 \quad .$$

From (3'), (11), and (20)

(B.22) $$G - G' = G\ \overleftarrow{\underline{\underline{D}}}\cdot\eta_1\cdot\overrightarrow{\underline{\underline{D}}}\ G' = -G'\ \overleftarrow{\underline{\underline{D}}}\cdot\eta_2\cdot\overrightarrow{\underline{\underline{D}}}\ G \quad .$$

Applying $\eta_2\cdot\overrightarrow{\underline{\underline{D}}}$ from the left,

(B.23) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}\ G = \eta_2\cdot\overrightarrow{\underline{\underline{D}}}_{(2)}G_{(1)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_1\cdot\overrightarrow{\underline{\underline{D}}}\ G' \quad ,$$

and $\overleftarrow{\underline{\underline{D}}}\cdot\eta_1$ from the right, and using (21)

(B.24) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}_{(1)}G_{(2)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_1 = \eta_2\cdot\overrightarrow{\underline{\underline{D}}}_{(2)}G_{(1)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_1 \equiv \eta_2\cdot\overrightarrow{\underline{\underline{D}}}\, G\, \overleftarrow{\underline{\underline{D}}}\cdot\eta_1 \equiv \underline{G}_{12} \quad ,$$

a symmetric $2N \times 2N$ kernel on $\partial\Omega$. Combining (24) with (11b') and (19), we obtain the discontinuity relation

(B.25) $$\overrightarrow{\underline{\underline{D}}}_{(2)}G_{(1)}\overleftarrow{\underline{\underline{D}}} - \overrightarrow{\underline{\underline{D}}}_{(1)}G_{(2)}\overleftarrow{\underline{\underline{D}}} = \eta$$

which also holds for G'. The relation analogous to (24) for G' is

(B.26) $$\eta_1\cdot\overrightarrow{\underline{\underline{D}}}_{(1)}G'_{(2)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = \eta_1\cdot\overrightarrow{\underline{\underline{D}}}_{(2)}G'_{(1)}\overleftarrow{\underline{\underline{D}}}\cdot\eta_2 \equiv \eta_1\cdot\overrightarrow{\underline{\underline{D}}}\, G'\, \overleftarrow{\underline{\underline{D}}}\cdot\eta_2 \equiv \underline{G}'_{12} \quad .$$

Applying on (B.23) $\overleftarrow{\underline{\underline{D}}}\eta_2$ from the right, we obtain with (19b), (24), and (26)

(B.27a) $$\eta_2\cdot\overrightarrow{\underline{\underline{D}}}\, G\, \overleftarrow{\underline{\underline{D}}}\cdot\eta_1\cdot\overrightarrow{\underline{\underline{D}}}\, G'\, \overleftarrow{\underline{\underline{D}}}\cdot\eta_2 = -\underline{G}_{21}\cdot\eta_2\cdot\underline{G}'_{12} = \eta_2 \quad .$$

Also, by transposition, or interchange of (N) and (D)

(B.27b) $$-\underline{G}'_{12}\cdot\eta_1\cdot\underline{G}_{21} = \eta_1 \quad .$$

Introducing the symmetric operators

$$-\eta_1\cdot\underline{G}_{21}\cdot\eta_2 = \underline{G}, \; -\eta_1\cdot\eta_2\cdot\underline{G}'_{12}\cdot\eta_1\cdot\eta_2 = \underline{G}' \quad ,$$

allows to write (27) simply

(B.28) $$\underline{G}\cdot(-\eta_1\cdot\eta_2)\cdot\underline{G}' = \underline{G}'\cdot(-\eta_1\cdot\eta_2)\underline{G} = -\eta_1\cdot\eta_2 \quad .$$

Thus, in the subspace determined by the projection

$-\eta_1 \cdot \eta_2 = \begin{pmatrix} (N) & 0 \\ 0 & (D) \end{pmatrix}$, $\underline{G}$ and $\underline{G}'$ are inverse to each other. Especially, for $(N) = \emptyset$, only N components are left:

$$\text{(B.29)} \qquad \underline{G} = -\overrightarrow{\underline{D}}\, G_D \overleftarrow{\underline{D}} \;, \quad \underline{G}' = \underline{G}^{-1} = \overrightarrow{\underline{D}}\, G_N \overleftarrow{\underline{D}} \;.$$

Then (23) becomes

$$\text{(B.30)} \qquad -\overrightarrow{\underline{D}}\, G_D = \underline{G} \cdot \overrightarrow{\underline{d}}\, G_N \;,$$

and (22), with use of (30)

$$\text{(B.31)} \qquad G_N = G_D + G_D \overleftarrow{D} \cdot \underline{G}^{-1} \cdot \overrightarrow{\underline{D}}\, G_D = G_D + G_N \overleftarrow{\underline{d}} \cdot \underline{G} \cdot \overrightarrow{\underline{d}}\, G_N \;.$$

Let G_o be any Green's function defined in a domain that includes Ω and may be the entire space, in which case G_o would be identical to (15). (3') gives with (11) and (20).

$$\text{(B.32)} \qquad G_o = G + G_o \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}\, G = G' - G' \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}\, G_o \;.$$

Then

$$\overrightarrow{\underline{\underline{D}}}\, G_o = (1 + \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_2) \cdot \overrightarrow{\underline{\underline{D}}}\, G$$

and with (19b)

$$\text{(B.33a)} \qquad \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 = \eta_2 + \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 \;.$$

By transposition

$$\text{(B.33b)} \qquad \eta_1 \cdot \overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_1 = \eta_1 + \eta_1 \cdot \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_1 \;.$$

Since from (32) and (24)

(B.33c) $$\eta_2 \cdot \overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_1 = \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_1$$

and from (32) and (26)

(B.33d) $$\eta_1 \cdot \overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 = \eta_1 \cdot \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \cdot \eta_2 \quad ,$$

we can combine (33) to

(B.34) $$\overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} - \overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} = \eta$$

which generalizes (25). If we define

$$\overrightarrow{\underline{\underline{D}}}_{(1)} G_{o(2)} \overleftarrow{\underline{\underline{D}}} + \overrightarrow{\underline{\underline{D}}}_{(2)} G_{o(1)} \overleftarrow{\underline{\underline{D}}} \equiv 2\overrightarrow{\underline{\underline{D}}}\, G_o \overleftarrow{\underline{\underline{D}}}$$

then from (32) and (34), with $\eta_2\eta_1\eta_2 = -\eta_2$,

(B.35) $$G = G_o - 2G_o \overleftarrow{\underline{\underline{D}}} \cdot (1 - 2\eta_2 \overrightarrow{\underline{\underline{D}}}\, G_o \overleftarrow{\underline{\underline{D}}})^{-1} \cdot \eta_2 \cdot \overrightarrow{\underline{\underline{D}}}\, G_o$$

which is the formal solution of the mixed problem obtained by appropriate adaption of the Fredholm method[46)*]. If Ω is the half space, the evaluation of (35) for G_D or G_N is elementary in Fourier space, as is also the evaluation of G if Ω is bounded by two parallel hyperplanes, with Dirichlet conditions on one and Neumann conditions on the other. This function will be will be needed in future work. Other formulae are obtained by

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* We do not give here the obvious modifications of (35) involving iterated kernels.

using (32) first to calculate G_D or G_N from G_o , and then to calculate G from G_o resp. G_N.

If (N) includes the n uppermost coverings of $\partial\Omega$, i.e. if $\vec{D}_{N-k}G = 0$ (K=1...n) on all $\partial\Omega$, then

$$\text{(B.36)} \qquad \lim_{\substack{m_i^2=\infty \\ i=1\ldots n}} \left[m_1^2 \ldots m_n^2 G\right] \equiv G_{red}$$

exists and is the Green's function to the "reduced" problem[47] where in (1) the first n factors are omitted, the (D) and (N) conditions on the remaining N-n coverings of $\partial\Omega$ being unchanged apart from the obvious redefinition of the operators $\vec{D}_o \ldots \vec{D}_{N-n-1}$.

We now give a few formulae that are immediate consequences (2), (3), (25), and (29):

$$\text{(B.37)} \qquad 2(J\Phi) - (\Phi K\Phi) = (JG_DJ) - 2\underline{a}\cdot\underline{\vec{D}}G_DJ -$$

$$- \underline{a}\cdot\underline{G}\cdot\underline{a} - 2\rho\underline{\overleftarrow{d}}\cdot\underline{G}\cdot\underline{a} - (\rho K\rho)$$

where $\Phi = G_DJ - G_D\underline{\overleftarrow{D}}\cdot\underline{a} + \rho$, ,

$$\text{(B.38)} \qquad 2(J\Phi) - (\Phi K\Phi) = (JG_NJ) - (\rho K\rho)$$

where $\Phi = G_NJ + \rho$,

$$\text{(B.39)} \qquad 2\underline{b}\cdot\underline{\vec{d}}\Phi - (\Phi K\Phi) = \underline{b}\cdot\underline{G}^{-1}\cdot\underline{b} - (\rho K\rho)$$

where $\Phi = G_N\underline{\overleftarrow{d}}\cdot\underline{b} + \rho$, and

(B.40) $(\Phi K \Phi) = \underline{a}\cdot\underline{G}\cdot\underline{a} + 2\rho\overleftarrow{\underline{d}}\cdot G\cdot\underline{a} + (\rho K \rho)$

where $\Phi = -G_D\overleftarrow{\underline{D}}\cdot\underline{a} + \rho$. While (37) with $\underline{a} \equiv 0$ and (38) have an obvious relation to (10), (38) and (39) give rise to the extremal principles

(B.41) $$\underline{b}\cdot\underline{G}^{-1}\cdot\underline{b} = \text{l.u.b.} \frac{(\underline{b}\cdot\overrightarrow{\underline{d}}\Phi)^2}{||\Phi||_N^2} \equiv |\underline{b}|^2_{-N+\frac{1}{2}}$$

and

(B.42) $$\underline{a}\cdot\underline{G}\cdot\underline{a} = \underset{\overrightarrow{\underline{d}}\Phi=\underline{a}}{\text{g.l.b.}} ||\Phi||_N^2 \equiv |\underline{a}|^2_{N-\frac{1}{2}} \quad .$$

$|\underline{b}|_{-N+\frac{1}{2}}$ and $|\underline{a}|_{N-\frac{1}{2}}$ are norms of N-component functions defined on $\partial\Omega$. The subscripts denote the lowest respectively highest derivative, which occur for b_o resp. a_o , and negative derivatives have the usual meaning analogous to the notation used in (10) already, while broken derivatives are best defined in Fourier space. The components b_i and a_i occur, as seen from (29) and Schwarz' inequality, essentially with derivatives of order[48] $- N + i + \frac{1}{2}$ and $N - i - \frac{1}{2}$, $i=0...N-1$, respectively. We may regard (42) and (41) as strong definitions of classes of basic functions and generalized functions, respectively, in the space of N-component functions on $\partial\Omega$.

This interpretation is relevant if we admit, as we must in our applications, for the functions $\underline{a}$ and $\underline{b}$ that appear in our formulas ideal functions defined only as elements of the

Hilbert spaces $|\underline{a}|_{N-\frac{1}{2}} < \infty$ and $|\underline{b}|_{-N+\frac{1}{2}} < \infty$. This situation by a few further formulae:

From $\rho = G_D\vec{K}\rho - G_D\overleftarrow{\underline{D}}\cdot\underline{\vec{d}}\rho$ and (2) we find easily

(B.43) $\quad (\rho K\rho) = (\rho \overleftarrow{K} G_D \vec{K}\rho) + \rho\overleftarrow{\underline{d}}\cdot\underline{G}\cdot\underline{\vec{d}}\rho \quad .$

Therefore, if for a sequence $\rho_n(n=1\ldots\infty)||\rho_n||_N \longrightarrow 0$, also $(\rho_n\overleftarrow{K}G_D\vec{K}\rho_n) \longrightarrow 0$ and $|\underline{\vec{d}}\rho_n|_{N-\frac{1}{2}} \longrightarrow 0$. On the other hand, from $\rho = G_N\vec{K}\rho + G_N\overleftarrow{d}.\underline{\vec{D}}\rho$ we derive $(\rho K\rho) = (\rho\overleftarrow{K}G_N\vec{K}\rho) - \rho\overleftarrow{\underline{D}}\cdot G^{-1}\underline{\vec{D}}\rho$

$+ 2\rho\overleftarrow{\underline{d}}\cdot\underline{\vec{D}}\rho$. Using (30) gives

$$\underline{\vec{d}}\rho = \underline{\vec{d}}G_N K\rho + \underline{G}^{-1}\cdot\underline{\vec{D}}\rho = \underline{G}^{-1}\cdot(\underline{\vec{D}}\rho - \underline{\vec{D}}G_D K\rho)$$

and finally

$$(\rho\overleftarrow{K}G_N\vec{K}\rho) = (\rho\overleftarrow{K}G_D\vec{K}\rho) + (\rho\overleftarrow{\underline{D}}-\rho\overleftarrow{\underline{d}}\cdot\underline{G})\cdot\underline{G}^{-1}\cdot(\underline{\vec{D}}\rho-\underline{G}\cdot\underline{\vec{d}}\rho) \quad .$$

Therefore, $(\rho_n\overleftarrow{K}G_N\vec{K}\rho_n) \longrightarrow 0$ implies $(\rho_n\overleftarrow{K}G_D\vec{K}\rho_n) \longrightarrow 0$ and $|\underline{\vec{D}}\rho_n - \underline{G}\cdot\underline{\vec{d}}\rho_n|_{-N+\frac{1}{2}} \longrightarrow 0$. If, in addition $||\rho_n||_N \longrightarrow 0$, $|\underline{\vec{d}}\rho_n|_{N-\frac{1}{2}} \longrightarrow 0$ and $|\underline{\vec{d}}\rho_n|_{-N+\frac{1}{2}} \longrightarrow 0$ separately.

As an application of these concepts, we have

Theorem A: If $(J\Phi) = 0$ for all J such that $(JG_NJ) < \infty$, and $\underline{a}\cdot\underline{\vec{D}}\Phi = 0$ for all $\underline{a}$ such that $|\underline{a}|_{N-\frac{1}{2}} < \infty$, and if both integrals are well determined in the L_2 sense, then $(\phi \overleftarrow{K} G_N \overrightarrow{K} \phi) = 0$, $|\overrightarrow{\underline{d}}\phi|_{N-\frac{1}{2}} = 0$, $|\overrightarrow{\underline{D}}\phi|_{-N+\frac{1}{2}} = 0$.

Proof: $|\overrightarrow{\underline{D}}\phi|_{-N+\frac{1}{2}} = 0$ follows immediately. Now $(J\phi) = (JG_N \overrightarrow{K}\phi) + JG_N \overleftarrow{\underline{d}} \cdot \overrightarrow{\underline{D}}\phi = 0$. Since $|\overrightarrow{\underline{d}} G_N J|^2_{N-\frac{1}{2}} = (JG_N J) - JG_D J) < \infty$ by (31), $(JG_N \overrightarrow{K}\phi) = 0$ and well determined. This requires $(\phi \overleftarrow{K} G_N \overrightarrow{K}\phi) = 0$. (44) then gives $|\overrightarrow{\underline{d}}\phi|_{N-\frac{1}{2}} = 0$.

We now give a more precise formulation of the Dirichlet problem and of its solution than we gave in (17),(18). Let $K\phi = J$ and $\overrightarrow{\underline{d}}\phi = \underline{a}$. If $(JG_D J) < \infty$ and $|\underline{a}|_{N-\frac{1}{2}} < \infty$ there exist sequences of well behaved functions* J_i and $\underline{a}_i$ such that $((J-J_i)G_D(J-J_i)) \longrightarrow 0$ and $|\underline{a}-\underline{a}_i|_{N-\frac{1}{2}} \longrightarrow 0$ as $i \longrightarrow \infty$. Then $\phi_i \equiv G_D J_i - G_D \overleftarrow{\underline{D}} \cdot \underline{a}_i \longrightarrow \phi$ in the sense $||\phi-\phi_i|| \longrightarrow 0$ due to (44).

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* This statement is nontrivial and does require proof. For smooth boundaries it has been shown by Schechter[47], but it is likely to hold more generally. The author is indebted to Prof. Schechter for a discussion of this point. The meaning of well-behavedness can easily be made more precise.

For completeness only we also reformulate the Neumann problem: Let $K\phi = J$ and $\vec{\underline{D}}\phi = \underline{b}$. If $(JG_NJ) < \infty$ and $|\underline{b}|_{-N+\frac{1}{2}} < \infty$, there exist sequences of well-behaved functions J_i and $\underline{b}_i$ such that $((J-J_i)G_N(J-J_i)) \rightarrow 0$ and $|\underline{b}-\underline{b}_i|_{-N+\frac{1}{2}} \rightarrow 0$ as $i \rightarrow \infty$. Then $\phi_i \equiv G_NJ_i + G_N\overleftarrow{\underline{d}}\cdot\underline{b}_i \rightarrow \phi$ in the sense $||\phi-\phi_i||_N \rightarrow 0$ due to (44) and (43).

Inserting $\phi = -G_D\overleftarrow{\underline{D}}\cdot\underline{a}$ in (8), we obtain with (25) used for G_D

$$(\underline{D}_1-\underline{D}_2)\psi = -\vec{\underline{D}}_{(1)}G_{D(2)}(\overleftarrow{\underline{D}}_1-\overleftarrow{\underline{D}}_2)\cdot\vec{\underline{d}}\psi \quad ,$$

or

$$\underline{K}_{12} = -\vec{\underline{D}}_{(1)}G_{D(2)}(\overleftarrow{\underline{D}}_1-\overleftarrow{\underline{D}}_2) \quad .$$

$\underline{K}_{12}$ is, of course, independent of the choice of $\vec{\underline{D}}$ since $\vec{d}G_D = 0$. Choosing $\vec{\underline{D}} = \vec{\underline{D}}_1$ and $\vec{\underline{D}} = \vec{\underline{D}}_2$, we obtain with (29)

(B.45) $$\underline{K}_{12} = \underline{G}_1 - \underline{G}_2 \quad .$$

We introduce the integral operators H by

(B.46) $$HH^T = G \quad .$$

Since G is positive definite, the general solution is the integral operator

(B.47a) $$H = G^{1/2}U$$

where $G^{1/2}$ is the positive square root and U, satisfying

(B.47b) $$UU^T = 1$$

is the general orthogonal operator. H^T maps $(fGf) < \infty$ functions into L_2 functions.

We shall also have to use the solution of

(B.48) $$\underline{F}^T\underline{F} = \underline{H}^{-1}$$

where f is a positive definite $N \times N$ integral operator (specifically, (55) below). The general solution is

(B.49a) $$\underline{F} = \underline{U} \cdot \underline{H}^{-1/2}$$

where $\underline{U}$ is the general $N \times N$ orthogonal integral operator.

(B.49b) $$\underline{U}^T \cdot \underline{U} = 1 \quad .$$

Finally, we derive "composition formulae*". Let the domain Ω be divided into domains Ω_1 and Ω_2 by a hypersurface such that $S = \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. Let $S_1 = \partial\Omega_1 - S$ and $S_2 = \partial\Omega_2 - S$, and let G_D be the Dirichlet function to $\Omega = \Omega_1 + \Omega_2$ and $G_{1,2D}$ the Dirichlet functions to $\Omega_{1,2}$. Let bracketed superscripts denote normal differentions on S outward from Ω_1 or

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* Formulae relating Green's functions to different domains and boundary conditions, of the variety considered here, are given for $N = 1$ in: S. Bergmann, M. Schiffer, "Kernel functions of Elliptic Differential Equations in mathematical physics", Acad. Press Inc. (New York, 1953).

Ω_2. Then, if $\underline{E}$ is the $N \times N$ matrix $\underline{E}_{ij} = (-1)^i \delta_{ij}$ $(i,j = 0 \ldots N-1)$,

(B.50) $$\underline{E}\cdot\underline{E} = 1 \; , \; \underline{E}\cdot\underline{\vec{d}}^{(1)} = \underline{\vec{d}}^{(2)} , \; -\underline{E}\cdot\underline{\vec{D}}^{(1)} = \underline{\vec{D}}^{(2)} \; .$$

For both arguments in Ω_1 , we have from (3) used in Ω_1

(B.51) $$G_D = G_{1D} - G_D^{(1)}\underline{d}\cdot(s)\cdot\underline{\vec{D}}^{(1)}G_{1D}$$

where (s) means restriction of the integration to S. From (51)

(B.52a) $$G_D = G_{1D} + G_{1D}{}^{(1)}\underline{\overleftarrow{D}}\cdot(s)\cdot\underline{\vec{d}}^{(1)}G_D{}^{(1)}\underline{\overleftarrow{d}}\cdot(s)\cdot\underline{\vec{D}}^{(1)}G_{1D} =$$

$$= G_{1D} + G_D{}^{(1)}\underline{\overleftarrow{d}}\cdot(s)\cdot(\underline{\vec{d}}^{(1)}G_D{}^{(1)}\underline{\overleftarrow{d}})_s^{-1}\cdot(s)\cdot\underline{\vec{d}}^{(1)}G_D$$

where the subscript s means that the inverse is taken on S. For both arguments in Ω_2,

(B.52b) $$G_D = G_{2D} + G_{2D}{}^{(2)}\underline{\overleftarrow{D}}\cdot(s)\cdot\underline{\vec{d}}^{(2)}G_D{}^{(2)}\underline{\overleftarrow{d}}\cdot(s)\cdot\underline{\vec{D}}^{(2)}G_{2D} \; ,$$

and for the left argument in Ω_1 , the right one in Ω_2 , from (3) used in Ω_1 and Ω_2.

(B.53) $$G_D = -G_{1D}{}^{(1)}\underline{\overleftarrow{D}}\cdot(s)\cdot\underline{\vec{d}}^{(1)}G_D = -G_D{}^{(2)}\underline{\overleftarrow{d}}\cdot(s)\cdot\underline{\vec{D}}^{(2)}G_{2D}$$

wherefrom, with (50),

(B.54) $$G_D = G_{1D}{}^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot\overrightarrow{\underline{d}}^{(1)}G_D{}^{(2)}\overleftarrow{\underline{d}}\cdot(s)\cdot\overrightarrow{\underline{D}}^{(2)}G_{2D}$$

$$= -G_{1D}{}^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot\overrightarrow{\underline{d}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{d}}\cdot(s)\cdot\overrightarrow{\underline{D}}^{(1)}G_{2D} \quad .$$

We calculate $\overrightarrow{\underline{D}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{D}}$ with both coordinates on S from (52a) and from (54). Comparison gives, on S ,

$$\overrightarrow{\underline{d}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{d}}\cdot\left[-\overrightarrow{\underline{D}}^{(1)}G_{1D}{}^{(1)}\overleftarrow{\underline{D}} - \overrightarrow{\underline{D}}^{(1)}G_{2D}{}^{(1)}\overleftarrow{\underline{D}}\right] = 1$$

where we have used that $-\overrightarrow{\underline{D}}^{(1)}G_{1D}{}^{(1)}\overleftarrow{\underline{D}}$, being positive definite, possesses an inverse on S. Therefore, on S , with (29),

(B.55) $$\overrightarrow{\underline{d}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{d}} = (\underline{G}_1 + \underline{E}\cdot\underline{G}_2\cdot\underline{E})_s^{-1} \equiv \underline{H}$$

which we may substitute in (52) - (54).

We also give a formula relating G_D and the functions G_1 and G_2 that satisfy Dirichlet conditions on S_1 resp. S_2 but Neumann conditions on S. Using a similar technique as before, we have, for both arguments in Ω_1

(B.56) $$G_D = G_1 - G_D{}^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot(\overrightarrow{\underline{D}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{D}})_s^{-1}\cdot(s)\cdot\overrightarrow{\underline{D}}^{(1)}G_D$$

where

(B.57a) $$\overrightarrow{\underline{D}}^{(1)}G_D{}^{(1)}\overleftarrow{\underline{D}} = (\overrightarrow{\underline{d}}^{(1)}G_1{}^{(1)}\overleftarrow{\underline{d}} + \underline{E}\cdot\overrightarrow{\underline{d}}^{(2)}G_2{}^{(2)}\overleftarrow{\underline{d}}\cdot\underline{E})_s^{-1}$$

and

(B.57b) $$\overrightarrow{\underline{d}}^{(1,2)}G_{1,2}{}^{(1,2)}\overleftarrow{\underline{d}} = (\underline{G}_{1,2})_s^{-1} \quad .$$

We now state

Theorem B: Let f_1 and f_2 be functions in Ω_1 resp. Ω_2 such that $\vec{K}f_{1,2} = g_{1,2}$, $\vec{\underline{d}}f_{1,2} = \underline{a}_{1,2}$ on $S_{1,2}$, and on S both conditions

$$\alpha)\vec{\underline{d}}^{(1)}f_1 = \underline{E}\cdot\vec{\underline{d}}^{(2)}f_2,\quad \beta)\vec{\underline{D}}^{(1)}f_1 = -\underline{E}\cdot\vec{\underline{D}}^{(2)}f_2$$

hold. Then f_1 and f_2 are the restrictions to Ω_1 resp. Ω_2 of the solution f of the boundary value problem*

$$Kf = \chi(\Omega_1)g_1 + \chi(\Omega_2)g_2 \quad \text{in } \Omega \ ,$$

$$\vec{\underline{d}}f = \chi(S_1)\underline{a}_1 + \chi(S_2)\underline{a}_2 \quad \text{on } S_1 + S_2 = \partial\Omega \ .$$

It suffices that $\alpha)$ hold up to vectors, restricted to S, of $||_{N-\frac{1}{2}}$-norm zero on $\partial\Omega_1$ or $\partial\Omega_2$, and that $\beta)$ hold up to vectors, restricted to S, of $||_{-N+\frac{1}{2}}$-norm zero on $\partial\Omega_1$ or $\partial\Omega_2$.

Proof: f is given by

$$f = G_D\,\chi(\Omega_1)g_1 + G_D\,\chi(\Omega_2)g_2 -$$

$$- G_D\overleftarrow{\underline{D}}\cdot\,\chi(S_1)\underline{a}_1 - G_D\overleftarrow{\underline{D}}\cdot\,\chi(S_2)\underline{a}_2 \ .$$

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* $\chi(\cdot)$ is the characteristic function of the set.

We use (3) in the first two terms and obtain in $\Omega_{1,2}$

$$f = f_{1,2} - G_D{}^{(1)}\overleftarrow{\underline{d}}\cdot(s)\cdot\overrightarrow{\underline{D}}^{(1)}f_1 - G_D{}^{(2)}\overleftarrow{\underline{d}}\cdot(s)\cdot\overrightarrow{\underline{D}}^{(2)}f_2 +$$

$$+ G_D{}^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot\overrightarrow{\underline{d}}^{(1)}f_1 + G_D{}^{(2)}\overleftarrow{\underline{D}}\cdot(s)\cdot\overrightarrow{\underline{d}}^{(2)}f_2 =$$

$$= f_{1,2} - G_D{}^{(1)}\overleftarrow{\underline{d}}\cdot(s)\cdot\left[\overrightarrow{\underline{D}}^{(1)}f_1 + \underline{E}\cdot\overrightarrow{\underline{D}}^{(2)}f_2\right] +$$

$$+ G_D{}^{(1)}\overleftarrow{\underline{D}}\cdot(s)\cdot\left[\overrightarrow{\underline{d}}^{(1)}f_1 - \underline{E}\cdot\overrightarrow{\underline{d}}^{(2)}f_2\right] \quad .$$

We need show only that the additional terms vanish. We use (52a), (55), and $|G_D(x,x) - G_{1,2D}(x,x)| < \infty$ for the first and (56), (57), and $|G_D(x,x) - G_{1,2}(x,x)| < \infty$ for the second term. This shows that it suffices that

(B.58a) $\underline{a}\cdot(s)(\underline{D}^{(1)}f_1 + \underline{E}\cdot\underline{D}^{(2)}f_2) = 0$

for all $\underline{a}$ such that

(B.58b) $\underline{a}\cdot(s)\cdot(\underline{G}_1 + \underline{E}\cdot\underline{G}_2\cdot\underline{E})\cdot(s)\cdot\underline{a} = 0$,

and that

(B.59a) $\underline{b}\cdot(s)\cdot\left[\overrightarrow{\underline{d}}^{(1)}f_1 - \underline{E}\cdot\overrightarrow{\underline{d}}^{(2)}f_2\right] = 0$

for all $\underline{b}$ such that

(B.59b) $\underline{b}\cdot(s)\cdot\left[(\underline{G}_1)_s^{-1} + \underline{E}\cdot(\underline{G}_2)_s^{-1}\cdot\underline{E}\right]\cdot(s)\cdot\underline{b} = 0$.

(58.b) restricts <u>a</u> to the intersection of classes of vectors restricted to S of finite $||_{N-\frac{1}{2}}$ norms, and (59b) restricts <u>b</u> to the intersection of classes of vectors restricted to S of finite norms in the duals of the $||_{N-\frac{1}{2}}$ -norms restricted to S. These intersections satisfy, due to the meaning of the conditions α) and β), (58a) and (59b).

Appendix C: Divergence of perturbation theory.

The perturbation theoretical expansion of $N\{\underline{a},J\}$ is given by (IV.13), and the graphical interpretation of its term were described. We add here that as the substitution $N\{\underline{a},J\} = \exp\left[N\{\underline{a},J\}^T\right]$ shows the terms proportional $G_o(0)$ only occur in the combination

$$\left[\delta^2/\delta J(x)^2\right] N\{\underline{a},J\}^T - G_o(0)$$

and serve here to cancel the nonregular part of the expression $G(x,x)$ contained in the first term.

For the reason explained in IV.G, we change to the interpretation α) of III.B. Then (IV.) becomes

$$N\{J\} = \exp\Big[-\tfrac{1}{4}g([\delta^4/\delta J(x)^4]) +$$

$$+ \tfrac{1}{2}\,3g\,G_o(0)([\delta^2/\delta J(x)^2])\Big]\exp\left[\tfrac{1}{2}(JG_oJ)\right] \quad .$$

The $G_o(0)$ -terms are now cancelled exactly, and all n^{th} order term in g of $N\{J\}$ have algebraically the sign $(-1)^n$. This also applies to the n^{th} order terms of $\ln N\{J\}$ and $\ln\left[N\{J\}/N\{0\}\right]$, since they are the subclass of connected terms of $N\{J\}$ with resp. without vacuum graphs. It suffices, of course, to prove the divergence of the expansion of $N\{J\}/N\{0\}$ to conclude it for the logarith also.

Now let $J(x)$ be nonnegative throughout Ω. We may choose an $\Omega' \subset \Omega$ such that $J(x) > j > 0$ in Ω'. Let $\min_{x,y\in\Omega'} G_o(x,y) = \gamma$ and $V_{\Omega'} = \omega$. Let

$$n(j,g,\gamma,\omega) = \exp\left[-\tfrac{1}{4}g\omega^{-3}(\partial/\partial j)^4 + \right.$$

$$\left. + \tfrac{1}{2}3g\gamma\omega^{-1}(\partial/\partial j)^2\right]\exp\left[\tfrac{1}{2}\gamma\omega^2 j^2\right] \quad .$$

Its g-expansion is term by term majorized by the g-expansion of $N\{J\}$, and the expansion of $n(j,g,\gamma,\omega)/n(0,g,\gamma,\omega)$ likewise by that of $N\{J\} / N\{0\}$, since the division effects only the elimination of pure vacuum graphs (i.e. j-independent terms in the first case). However, $n(j,g,\gamma,\omega)$ is the formal expansion of

$$\text{(C.1)} \qquad n(j,g,\gamma,\omega) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} dp \exp\left[-\tfrac{1}{2}p^2 - \right.$$

$$\left. - \tfrac{1}{4}g\omega\gamma^2 p^4 + \tfrac{1}{2}3g\omega\gamma^2 p^2 + j\omega\gamma^{1/2}p\right] =$$

$$= (2\pi\alpha)^{-1/2} \int dp \exp\left[-\tfrac{1}{4}p^4 + \tfrac{1}{2}3\alpha p^2 - \right.$$

$$\left. - \tfrac{1}{2}\alpha^{-1}p^2 + \alpha^{-1/2}\beta p\right] \equiv n(\alpha,\beta)$$

with $\alpha = \gamma(\omega g)^{1/2}$, $\beta = \gamma^{1/2}\omega j$. $n(\alpha,\beta)$ is analytic to the α-plane except for essential singularities at zero and infinity. $n(\alpha,\beta)/n(\alpha,0)$ for $\beta \neq 0$ has, in addition, poles in part of the plane. It is easily seen that this ratio is not analytic in α^2 at the origin, which proves the divergence of the perturbation series as stated in IV.G, even without examining the singularity at $\alpha = 0$ further, which could be done using e.g. the saddle point method.

A slight modification of the proof gives divergence also if only $\int G_o(x-y)J(y)dy$ to be nonnegative is assumed. For $\Omega = \infty$, the proof can also be carried out in momentum space if $\int \exp[iKx]J(x)dx$ is nonnegative. For the functions $S(x_1 \ldots x_n)$ and, if $\Omega = \infty$, their Fourier transforms divergence proofs can also be given.

We have not considered here renormalization but regularized instead. We feel that existing divergence proofs[49] in other (physically not acceptable) models, which take renormalization into account, could be modified such as to apply to the present model also.

We have shown that in general the perturbation expansion will be divergent however small Ω is chosen and however strongly is being regularized. This is characteristic for a Bose field theory. A theory of Fermi- and Bose fields in Yukawa coupling leads under slightly stronger modification than we introduced to convergence[50] of the perturbation theoretical expansion, the different behavior being due to cancellations within a given order, which do not occur in the pure Bose field case. However, the cancellation of the Ω -dependence corresponding to that we have been discussing for $N\{\underline{a},J\}/N\{\underline{a},0\}$ has not been shown yet also.

Appendix D: The classical model.

In section V.C we were led to consider the extremal problem*

(D.1) $$E\{\underline{a},J\} = \underset{\phi,\vec{\underline{d}}\phi=\underline{a} \text{ on } \partial\Omega}{\text{l.u.b.}} \left[(J\phi) - \tfrac{1}{2}(\phi K\phi) - \tfrac{1}{4}g(\phi^4)\right] .$$

The Euler equation is

(D.2a) $$\vec{K}\phi(x) + g\phi(x)^3 = J(x) ,$$

(D.2b) $$\vec{\underline{d}}\phi = \underline{a} \text{ on } \partial\Omega .$$

Its solution is a functional of J and actually

$$\phi(x) = [\delta/\delta J(x)]E\{J\} .$$

Substituting this into (2) converts (2) into the Hamilton-Takobi equation of a classical problem that stands to the quantum theoretical problem solved in III in the same relation as classical point mechanics stands to the elementary Schrödinger equation. In fact, (2) results by substituting in (III.22) $\exp E\{\underline{a},J\}$ for $N\{\underline{a},J\}$ and neglecting derivatives of $E\{\underline{a},J\}$ higher than the first[1], a procedure analogous to that in the

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* We omit here the term containing $S(xx) - G_o(0)$, which is insignificant for the present consideration.

[1] Also to be omitted is the term containing $G_o(0)$, which is a purely quantum mechanical correction term whose role is essentially to cancel the likewise omitted term containing the J-independent part of $[\delta^2/\delta J(x)^2]E\{\underline{a},J\}$.

elementary case[51].

Constructing a minimal sequence by e.g. a direct method of variational calculus[52] it can be shown that (2) possesses a weak solution, and $E\{\underline{a},J\}$ is finite, if

(D.3) $$(JG_N J) < \infty \ , \ \underline{a}\cdot\underline{G}\cdot\underline{a} < \infty \ ,$$

although this is not a necessary condition. We have the inequalities

(D.4) $$E_o\{\underline{a},J\} - \tfrac{1}{4}g(f^4) < E\{\underline{a},J\} < \ln N_o\{\underline{a},J\} \equiv E_o\{\underline{a},J\}$$

with (III.8b) and (III.10),

(D.5) $$E\{\underline{a},J\} < E_o\{\underline{a},J\} - \tfrac{1}{4}g(f^4) + \tfrac{1}{2}g^2(f^3Gf^3) \ ,$$

and

(D.6) $$(\phi J) - \tfrac{1}{2}(\phi K\phi) - \tfrac{g}{4}(\phi^4) < E\{\underline{a},J\} < \tfrac{1}{4}3g^{-1/3}(|J|^{4/3})$$

where $\phi = (J/g)^{1/3} - G\overleftarrow{\underline{D}}\cdot[\underline{a}-\overrightarrow{\underline{d}}(J/g)^{1/3}]$.

(5) comes from

$$E\{\underline{a},J\} = E_o\{\underline{a},J\} - \underset{\psi}{\text{l.u.b.}}\Big\{g(f[G\psi+f]^3) -$$

$$- \tfrac{1}{4}3g([G\psi+f]^4) - \tfrac{1}{2}g^2([G\psi+f]^3G[G\psi+f]^3)\Big\} \ .$$

The first inequality in (6) together with (IV.1a), (V.9) shows that the order of the entire functional $N\{\underline{a},J\}$ is precisely 4/3.

It is instructive to compare the properties of $\ln N\{\underline{a},J\} = N\{\underline{a},J\}^T$ and $E\{\underline{a},J\}$ further. While $N\{\underline{a},J\}$ is entire in J (and $\underline{a}$), $\ln N\{\underline{a},J\}$ is not since its order in J is no integer, and also $E\{\underline{a},J\}$ is not although its Volterra series expansion has (at least for $\underline{a}$ small enough) a finite convergence radius. $N\{\underline{a},J\}$ and, therefore, $\ln N\{\underline{a},J\}$ are nonanalytic in g at the origin (at least in the slightly altered definition described in appendix C; and which may be copied for the classical functional) although analytic in the right half plane, whereas $E\{\underline{a},J\}$ has a finite convergence radius in g. This is easiest to see by rewriting (1)

$$E\{\underline{a},J\} = E_o\{\underline{a},J\} + \underset{\psi}{\text{l.u.b.}}\left[-\tfrac{1}{2}(\psi G\psi) - \tfrac{1}{4}g([G\psi+f]^4)\right]$$

$$= E_o\{\underline{a},J\} + g^{-1}\,\text{l.u.b}\left[-\tfrac{1}{2}(\psi G\psi) - \tfrac{1}{4}([G\psi+g^{1/2}f]^4)\right].$$

wherefrom it follows that the Volterra series expansion in f is identical with the power series expansion in g. We prove the convergence of the g-expansion by examining the power series expansion of Φ, which satisfies from (1)

$$\Phi = f - g\,G\Phi^3 \quad . \qquad \text{(D.7)}$$

Insertion of the expansion, if convergent, into (1) disregarding the l.u.b. sign gives the expansion in question. We generate the expansion of Φ by iterating (7):

$$\Phi_o = f \;,\; \Phi_{n+1} = f - g\,G\Phi_n^3 \quad .$$

We hereby proceed not in powers but in polynomials in g. The series $\phi_o + (\phi_1 - \phi_o) + \ldots$ will, however, converge also after reordering as a g-power series, if we obtain a majorication from the solution of

$$U = \text{Max}\ |f| + C|g|U^3$$

where $C = \text{Max}_x \int_\Omega |G(x,y)|dy$. We have a positive root for $|g| < 4[27C(\text{Max}|f|)^2]^{-1}$, which is a lower bound for the convergence radius.

$E\{\underline{a},J\}$ is converse in J as is ln $N\{\underline{a},J\}$. We did not investigate if $E\{\underline{a},J\}$ possesses higher conversity properties similar to those one obtains from (IV.2) with $k \geq 3$. However, $E\{\underline{a},J\}$ has refined conversity properties that do depend on the form of the integrand in (1), e.g. with

$$(\mu+\nu)^{-1/2}\left[1\pm(\nu/\mu)^{1/2}(\mu+\nu-1)^{1/2}\right]^{1/2} \equiv \alpha_\pm \quad ,$$

$$(\mu+\nu)^{-1/2}\left[1\pm(\mu/\nu)^{1/2}(\mu+\nu-1)^{1/2}\right]^{1/2} \equiv \beta_\pm$$

we have

$$\mu E\{\alpha_\pm \underline{a}, J_1\} + \nu E\{\beta_\mp \underline{a}, J_2\} \geq$$

$$\geq E\{\underline{a}, \mu\alpha_\pm J_1 + \nu\beta_\mp J_2\}$$

for $\mu + \nu \geq 1$, $\mu > 0$, $\nu > 0$, and the upper sign if $\mu < 1$, the lower sign if $\nu < 1$. However, we found no analogy of this property for $N\{\underline{a},J\}$.

The "classical analog" of the zero-dimensional model considered in IV.B and appendix C, and to be compared with $\ln[n(\alpha,\beta)/n(\alpha,0)]$ of (C.1) is the function

$$e(\alpha,\beta) = \underset{p}{\mathrm{Max}}\left[-\tfrac{1}{4}p^4 - \tfrac{1}{2}\alpha^{-1}p^2 + \alpha^{-1/2}\beta p\right] =$$

$$= \frac{1}{3}\, 2\alpha^{-2}(\sinh\theta)^2 \cosh 2\theta \quad ,$$

with $\theta = \frac{1}{3}\sinh^{-1}(\frac{1}{2}\, 3^{3/2}\alpha\beta)$.

which is an analytic function of α and β , with a branch point at $27\alpha^2\beta^2 = -4$, or $g = -4(3\omega\gamma)^{-3}\, j^{-2}$.

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