Quantum Theory for Scalar Bosons [1]
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by Arthur Jaffe

Contents

I One Particle States in Quantum Theory **2**
I.1 Continuum States 3
I.2 The Momentum-Space Representation 3
I.3 Dirac Notation 5
I.4 Continuum Eigen-Bras and Kets 6
I.5 Configuration-Space Eigenkets 7
I.6 Energy, Momentum, and Mass 8
I.7 Group Representations 9
I.8 Lorentz Transformations 10
I.9 States and Transition Amplitudes 14

II Two Particles **16**
II.1 The Tensor Product 16
II.2 Dirac Notation for Two-Particle States 17
II.3 Tensor Products of Operators 18
II.4 An Operator on $\mathcal{H}_1$ Gives Operators on $\mathcal{H}_2$ 19
II.5 Two Identical Bosons 19
II.6 Dirac Notation for Two Identical Bosons 21
II.7 Operators Acting on States of Two Identical Bosons 22

III n-Particles **23**
III.1 Wave Functions for n-Identical Bosons 23
III.2 Permanents, Scalar Products, and Recursion Relations 24

IV Fock Space **26**
IV.1 States in Fock Space 26
IV.2 Operators Preserving Particle Number 27
IV.3 Creation and Annihilation Operators 28
IV.4 The Canonical Commutation Relations 31

V The Free Quantum Field **33**
V.1 Poincaré Covariance 34
V.2 Canonical Commutation Relations for the Fields 35
V.3 Consequences of Positive Energy 36
V.4 Local Commutation Relations 36
V.5 Properties of the Free-Field Two Point Wightman Function 37

[1]The author is currently writing a book, *Introduction to Quantum Field Theory*, which contains a version of these and other of his notes posted on the course website.

V.6 Properties of the Free-Field Commutator Function 39
V.7 The Time-Ordered Product 40
V.8 The Retarded Commutator 42
V.9 The Minkowski-Euclidean Correspondence 43

I One Particle States in Quantum Theory

Consider a one-particle, configuration-space wave function $f(\vec{x})$ which is a function of the spatial variable $\vec{x} \in \mathbb{R}^3$. A wave function is also called a *state*, and it is a vector in a Hilbert space $\mathcal{H}_1$ of possible particle states. (We denote a wave function with the letter f, rather than the usual ψ of non-relativistic quantum theory texts, as we reserve the letter ψ for a Dirac field.) For now we interpret $|f(\vec{x})|^2$ as the probability density for a particle being located at the position $\vec{x}$. Here we assume that the position is a vector in Euclidean 3-space, but we may at other times replace $\mathbb{R}^3$ by an appropriate configuration space, such as a torus or a lower (or higher) dimensional space.

Two states f, g will have the scalar product

$$\langle f, g\rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^3} \overline{f(\vec{x})} g(\vec{x}) d\vec{x} \ . \tag{I.1}$$

Here the notation $\overline{f(\vec{x})}$ denotes complex conjugation of the function $f(\vec{x})$. We call this the *configuration-space* representation. This space of one-particle states is the Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}^3)$ of *square-integrable* functions, obtained by completing the linear space of smooth functions that vanish outside a bounded region. (Vanishing outside a bounded region is sometimes called compact support. The completion is with respect to the norm $\|f\| = \langle f, f\rangle^{1/2}$ determined by this scalar product.) The subscript "1" on $\mathcal{H}_1$ specifies that wave functions $f \in \mathcal{H}_1$ describe one particle.

A linear transformation T (or *operator* for short) transforms $\mathcal{H}_1$ into $\mathcal{H}_1$ and satisfies for any $f, f' \in \mathcal{H}_1$ and complex λ,

$$T(f + \lambda f') = Tf + \lambda T f' \ . \tag{I.2}$$

Note that most operators T we encounter in quantum theory (such as position, momentum, angular momentum, energy, etc.) are only defined for a subset of wave-functions f. In such a situation one requires the operator to be defined for a *dense* subset of wave functions, meaning a set which for any given state f contains a sequence $\{f_n\}$ of states for which $f_n \to f$, as $n \to \infty$.

Polarization: In the case of any Hilbert space $\mathcal{H}$, one can express the hermitian scalar product $\langle f, f'\rangle = \overline{\langle f', f\rangle}$ between two distinct vectors f, f' as a linear combination of squares of lengths of vectors. In particular,

$$\langle f, f'\rangle = \frac{1}{4} \sum_{\epsilon^4=1} \bar{\epsilon} \left\|f + \epsilon f'\right\|^2 , \tag{I.3}$$

where the sum extends over the fourth roots of unity, denoted by ϵ. This relies on the fact that $\sum_{\epsilon^4=1} \epsilon^n = 0$, unless $n = 0 \pmod 4$, in which case the sum is 4. When the functions f, f'

are both real, then the inner product $\langle f, f' \rangle = \langle f', f \rangle$ is symmetric under the interchange of f with f', and one can reduce the polarization identity to a sum over the two terms:

$$\langle f, f' \rangle = \frac{1}{4} \sum_{\epsilon^2=1} \epsilon \left\| f + \epsilon f' \right\|^2 . \tag{I.4}$$

Similar formulas hold for matrix elements of a linear transformation T on $\mathcal{H}$. One can reduce the computation of arbitrary matrix elements $\langle f, Tf' \rangle$ of a linear operator T on $\mathcal{H}$, to the computation of expectations of T; the same argument leading to (I.3) and (I.4) also shows that

$$\langle f, Tf' \rangle = \frac{1}{4} \sum_{\epsilon^4=1} \bar{\epsilon} \left\langle f + \epsilon f', T\left(f + \epsilon f'\right) \right\rangle , \tag{I.5}$$

and for real, self-adjoint T,

$$\langle f, Tf' \rangle = \frac{1}{4} \sum_{\epsilon^2=1} \epsilon \left\langle f + \epsilon f', T\left(f + \epsilon f'\right) \right\rangle . \tag{I.6}$$

I.1 Continuum States

Continuum states can occur in a scalar product, but they are not normalizable. The Dirac delta "function" is a very important continuum state. This function $\delta_{\vec{x}}(\vec{x}\,') = \delta^3(\vec{x}\,' - \vec{x}\,)$ of $\vec{x}\,'$ is zero everywhere but at the one point $\vec{x}$, and has total integral one. One says it is localized at $\vec{x}$. (Mathematically it is a measure, not a function, and sometimes this is called a generalized function.) The normalization condition is

$$\int \delta_{\vec{x}}(\vec{x}\,') d\vec{x}\,' = 1 . \tag{I.7}$$

The delta function localized at $\vec{x}$ has the property that for a continuous wave function f,

$$\langle \delta_{\vec{x}}, f \rangle = f(\vec{x}\,) . \tag{I.8}$$

The delta function $\delta_{\vec{a}}$ behaves in many ways like an eigenstate of the coordinate operator with the eigenvalue $\vec{a}$. In other words the operator $\vec{x}\,'$ (the operator of multiplication by the coordinate function $\vec{x}\,'$) satisfies the eigenvalue equation,

$$\vec{x}\,'\delta_{\vec{a}}(\vec{x}\,') = \vec{a}\,\delta_{\vec{a}}(\vec{x}\,') . \tag{I.9}$$

But the square of $\delta_{\vec{a}}$ does not exist, so $\delta_{\vec{a}}$ is not a true eigenvector. While there is an appropriate mathematical theory of the delta function as a *generalized* eigenfunction, in this course we will work with the delta function as if it were an eigenfunction.

I.2 The Momentum-Space Representation

The *momentum-space* representation is related to the configuration-space representation by Fourier transform. We only indicate the subscript $\mathcal{H}_1$ on the scalar product when there may be a confusion.

The function f is related to its Fourier transform $\tilde{f}$, by

$$\tilde{f}(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{x}\,) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} \ . \tag{I.10}$$

The Fourier inversion formula says that

$$f(\vec{x}\,) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(\vec{k}\,) e^{i\vec{k}\cdot\vec{x}}\, d\vec{k} \ . \tag{I.11}$$

Note that the sign of the exponential is not arbitrary. We choose this in order that the momentum $\vec{P}$ is the operator

$$\vec{P} = -i\nabla_{\vec{x}} \ . \tag{I.12}$$

Here, as everywhere, we use units for which $\hbar = 1$. We will also use units for which the velocity of light $c = 1$. One must re-introduce factors of $\hbar$ and c when one wants to recover numerical answers. Note that the Fourier transform of the Dirac delta function $\delta_{\vec{x}}(\vec{x}\,')$ is a plane-wave function $\vec{k}$,

$$\tilde{\delta}_{\vec{x}}(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} \ . \tag{I.13}$$

Note also that the factor $(2\pi)^{-3/2}$ differs from Coleman's notes but agrees with the book by Srednicki. For a Fourier transform on $\mathbb{R}^d$, one would replace $(2\pi)^{-3/2}$ by $(2\pi)^{-d/2}$. One makes this choice in order to give Fourier transformation especially nice properties.

In fact Fourier transformation is a linear transformation $\mathfrak{F}$ on the Hilbert space of square-integrable functions. This means one can write

$$\boxed{\tilde{f} = \mathfrak{F} f}\,, \qquad \text{or} \qquad \boxed{(\mathfrak{F} f)\,(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{x}\,) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x}}\ . \tag{I.14}$$

Linearity means that for any complex constants λ, μ and any two functions f, g,

$$\mathfrak{F}(\lambda f + \mu g) = \lambda\, \mathfrak{F} f + \mu\, \mathfrak{F} g \ . \tag{I.15}$$

With our normalization, $\mathfrak{F}$ is a unitary transformation,

$$\mathfrak{F}^* \mathfrak{F} = \mathfrak{F} \mathfrak{F}^* = I \ . \tag{I.16}$$

Unitarity also could be written

$$\langle f, g \rangle = \left\langle \tilde{f}, \tilde{g} \right\rangle = \langle \mathfrak{F} f, \mathfrak{F} g \rangle \ , \tag{I.17}$$

for any square-integrable f, g. In fact the Fourier inversion formula says that

$$\mathfrak{F}^* = \mathfrak{F}^{-1} = \mathfrak{F}\mathfrak{P} \ , \tag{I.18}$$

where $\mathfrak{P}$ denotes the inversion (parity) transformation

$$(\mathfrak{P} f)(\vec{x}\,) = f(-\vec{x}\,) \ . \tag{I.19}$$

It turns out that the unitarity of $\mathfrak{F}$ and the form of its inverse (I.18) can be derived from (and are equivalent to) the completeness of the set of eigenfunctions of the quantum-mechanical harmonic oscillator! If you are interested to read a full and concise explanation of these facts, look at the handout called *Fourier-Oscillator* on the course web site.

I.3 Dirac Notation

In Dirac notation, one writes the Hilbert space of states $\mathcal{H}_1$ in terms of vectors $|f'\rangle$ called *ket-vectors* or "kets" and a dual space of vectors $\langle f|$ called *bras*. Wave functions f and ket vectors $|f\rangle$ correspond to another exactly,

$$f \quad \leftrightarrow \quad |f\rangle\ . \tag{I.20}$$

Ket vectors are dual (in a conjugate-linear fashion) to bra vectors,

$$|f\rangle \quad \leftrightarrow \quad \langle f|\ . \tag{I.21}$$

This latter can be written as a hermitian adjoint

$$\langle f| \;=\; |f\rangle^*\ , \tag{I.22}$$

denoted by *. In terms of the original wave functions $f(\vec{x}\,)$, the duality between a ket $|f\rangle$ and bra $\langle f|$ just amounts to complex conjugation,

$$f(\vec{x}\,) \leftrightarrow |f\rangle \quad \longrightarrow \quad \overline{f(\vec{x}\,)} \leftrightarrow \langle f|\ . \tag{I.23}$$

One pairs a bra $\langle f|$ with a ket $|f'\rangle$ to form a scalar product or *bracket* $\langle f|f'\rangle$. This is Dirac's elegant notation for the original scalar product,

$$\boxed{\langle f, f'\rangle = \langle f|f'\rangle}\ . \tag{I.24}$$

One requires that the inner product be hermitian, namely

$$\langle f|f'\rangle = \langle f'|f\rangle^* = \overline{\langle f'|f\rangle}\ . \tag{I.25}$$

Here * or $^-$ denotes complex conjugation of the scalar product. (We use * to denote the hermitian adjoint of a transformation, and acting on functions the two conjugations agree.)

Suppose that T is a linear transformation T acting on the space of (ket) vectors. The natural correspondence between the ordinary notation and the Dirac notation is to set

$$T|f\rangle = |Tf\rangle\ . \tag{I.26}$$

The ordinary *hermitian-adjoint* transformation T^* is defined by

$$\langle f', T^*f\rangle = \langle Tf', f\rangle\ \ . \tag{I.27}$$

In Dirac notation, the corresponding definition is

$$\langle f'|T^*f\rangle = \langle f'|T^*|f\rangle = \langle Tf'|f\rangle\ . \tag{I.28}$$

Therefore the bra corresponding to $T|f\rangle$ is

$$\boxed{T|f\rangle \quad \longrightarrow \quad \langle Tf| = \langle f|\,T^*}\ . \tag{I.29}$$

I.4 Continuum Eigen-Bras and Kets

Consider the one-particle momentum eigen-kets $|\vec{k}\rangle$ of the momentum, for which

$$\vec{P}|\vec{k}\,'\rangle = \vec{k}\,'|\vec{k}\,'\rangle \ . \tag{I.30}$$

This continuum eigenfunction corresponds to the ordinary momentum-space wave function $\delta_{\vec{k}\,'}(\vec{k}\,) = \delta(\vec{k}\, - \vec{k}\,')$.

Each ket vector $|\vec{k}\,'\rangle$ is associated with a dual vector, the bra vector $\langle\vec{k}\,'|$. The scalar product between two kets $|\vec{k}\rangle$ and $|\vec{k}'\rangle$ is given by the bracket

$$\langle\vec{k}|\vec{k}'\rangle = \langle\vec{k}'|\vec{k}\rangle^* \ . \tag{I.31}$$

These kets are continuum eigenfunctions, so they cannot be normalized in the usual sense, for they have infinite length. One normalizes the bras and kets so that

$$\langle\vec{k}|\vec{k}'\rangle = \delta^3(\vec{k} - \vec{k}') \ . \tag{I.32}$$

as one would expect from

$$\langle\vec{k}\,|\vec{k}\,'\rangle = \left\langle\delta_{\vec{k}}\,, \delta_{\vec{k}\,'}\right\rangle = \int \delta(\vec{k}\,'' - \vec{k}\,)\, \delta(\vec{k}\,'' - \vec{k}\,')\, d\vec{k}\,'' = \delta(\vec{k}\, - \vec{k}\,') \ . \tag{I.33}$$

In the last identity we use $\delta(\vec{k}\,) = \delta(-\vec{k}\,)$.

The Fourier representation of the 3-dimensional delta function is

$$\delta^3(\vec{x}\,) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{x}}\, d^3\vec{k} \ . \tag{I.34}$$

We also assume that the ket vectors $|\vec{k}\rangle$ form a complete set of one-particle states. Completeness can be written as the condition

$$\boxed{\int |\vec{k}\rangle\langle\vec{k}|\ d^3\vec{k} = I} \ . \tag{I.35}$$

In Dirac notation, one can use a wave packet $\tilde{f}(\vec{k})$, i.e. a momentum space wave function that is a square-integrable function of $\vec{k}$, to define a ket vector $|\tilde{f}\rangle$ with finite length. The wave packet defines a ket

$$|\tilde{f}\rangle = \int \tilde{f}(\vec{k})\, |\vec{k}\,\rangle\, d^3\vec{k} \ . \tag{I.36}$$

The corresponding bra vector is

$$\langle\tilde{f}| = \int \tilde{f}(\vec{k}\,)^*\, \langle\vec{k}\,|\, d\vec{k} \ \ . \tag{I.37}$$

The completeness relation gives us the relation between the notation in the standard (momentum) representation and the Dirac notation. Pairing the ket (I.36) with the bra $\langle\vec{k}\,|$ gives

$$\boxed{\tilde{f}(\vec{k}) = \langle\vec{k}|\tilde{f}\rangle} \ . \tag{I.38}$$

The Dirac scalar product of a bra and a ket defined by two wave packets $\tilde{f}$ and $\tilde{g}$ coincides exactly with the scalar product of the wave packets in terms of the standard wave functions. The standard momentum space inner product is

$$\left\langle \tilde{f}, \tilde{f}' \right\rangle = \int \tilde{f}(\vec{k}\,)^* \, f'(\vec{k}\,) d\vec{k} \ , \tag{I.39}$$

which in Dirac notation is

$$\langle \tilde{f} | \tilde{f}' \rangle = \int \langle f | \vec{k}\,\rangle \langle \vec{k}\,| f' \rangle d\vec{k} \ = \int \overline{\tilde{f}(\vec{k}\,)} \, f'(\vec{k}\,) d\vec{k} \ . \tag{I.40}$$

I.5 Configuration-Space Eigenkets

Given the wave function $\tilde{f} = \mathfrak{F} f$, and the ket $|\tilde{f}\rangle$ of the form (I.36), we look for a ket $|\vec{x}\,\rangle\!\rangle$ such that[2]

$$\boxed{f(\vec{x}\,) = \langle\!\langle \vec{x}\,| \tilde{f} \rangle} \ . \tag{I.41}$$

In fact

$$f(\vec{x}\,) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{x}\,\cdot\vec{k}} \, \tilde{f}(\vec{k}\,) d\vec{k} \ = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{x}\,\cdot\vec{k}} \, \langle \vec{k}\,| \tilde{f} \rangle d\vec{k} \ , \tag{I.42}$$

so that

$$\boxed{\langle\!\langle \vec{x}\,| = \frac{1}{(2\pi)^{3/2}} \int \langle \vec{k}\,| \, e^{i\vec{k}\cdot\vec{x}} \, d^3\vec{k}} \ , \qquad \text{and} \ \boxed{|\vec{x}\,\rangle\!\rangle \ = \frac{1}{(2\pi)^{3/2}} \int |\vec{k}\,\rangle \ e^{-i\vec{k}\cdot\vec{x}} \, d^3\vec{k}} \ . \tag{I.43}$$

With these configuration-space eigen-kets, one infers from (I.32) that

$$\boxed{\langle\!\langle \vec{x}\,| \vec{k}\,\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}} \ , \qquad \text{and} \ \boxed{\langle\!\langle \vec{x}\,| \vec{x}\,'\rangle\!\rangle = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\,\cdot(\vec{x}\,-\vec{x}\,')} d\vec{k} \ = \delta^3(\vec{x}\, - \vec{x}\,')} \ . \tag{I.44}$$

We also have a completeness relation expressed in terms of the configuration space vectors. In fact,

$$\boxed{\int |\vec{x}\,\rangle\!\rangle \langle\!\langle \vec{x}\,| \, d\vec{x} \ = \int |\vec{k}\,\rangle \langle \vec{k}\,| \, d\vec{x} \ = I} \ . \tag{I.45}$$

From the completeness relations, one has $|f\rangle\!\rangle \ = \int f(\vec{x}\,) |\vec{x}\,\rangle\!\rangle \ d\vec{x} \ = \int |\vec{x}\,\rangle\!\rangle \ \langle\!\langle \vec{x}\,| \tilde{f} \rangle d\vec{x} \ = |\tilde{f}\rangle$, or

$$\boxed{|f\rangle\!\rangle \ = \int f(\vec{x}\,) |\vec{x}\,\rangle\!\rangle \ d\vec{x} \ = \int \tilde{f}(\vec{k}\,) |\vec{k}\,\rangle d\vec{k} \ = |\tilde{f}\rangle} \ . \tag{I.46}$$

In other words, we express the vector $|f\rangle\!\rangle$ in terms of two bases $|\vec{x}\,\rangle$ or $|\vec{k}\,\rangle$, with coefficents $f(\vec{x}\,)$ and $\tilde{f}(\vec{k}\,)$ respectively.

[2]Here we distinguish position space kets from momentum space kets by using a double bracket $| \cdot \rangle\!\rangle$ in place of $| \cdot \rangle$. In case there is no chance of confusion, we drop this double bracket.

I.6 Energy, Momentum, and Mass

The *mass* m of the free particle under consideration enters when we introduce the time evolution. We take for the energy (or Hamiltonian H) the expression

$$H = \left(\vec{P}^{\,2} + m^2\right)^{1/2} . \tag{I.47}$$

This is the standard energy-momentum relation in special relativity, when we take units with $c = 1$.

Using the quantum-mechanical form of the momentum as the operator $\vec{P} = -i\nabla_{\vec{x}}$ as given in (I.12), one can write

$$H = \left(\vec{P}^2 + m^2\right)^{1/2} = \left(-\vec{\nabla}_{\vec{x}}^2 + m^2\right)^{1/2} = \omega . \tag{I.48}$$

We use the symbol ω to denote the energy H restricted to the space of a single particle. One defines the time evolution of the free one-particle wave function as the solution to Schrödinger's equation (with $\hbar = 1$),

$$f_t = e^{-itH} f = e^{-it\omega} f . \tag{I.49}$$

Here f is the time-zero wave function.

One can solve this equation using Fourier transformation. In the momentum representation, the momentum operator multiplies the wave function $\tilde{f}(\vec{k}\,)$ by the function $\vec{k}$, and the energy operator multiplies the wave function by the function $\omega(\vec{k}\,) = \left(\vec{k}^{\,2} + m^2\right)^{1/2}$. Thus the Fourier transform on $f_t(\vec{x}\,)$ is

$$\tilde{f}_t(\vec{k}\,) = e^{-it\omega(\vec{k}\,)} \tilde{f}(\vec{k}\,) . \tag{I.50}$$

It is also natural to define the "mass" operator as the positive square root,

$$M = \sqrt{H^2 - \vec{P}^2} . \tag{I.51}$$

The kets $|\vec{k}\,\rangle$ are eigenstates of M, namely

$$M|\vec{k}\,\rangle = m|\vec{k}\,\rangle . \tag{I.52}$$

As a consequence, the vectors

$$|\tilde{f}\rangle = \int \tilde{f}(\vec{k}\,)|\vec{k}\,\rangle\, d\vec{k} , \tag{I.53}$$

are also eigenvectors of M, as are the time-dependent vectors $U(t)^*|\tilde{f}\rangle = e^{-it\omega}|\tilde{f}\rangle$. Similarly

$$M|\vec{x}\,\rangle\!\rangle = m|\vec{x}\,\rangle\!\rangle . \tag{I.54}$$

The value of the rest mass m enters the calculation through the definition of H for a free particle. If H had a more complicated form, it might not be so easy to identify eigenstates of the mass M.

I.7 Group Representations

Let $\mathfrak{G}$ denote a group with elements g and identity e. A representation $U(g)$ of the group on the Hilbert space $\mathcal{H}_1$ is a mapping $g \mapsto U(g)$ from the group to linear transformations on $\mathcal{H}_1$ with the properties,

$$U(g_1)U(g_2) = U(g_1 g_2) \;, \qquad \text{and } U(e) = I \;. \tag{I.55}$$

As a consequence, $U(g)^{-1} = U(g^{-1})$, and in case $U(g)$ is unitary, then

$$\boxed{U(g)^* = U(g^{-1})} \;. \tag{I.56}$$

In quantum theory we generally consider unitary representations, as they conserve probabilities and transition amplitudes. Furthermore, when the group $\mathfrak{G}$ is parameterized by some continuous parameters, we want the representation $U(g)$ to depend continuously on these parameters. (Technically we require strong continuity.)

In some cases the group $\mathfrak{G}$ acts on configuration space according to $\vec{x} \mapsto g\vec{x}$. A couple of elementary cases of this phenomenon are spatial translations and spatial rotations. If the measure $d\vec{x}$ is also invariant under the group, namely $gd\vec{x} = d\vec{x}$ for all $g \in \mathfrak{G}$, then the transformations $U(g)$ defined by

$$\boxed{(U(g)f)(\vec{x}) = f(g^{-1}\vec{x})} \;, \tag{I.57}$$

give a unitary representation of $\mathfrak{G}$ on $\mathcal{H}_1$. We now show that

$$\boxed{U(g)|\vec{x}\,\rangle\!\rangle = |g\vec{x}\,\rangle\!\rangle} \;. \tag{I.58}$$

Equivalently, using the relation (I.29), it is sufficient to verify that

$$\langle\!\langle g\vec{x}\,| = \langle\!\langle \vec{x}\,|U(g)^* \;. \tag{I.59}$$

From (I.41) one has

$$f(g^{-1}\vec{x}\,) = \langle\!\langle g^{-1}\vec{x}\,|f\rangle\!\rangle \;, \tag{I.60}$$

while from (I.57) one has

$$f(g^{-1}\vec{x}\,) = \langle\!\langle \vec{x}\,|U(g)f\rangle\!\rangle \;. \tag{I.61}$$

This is true for any f, so $\langle\!\langle g^{-1}\vec{x}\,| = \langle\!\langle \vec{x}\,|U(g)$. Replacing g by g^{-1}, and using the unitarity of $U(g)$, one has $U(g^{-1}) = U(g)^*$, so

$$\langle\!\langle g\vec{x}\,| = \langle\!\langle \vec{x}\,|U(g^{-1}) = \langle\!\langle \vec{x}\,|U(g)^* \;. \tag{I.62}$$

But this is just the claimed relation (I.59), so (I.58) holds. Similarly one has

$$\boxed{U(g)|\vec{k}\,\rangle = |g\vec{k}\,\rangle} \;. \tag{I.63}$$

It may be possible to find a unitary representation of $\mathfrak{G}$ on $\mathcal{H}_1$, even if $\mathfrak{G}$ does not act on configuration space with $d\vec{x}$ invariant. This is the case for a Lorentz boost. That unitary representation is given in the second homework problem. We have shown in class that every proper, time-direction preserving Lorentz transformation Λ can be written uniquely as $\Lambda = BR$, where R is proper rotation and B is a pure boost. As a result, one recovers a unitary representation $U(a,\Lambda)$ of the Poincaré group on $\mathcal{H}_1$. This representation is *irreducible*; namely it leaves no proper subspace of $\mathcal{H}_1$ invariant. This is the standard *mass-m, spin*-0 irreducible representation of the Poincaré group.

For the space-time translation subgroup (a, I) of the Poincaré group, we define

$$U(a) = U(a,I) = e^{ia_0 H - i\vec{a}\cdot\vec{P}} \ . \tag{I.64}$$

Acting on the one-particle space, this gives

$$U(a)|\vec{k}\,\rangle = e^{ia_0\omega(\vec{k}\,) - i\vec{a}\cdot\vec{k}}|\vec{k}\,\rangle \ . \tag{I.65}$$

In configuration space, (I.43) shows that space-time translations have the effect,

$$\boxed{U(a)|\vec{x}\,\rangle\!\rangle \ = e^{ia_0\omega}|\vec{x}+\vec{a}\,\rangle\!\rangle} \ . \tag{I.66}$$

We use this as our standard sign convention for the direction of space-time translations. In terms of the Schrödinger equation, it is the adjoint $U(t)^* = U(t,\vec{0})^*$ that gives the standard solution $f_t = U(t)^* f = e^{-it\omega} f$. As a consequence of the convention (I.66), the relation (I.57) shows that

$$\boxed{(U(t,\vec{a}\,)f)(\vec{x}\,) = f_{-t}(\vec{x}-\vec{a}\,)} \ . \tag{I.67}$$

I.8 Lorentz Transformations

Here we analyze proper Lorentz transformations Λ that preserve the direction of time. A Lorentz transformation Λ is a real, 4×4-matrix that preserves the Minkowski square of the four-vector $x = (t,\vec{x}\,)$. Denote the components of x by x_μ with $t = x_0$, and the Minkowski square as $x_M^2 = t^2 - \vec{x}\,^2 = x^\mu g_{\mu\nu} x^\nu$. The Lorentz transformation acts on space-time as

$$x^\mu = \sum_\nu \Lambda_{\mu\nu} x^\nu \ . \tag{I.68}$$

Thus as a matrix equation the condition of preserving the Minkowski length is

$$\langle x, gx\rangle = \langle \Lambda x, g\Lambda x\rangle \ , \tag{I.69}$$

or

$$\Lambda^T g \Lambda = g \ . \tag{I.70}$$

Here Λ^T denotes the transpose of Λ, and g is the metric,

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{I.71}$$

From this identity we see that $\det \Lambda^2 = 1$. Those Λ with determinant $+1$ are called *proper*, and those that perserve the direction of time have $\Lambda_{00} > 0$.

Proposition I.1. *Every proper, time-preserving Lorentz transformation has a unique decomposition $\Lambda = BR$, where R is a pure rotation in 3-space about an axis $\vec{n}$ by angle θ and B is a Lorentz boost along some 3-space direction $\vec{n}'$ with rapidity χ.*

One can see this in many ways. One convenient way involves the beautiful relation between the quaternions and the geometry of Minkowski space. Quaternions were discovered in 1843 by William Hamilton, perhaps Ireland's most famous scientist. He was looking for a generalization of complex numbers to a higher dimension. He found the algebra $i^2 = j^2 = k^2 = ijk = -1$. From a modern point of view we represent these relations by four self-adjoint 2×2-matrices τ_μ taken as the four-vector. Up to a factor $\sqrt{-1}$, they also equal the Pauli matrices σ_j, complemented by the identity τ_0.

$$\begin{aligned} \tau &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ &= (\tau_0, \tau_1, \tau_2, \tau_3) = (I, \sigma_1, \sigma_2, \sigma_3) \ . \end{aligned} \tag{I.72}$$

There is a 1-1 correspondence between points x in Minkowski space corresponds and 2×2-hermitian matrices $\hat{x}$. This is given by

$$\hat{x} = \sum_{\mu=0}^{3} x^\mu \tau_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} , \tag{I.73}$$

with the inverse relation

$$x^\mu = \frac{1}{2} \mathrm{Tr} \left(\tau_\mu \hat{x} \right) \ . \tag{I.74}$$

Note that

$$\det \hat{x} = x_M^2 \ . \tag{I.75}$$

The Lorentz transformations act on the matrices $\hat{x}$. Three properties characterize maps $\hat{x} \mapsto \hat{x}'$ that arise from Lorentz transformations:

i. The transformation must be linear in $\hat{x}$.

ii. The transformation must map hermitian $\hat{x}$ to hermitian $\hat{x}'$.

iii. The transformation must preserve the determinant $\det \hat{x} = x_M^2 = \det \hat{x}'$.

The most general transformation satisfying (i) has the form

$$\hat{x} \quad \longrightarrow \quad \hat{x}' = \widehat{\Lambda x} = A \hat{x} B^* , \tag{I.76}$$

where A and B are non-singular 2×2-matrices. Not every non-singular A and B leads to properties (ii–iii). The restriction that $\hat{x}'$ is hermitian (for hermitian $\hat{x}$) means that

$$(A \hat{x} B^*)^* = B \hat{x} A^* = A \hat{x} B^* \ . \tag{I.77}$$

In other words, the transformation $T = A^{-1}B$ must satisfy $T\hat{x} = \hat{x}T^*$, for all hermitian $\hat{x}$. Taking $\hat{x} = I$ ensures that $T = T^*$. Therefore $T\hat{x} = \hat{x}T$ for all 2×2, hermitian matrices. Such T must be a real multiple λ of the identity and (I.76) becomes

$$\hat{x} \quad \longrightarrow \quad \hat{x}' = \lambda A \hat{x} A^* = \widehat{\Lambda x} \ . \tag{I.78}$$

Furthermore, any time-like vector x can be reduced by a boost to $(x_0, \vec{0})$, and in this case

$$x_0' = \lambda x_0 \frac{1}{2} \mathrm{Tr}(AA^*) \ . \tag{I.79}$$

As $\mathrm{Tr}(AA^*) > 0$, we infer that x_0' is a positive multiple of x_0 for $\lambda > 0$, and the direction of time is reversed for $\lambda < 0$. As we are analyzing the time-direction preserving case, we are considering $\lambda > 0$, so there is no loss of generality in absorbing $\lambda^{1/2}$ into A. Thus we are reduced to the relation

$$\hat{x} \quad \longrightarrow \quad \hat{x}' = A\hat{x}A^* \ . \tag{I.80}$$

Condition (iii) ensures that $|\det A|^2 = 1$, so after multiplying A by a phase, which does not change the transformation (I.80), we are reduced to the case $\det A = 1$.

In other words A is an element of the group $SL(2, \mathbb{C})$ of non-singular 2×2 complex matrices with determinant 1. Every element A of $SL(2, \mathbb{C})$ has a unique polar decomposition,

$$A = HU \ , \tag{I.81}$$

where H is a positive hermitian matrix and U is unitary. In fact H is the positive square root,

$$H = (AA^*)^{1/2} \ , \qquad \text{and } \ U = H^{-1/2}A \ . \tag{I.82}$$

We complete the proof of Proposition I.1 by showing that a unitary matrix A leads to a rotation (which preserves $x_0 = t$), while a positive, hermitian matrix A leads to a Lorentz boost along some axis.

In fact, if $A = U$ is unitary, then (I.74) says that

$$x_0' = \frac{1}{2}\mathrm{Tr}(IU\hat{x}U^*) = \frac{1}{2}\mathrm{Tr}(U\hat{x}U^*) = \frac{1}{2}\mathrm{Tr}(\hat{x}) = x_0 \ . \tag{I.83}$$

In other words, $\Lambda_{0\nu} = \delta_{0\nu}$, which characterizes a pure rotation. In particular, the unitary

$$U = e^{i(\theta/2)\sigma_3} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \ , \tag{I.84}$$

gives rise to a rotation by angle θ in the (x_1, x_2) plane. One sees that

$$x_3' = \frac{1}{2}\mathrm{Tr}\left(\sigma_3 U\hat{x}U^*\right) = \frac{1}{2}\mathrm{Tr}\left(U\sigma_3\hat{x}U^*\right) = \frac{1}{2}\mathrm{Tr}\left(\sigma_3\hat{x}\right) = x_3 \ , \tag{I.85}$$

so the rotation is about the axis x_3. Then as $\sigma_1\sigma_3 = -\sigma_3\sigma_1$, also

$$\begin{aligned} x_1' &= \frac{1}{2}\mathrm{Tr}\left(\sigma_1 U\hat{x}U^*\right) = \frac{1}{2}\mathrm{Tr}\left(U^*\sigma_1 U\hat{x}\right) = \frac{1}{2}\mathrm{Tr}\left(\sigma_1 U^2\hat{x}\right) \\ &= x_1\frac{1}{2}\mathrm{Tr}(U^2) + x_2\frac{1}{2i}\mathrm{Tr}(U^2\sigma_3) \\ &= x_1\cos\theta + x_2\sin\theta \ . \end{aligned} \tag{I.86}$$

Here the x_3 term vanishes, as $\mathrm{Tr}(\sigma_1 U^2\sigma_3) = i\mathrm{Tr}(U\sigma_3)$.

On the other hand, if $A = H$ is positive and hermitian, then A can be diagonalized by a unitary $A = UDU^*$, where D is diagonal, has positive eigenvalues, and determinant one (as A and U have determinant 1). Thus

$$D = \begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix} = e^{(\chi/2)\sigma_3} \ , \tag{I.87}$$

for some real χ. We claim that $\hat{x} \mapsto \hat{x}' = D\hat{x}D$ is the boost along the z axis, given by

$$x_0' = x_0 \cosh\chi + x_3 \sinh\chi \ , \quad x_1' = x_1 \quad x_2' = x_2 \ , \qquad \text{and } \ x_3' = x_0 \sinh\chi + x_3 \cosh\chi \ . \tag{I.88}$$

Using $\mathrm{Tr}(D\sigma_1) = \mathrm{Tr}(D\sigma_2) = 0$, the value of x_0' follows from (I.74) as

$$x_0' = \frac{1}{2}\mathrm{Tr}(D\hat{x}D) = \frac{1}{2}\mathrm{Tr}(D^2\hat{x}) = \frac{x_0}{2}\mathrm{Tr}(D^2) + \frac{x_3}{2}\mathrm{Tr}(D^2\sigma_3) = x_0\cosh\chi + x_3\sinh\chi \ . \tag{I.89}$$

Also the Pauli matrices satisfy $\sigma_3\sigma_1 = -\sigma_1\sigma_3$. Thus the relation (I.74) gives

$$x_1' = \frac{1}{2}\mathrm{Tr}(\sigma_1 D\hat{x}D) = \frac{1}{2}\mathrm{Tr}(D\sigma_1 D\hat{x}) = \frac{1}{2}\mathrm{Tr}(\sigma_1 D^{-1}D\hat{x}) = \frac{1}{2}\mathrm{Tr}(\sigma_1\hat{x}) = x_1 \ . \tag{I.90}$$

Checking the values of x_2' and x_3' is similar.

Poincaré Transformations: The Poincaré group is a (semi-direct) product of the Lorentz group (given by matrices Λ) with space-time translations b indexed by vectors in Minkowski 4-space, $b \in M^4$. One writes an element of the group as (b, Λ), and the group acts on Minkowski space by first making a Lorentz transformation, then a space-time translation,

$$(b, \Lambda)x = \Lambda x + b \ . \tag{I.91}$$

The multiplication law for the Poincaré group is

$$(b_1, \Lambda_1)\,(b_2, \Lambda_2) = (b_1 + \Lambda_1 b_2, \Lambda_1\Lambda_2) \ . \tag{I.92}$$

The inverse element to (b, Λ) is

$$(b, \Lambda)^{-1} = (-\Lambda^{-1}b, \, \Lambda^{-1}) \ , \qquad \text{or } \ x = (b, \Lambda)\left(\Lambda^{-1}(x - b)\right) \ . \tag{I.93}$$

The non-trivial unitary representations of the Poincaré group on a Hilbert space are infinite dimensional. The space-time translation subgroup $U(b) = U(b, I)$ can be written in the form $U(b) = e^{ib_0 H - i\vec{b}\cdot\vec{P}}$, defining an energy H and momentum $\vec{P}$ as its infinitesimal generators. The four components of the energy-momentum vector $(H, \vec{P})$ commute, as the space-time translation group is abelian. The *mass* operator M is defined as the positive square root $M = \left(H^2 - \vec{P}^2\right)^{1/2}$. Likewise the representation of the subgroup of rotations of three space about an axis $\vec{n}$ by an angle θ can be written as $U(\vec{n}, \theta) = e^{i\theta\vec{n}\cdot\vec{J}}$, defining the components of the angular momentum operators $\vec{J}$.

The irreducible, positive energy representations of the Poincaré group were classified in a famous paper by Wigner. They are characterized by two quantum numbers: (m, s), namely *mass* and *spin*. The mass m is a non-negative number in $[0, \infty)$ which is an eigenvalue of M, while the spin is a non-negative half-integer s for which $s(s+1)$ is an eigenvalue of $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$. The scalar representation corresponds to a particle of spin-0 and mass $m \geq 0$. There are two inequivalent spin-1/2 representations of the proper group, and these are mixed by the reflections of the full Poincaré group.

Vectors in Minkowski Space: There are three types of 4-vectors in Minkowski space:

- **Spacelike Vectors:** A vector x is space like if $x_M^2 < 0$.
- **Null Vectors:** A vector x is null if $x_M^2 = 0$.
- **Time-Like Vectors:** A vector x is time-like if $x_M^2 > 0$.

Two points x, x' are said to be space-like separated if $x - x'$ is a space-like vector.

Proposition I.2. *If x, x' are space-like separated, then there is a Lorentz transformation Λ_B that brings x, x' to the same time, a Lorentz transformation Λ that inverts $x - x'$, and a Poincaré transformation (b, Λ) that interchanges x with x'. In other words*

$$(\Lambda_B x)_0 = (\Lambda_B x')_0 \ , \qquad \Lambda(x - x') = (x' - x) \ , \qquad (b, \Lambda)x = x' \ , \qquad \text{and } (b, \Lambda)x' = x \ . \tag{I.94}$$

These transformations depend on x and x'.

Proof. The condition that $x - x'$ is space-like means that $\left|\vec{x} - \vec{x}\,'\right| > |x_0 - x_0'|$. Therefore these exists a real number χ such that $\tanh\chi = (x_0 - x_0')/\left|\vec{x} - \vec{x}\,'\right|$. Let Λ_B denote a Lorentz boost with rapidity $-\chi$ along $\vec{x} - \vec{x}\,'$. Then $\Lambda_B(x - x')$ has zero time component, or $x_0 = x_0'$. Let $\Lambda_B x = y$ and $\Lambda_B x' = y'$, so the vector $y - y' = (\vec{y} - \vec{y}\,', 0)$ has zero time component.

Let $\vec{n}$ be any 3-vector orthogonal to $\vec{y} - \vec{y}\,'$, and let Λ_R denote the Lorentz transformation of rotation by angle π about the axis n. This reverses the vector $\vec{y} - \vec{y}\,'$, so the transformation $\Lambda = \Lambda_B^{-1}\Lambda_R\Lambda_B$ reverses $x - x'$. In order to exchange x with x', one must perform these operations in a coordinate system translated os the rotation takes place midway between $\vec{y}$ and $\vec{y}'$; therefore the rotation Λ_R exchanges $\vec{y}$ with $\vec{y}'$. In order to do this, let $c = ((\vec{y} - \vec{y}'\,)/2, 0)$. Then the composition of four Poincaré transformations that interchanges x with x' is

$$(0, \Lambda_B^{-1})\,(0, \Lambda_R)\,(c, I)\,(0, \Lambda_B) = (0, \Lambda_B^{-1})\,(\Lambda_R c, \Lambda_R\Lambda_B) = (\Lambda_B^{-1}\Lambda_R c, \Lambda) \ . \tag{I.95}$$

Thus $b = \Lambda_B^{-1}\Lambda_R c = \Lambda(x - x')/2$. □

I.9 States and Transition Amplitudes

Consider the state

$$|x\rangle\!\rangle \ = |\vec{x}\,, t\rangle\!\rangle \ = e^{it\omega}|\vec{x}\,\rangle\!\rangle \ . \tag{I.96}$$

In other words, $|x\rangle\!\rangle \ = U(t)|\vec{x}\,\rangle\!\rangle$ where $U(t) = e^{it\omega}$ is the time translation subgroup for the free, scalar particle of mass m, generated by the Hamiltonian $H = \omega$. It has the initial value (time-zero value) equal to $|\vec{x}\,\rangle\!\rangle$. It gives rise to the usual solution to the Schrödinger equation,

$$f_t(\vec{x}\,) = \left(e^{-it\omega} f\right)(\vec{x}\,) = \ \langle\!\langle\vec{x}\,|\, e^{-it\omega}|f\rangle\!\rangle \ = \ \langle\!\langle x|f\rangle\!\rangle \ . \tag{I.97}$$

This state also satisfies the massive wave equation

$$\boxed{\left(\Box + m^2\right)|\vec{x}\,, t\rangle\!\rangle \ = 0} \ . \tag{I.98}$$

Here $\Box$ denotes the wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2_{\vec{x}} = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \ . \tag{I.99}$$

The massive wave equation is a linear wave equation for the states of a relativistic particle of mass m. The equation (I.98) is also known as the *Klein-Gordon* equation.

Define the transition amplitude

$$\boxed{K(x;x') = \langle\!\langle x|x'\rangle\!\rangle} \ . \tag{I.100}$$

We could also write K as a function of the difference vector $x - x'$ and define it as a function of a single four vector $\xi = x - x'$, namely

$$K(x-x') = K(x;x') = \langle\!\langle \vec{x}\,|\,e^{-i(t-t')H}|\vec{x}\,'\rangle\!\rangle \ = \ \langle\!\langle 0|e^{-i(t-t')H+i(\vec{x}-\vec{x}\,')\cdot\vec{P}}|0\rangle\!\rangle \ . \tag{I.101}$$

We now study

$$\boxed{K(x) = \langle\!\langle 0|\,e^{-it\omega+i\vec{x}\cdot\vec{P}}|\vec{0}\rangle\!\rangle} \ . \tag{I.102}$$

Since the energy $\omega > 0$, the function $K(x)$ extends to a function analytic in the lower-half t-plane. Let $K_\epsilon(x) = K(\vec{x}, t - i\epsilon)$. Then

$$K(x) = \lim_{\epsilon\to 0+} K_\epsilon(x) = \lim_{\epsilon\to 0+} \frac{1}{(2\pi)^3} \int e^{-i(t-i\epsilon)\omega(\vec{k})+i\vec{k}\cdot\vec{x}}\, d^3\vec{k} \ . \tag{I.103}$$

One can find another representation of K for $r = |\vec{x}\,| > |t|$, namely for x a space-like 4-vector. In this case one can evaluate (I.103). Take $k = \left|\vec{k}\right| \geq 0$, so

$$\begin{aligned} K(r,t) &= \lim_{\epsilon\to 0+} \frac{1}{(2\pi)^2} \int_0^\infty dk\, k^2 \int_0^\pi d\theta\, \sin\theta\, e^{-i(t-i\epsilon)\omega(k)+ikr\cos\theta} \\ &= \lim_{\epsilon\to 0+} \frac{-i}{(2\pi)^2 r} \int_0^\infty dk\, k e^{-i(t-i\epsilon)\omega(k)} \left(e^{ikr} - e^{-ikr}\right) \\ &= \lim_{\epsilon\to 0+} \frac{-i}{(2\pi)^2 r} \int_{-\infty}^\infty e^{-i(t-i\epsilon)\omega(k)} e^{ikr}\, k\, dk \ . \end{aligned} \tag{I.104}$$

One can transform this integral into the integral of a positive function, times a phase, by deforming the k integration from the real axis to a contour in the upper-half complex k-plane (where the integrand decays). Ultimately one can move it around the interval $i[m,\infty)$ on the positive, imaginary axis. At the point $k = is$, with $s > m$, the function $\omega(k) = \omega(is) = i\sqrt{s^2 - m^2}$ on the right side of the cut and $-i\sqrt{s^2-m^2}$ on the left. Thus

$$K(r,t) = -i\frac{1}{2\pi^2 r} \int_m^\infty \sinh\left(t\sqrt{s^2-m^2}\right) e^{-sr} s\, ds \ . \tag{I.105}$$

Note the condition $r > |t|$ ensures convergence of the integral in (I.105), and $iK(r,t) > 0$.

For $r > |t|$, it is the case that $K(r,t) \to 0$ as $t \to 0$. (On the other hand, the $t = 0$ value of $K(\vec{x}, 0)$ is not identically zero, since $K(\vec{x}\,, 0) = \delta^3(\vec{x}\,)$.) Rhe initial value of the time derivative (also needed to specify a solution to the Klein-Gordon equation) is a nowhere-vanishing function of $\vec{x}$, namely

$$\lim_{t\to 0} \frac{\partial K(r,t)}{\partial t} = -i\frac{1}{2\pi r}\int_m^\infty \sqrt{s^2-m^2}\, e^{-rs} s\, ds \neq 0 \; . \tag{I.106}$$

This explains the instantaneous spread of the transition amplitude to all non-zero $\vec{x}$ when $t \neq 0$. One should not attribute this effect to acausal propagation, nor the necessity to study multi-particle production, as one finds in many books! It arises from the positive-energy nature of the solution $K(r,t)$ and the non-local character of the energy operator ω.

II Two Particles

The interpretation of particles as arising from a field includes the possibility of multi-particle states. Thus we need not only to consider the wave function for one particle, but also the wave function for multi-particle states. Let us begin with two scalar particles. So we need to define a Hilbert space of two-particle wave functions $\mathcal{H}_2$. The vectors in this space are functions f of two vector variables $\vec{x}_1, \vec{x}_2$, and the scalar product in the two particle space given by

$$\langle f, f' \rangle_2 = \int \overline{f(\vec{x}_1, \vec{x}_2)}\, f'(\vec{x}_1, \vec{x}_2)\, d\vec{x}_1\, d\vec{x}_2 \; . \tag{II.1}$$

Here we use a subscript 2 on the inner product $\langle \cdot, \cdot \rangle_2$ to indicate that we are considering the two-particle space $\mathcal{H}_2$; we also write $\langle \cdot, \cdot \rangle_1$ for the scalar product in $\mathcal{H}_1$. When there is no chance of ambiguity, we may omit these subscripts.

In the momentum representation one has similar expressions. Given a two-particle configuration-space wave function $f(\vec{x}_1, \vec{x}_2)$, the two-particle wave function in momentum space has the form

$$\tilde{f}(\vec{k}_1, \vec{k}_2) = (\mathfrak{F}f)(\vec{k}_1, \vec{k}_2) = \frac{1}{(2\pi)^3}\int f(\vec{x}_1, \vec{x}_2) e^{-i\left(\vec{k}_1\cdot\vec{x}_1 + \vec{k}_2\cdot\vec{x}_2\right)} d\vec{x}_1\, d\vec{x}_2 \; , \tag{II.2}$$

and the inverse Fourier transform is

$$f(\vec{x}_1, \vec{x}_2) = \frac{1}{(2\pi)^3}\int \tilde{f}(\vec{k}_1, \vec{k}_2) e^{i\left(\vec{k}_1\cdot\vec{x}_1 + \vec{k}_2\cdot\vec{x}_2\right)} d\vec{k}_1\, d\vec{k}_2 \; . \tag{II.3}$$

The Fourier transform is unitary for the two-particle states, and

$$\langle f, f' \rangle_2 = \left\langle \tilde{f}, \tilde{f}' \right\rangle_2 \tag{II.4}$$

II.1 The Tensor Product

Given two single-particle wave functions, there is a natural way to combine them to obtain a two-particle wave function. Given $f_1(\vec{x}_1)$ and $f_2(\vec{x}_2)$, the simplest combination is the product wave function

$$f(\vec{x}_1, \vec{x}_2) = f_1(\vec{x}_1)\, f_2(\vec{x}_2) \; . \tag{II.5}$$

This is called the *tensor product*. One denotes the tensor product wave function by the multiplication sign multiplication sign "⊗", so the tensor product of the wave function f_1 with the wave function f_2 is $f_1 \otimes f_2$. The values of the tensor-product wave function are

$$(f_1 \otimes f_2)\,(\vec{x}_1, \vec{x}_2) = f_1(\vec{x}_1)\, f_2(\vec{x}_2)\ . \tag{II.6}$$

The inner product of two different tensor-product wave functions $f_1 \otimes f_2$ and $f_1' \otimes f_2'$ is

$$\boxed{\langle f_1 \otimes f_2, f_1' \otimes f_2' \rangle_2 = \langle f_1, f_1' \rangle_1\ \langle f_2, f_2' \rangle_1}\ . \tag{II.7}$$

Note that $f_1 \otimes f_2 \neq f_2 \otimes f_1$. Furthermore, it is clear that not every two-particle wave function $f(\vec{x}_1, \vec{x}_2)$ can be written as a tensor product of two one-particle wave functions. However we can obtain a general two-particle wave function as a limit of sums of product wave functions. In particular, if $\{e_j\}$ for $j = 0, 1, 2, \ldots$ is an ortho-normal basis for the one-particle wave functions in $\mathcal{H}_1$, then the vectors $e_i \otimes e_j$ are a basis for $\mathcal{H}_2$. Note that

$$\langle e_i \otimes e_j, e_{i'} \otimes e_{j'} \rangle_2 = \delta_{ii'} \delta_{jj'}\ , \tag{II.8}$$

so the vectors $e_i \otimes e_j$ are ortho-normal. Thus every general two-particle wave function can be written as a limit,

$$f = \sum_{i,j=0}^{\infty} c_{ij}\ e_i \otimes e_j\ , \tag{II.9}$$

where the coefficients can be obtained by taking the inner product with $e_i \otimes e_j$, namely

$$c_{ij} = \langle e_i \otimes e_j\, , f \rangle_2\ . \tag{II.10}$$

For two different two-particle functions f and f', one can express the inner product as

$$\langle f, f' \rangle_2 = \sum_{i,j=0}^{\infty} \overline{c_{ij}}\, c'_{ij}\ . \tag{II.11}$$

Thus the norm squared of a general two-particle state f is

$$\|f\|_2^2 = \langle f, f \rangle_2 = \sum_{ij} |c_{ij}|^2\ . \tag{II.12}$$

A square-integrable two-particle state f corresponds to a square summable sequence $\{c_{ij}\}$.

II.2 Dirac Notation for Two-Particle States

The Dirac notation for the ket vector corresponding to the tensor product is

$$f_1 \otimes f_2 \quad \longrightarrow \quad |f_1\rangle |f_2\rangle\ . \tag{II.13}$$

The bra vector corresponding to this ket is

$$|f_1\rangle |f_2\rangle \quad \longrightarrow \quad \langle f_2| \langle f_1|\ . \tag{II.14}$$

Notice that the order of the labels has been reversed, in a similar fashion to taking the hermitian adjoint of a matrix. This reversal is natural, as it gives rise to the scalar product (II.7), namely

$$\boxed{\langle f_2| \langle f_1 | f_1' \rangle | f_2' \rangle = \langle f_1 | f_1' \rangle\, \langle f_2 | f_2' \rangle}\ . \tag{II.15}$$

II.3 Tensor Products of Operators

Given two operators T and S, each acting on the one-particle space $\mathcal{H}_1$, we can define the tensor product operator $T \otimes S$ that acts on the two-particle space. On any product wave function $f_1 \otimes f_2 \in \mathcal{H}_2$, with $f_1, f_2 \in \mathcal{H}_1$, define the linear transformation $T \otimes S$ as

$$(T \otimes S)\,(f_1 \otimes f_2) = Tf_1 \otimes Sf_2 \; . \tag{II.16}$$

In particular, $(T \otimes I)\,(f_1 \otimes f_2) = Tf_1 \otimes f_2$, and also $(I \otimes S)\,(f_1 \otimes f_2) = f_1 \otimes Sf_2$. For a general vector $f \in \mathcal{H}_2$ of the form (II.9), one has as a consequence of linearity,

$$\boxed{(T \otimes S)\, f = \sum_{i,j} c_{ij}\,(Te_i \otimes Se_j)} \; . \tag{II.17}$$

The operator $T \otimes I$ is a special case of a tensor product operator that acts on the first wave function only. Likewise $I \otimes S$ acts on the second wave function only. In general two tensor product operators obey the multiplication law

$$(T_1 \otimes S_1)\,(T_2 \otimes S_2) = (T_1 T_2 \otimes S_1 S_2) \; . \tag{II.18}$$

Accordingly, any tensor product operator can be written as a product of an operator acting on each variable,

$$T \otimes S = (T \otimes I)\,(I \otimes S) \; . \tag{II.19}$$

The matrix elements on the basis $\{e_i \otimes e_j\}$ of a tensor product operator $T \otimes S$ on $\mathcal{H}_2$ are given by the tensor with four indices. This tensor is the product of the matrices for the two one-particle operators. Namely $(T \otimes S)_{ij\;i'j'} = \langle e_i \otimes e_j\,, (T \otimes S)\,(e_{i'} \otimes e_{j'})\rangle_2$, so that

$$\boxed{(T \otimes S)_{ij\;i'j'} = T_{ii'}\, S_{jj'}} \; . \tag{II.20}$$

A general operator X on $\mathcal{H}_2$ cannot be expressed as a tensor product of two one-particle operators. (For example, the operator $T \otimes I + I \otimes T$ is not a tensor product operator, but a sum of two.) However, a general operator X is a limit of finite sums of tensor product operators, and it has the form

$$X = \sum_{\alpha\beta} x_{\alpha\beta}\, T_\alpha \otimes S_\beta \; , \tag{II.21}$$

for some coefficients $x_{\alpha\beta}$. Such a general operator acts on a tensor product state as

$$Xf_1 \otimes f_2 = \sum_{\alpha\beta} x_{\alpha\beta}\, T_\alpha f_1 \otimes S_\beta f_2 \; . \tag{II.22}$$

We can express this relation in Dirac notation (I.26) and (II.13) as

$$X|f_1\rangle|f_2\rangle = \sum_{\alpha\beta} x_{\alpha\beta}\, |T_\alpha f_1\rangle|S_\beta f_2\rangle \; . \tag{II.23}$$

II.4 An Operator on $\mathcal{H}_1$ Gives Operators on $\mathcal{H}_2$

Rather than considering the tensor product of two general operators as in §II.3, there are two very standard ways to take a single operator T acting on $\mathcal{H}_1$ and to produce an operator T_2 on $\mathcal{H}_2$.

The first standard method is the *multiplicative action*,

$$T_2 = T \otimes T \ . \tag{II.24}$$

In case that T is unitary on $\mathcal{H}_1$, then $T \otimes T$ is a unitary on $\mathcal{H}_2$. For example, this gives the unitary Fourier transform on $\mathcal{H}_2$

$$\mathfrak{F}_2 = \mathfrak{F} \otimes \mathfrak{F} \ , \tag{II.25}$$

or the unitary Poincaré symmetry on $\mathcal{H}_2$

$$U_2(a,\Lambda) = U(a,\Lambda) \otimes U(a,\Lambda) \ . \tag{II.26}$$

The second standard method is the *additive action.* If S is an operator on $\mathcal{H}_1$ define the additive operator,

$$S_2 = S \otimes I + I \otimes S \ . \tag{II.27}$$

on $\mathcal{H}_2$. This method is suitable if S is a quantity such as the momentum $\vec{P}$, angular momentum $\vec{L}$, or energy ω of a non-interacting pair of particles. For example,

$$\vec{P}_2 = \vec{P} \otimes I + I \otimes \vec{P} \ , \tag{II.28}$$

and

$$\vec{P}_2\,(f_1 \otimes f_2) = \left(\vec{P} \otimes I + I \otimes \vec{P}\right)(f_1 \otimes f_2) = \vec{P} f_1 \otimes f_2 + f_1 \otimes \vec{P} f_2 \ . \tag{II.29}$$

In particular, if f_1, f_2 are momentum eigenkets $|\vec{k}_1\rangle$ and $|\vec{k}_2\rangle$ respectively, then $\vec{P}_2$ agrees with its interpretation as the total momentum,

$$\boxed{\vec{P}_2\,|\vec{k}_1\rangle|\vec{k}_2\rangle = \left(\vec{k}_1 + \vec{k}_2\right)|\vec{k}_1\rangle|\vec{k}_2\rangle} \ . \tag{II.30}$$

II.5 Two Identical Bosons

%subsectionWave Functions When considering two particles which are identical bosons, it is useful to build into the space of wave functions the symmetry under interchange of the particles. This means that we restrict $\mathcal{H}_2$ to a bosonic subspace by restricting attention to the two-particle wave functions $f(\vec{x}_1, \vec{x}_2)$ that obey the symmetry condition,

$$f(\vec{x}_1, \vec{x}_2) = f(\vec{x}_2, \vec{x}_1) \ . \tag{II.31}$$

Clearly a linear combination of two symmetric wave functions is symmetric, so the symmetric two-particle wave functions form a linear subspace of the two-particle wave functions. We denote the symmetric (bosonic) subspace by

$$\mathcal{H}_2^b \subset \mathcal{H}_2 \ . \tag{II.32}$$

In general, a tensor product $f_1 \otimes f_2$ of wave functions is a bosonic symmetric state only if f_1 is a multiple of f_2. Thus it is useful to introduce a symmetrized tensor product $\otimes_s$ that ensures (II.31). All quantum field theory books use the two-particle *symmetric* (or *bosonic*) tensor product

$$(f_1 \otimes_s f_2)(\vec{x}_1, \vec{x}_2) = \frac{1}{\sqrt{2}} \left(f_1(\vec{x}_1) f_2(\vec{x}_2) + f_1(\vec{x}_2) f_2(\vec{x}_1) \right) . \tag{II.33}$$

Alternatively one can write in tensor product notation

$$\boxed{f_1 \otimes_s f_2 = \frac{1}{\sqrt{2}} (f_1 \otimes f_2 + f_2 \otimes f_1)} , \qquad \text{so} \quad \boxed{f_1 \otimes_s f_2 = f_2 \otimes_s f_1 \in \mathcal{H}_2^b} . \tag{II.34}$$

Two symmetric tensor product wave functions have the scalar product[3]

$$\boxed{\langle f_1 \otimes_s f_2, f_1' \otimes_s f_2' \rangle_2 = \langle f_1, f_1' \rangle_1 \langle f_2, f_2' \rangle_1 + \langle f_1, f_2' \rangle_1 \langle f_2, f_1' \rangle_1} , \tag{II.36}$$

in place of (II.7). The space $\mathcal{H}_2^b$ is spanned by the symmetrized basis vectors

$$e_i \otimes_s e_j = \frac{1}{\sqrt{2}} (e_i \otimes e_j + e_j \otimes e_i) . \tag{II.37}$$

Note that these vectors are orthogonal, but not normalized, as

$$\langle e_i \otimes_s e_j \, , e_{i'} \otimes_s e_{j'} \rangle_2 = \delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{i'j} . \tag{II.38}$$

This vanishes unless the unordered pair of values $\{i, j\}$ equals the unordered pair of values $\{i', j'\}$. But in case it is nonzero, it equals 1 in case $i \neq j$, and it equals 2 in case $i = j$. Thus we can define the ortho-normal basis elements for $\mathcal{H}_2^b$ as

$$e_{ij} = \frac{1}{\sqrt{1 + \delta_{ij}}} \, e_i \otimes_s e_j . \tag{II.39}$$

An arbitrary element $F \in \mathcal{H}_2^b$ is a symmetric, square integrable wave function $F(\vec{x}_1, \vec{x}_2) = F(\vec{x}_2, \vec{x}_1)$ has expansion in term of the basis vectors as

$$F = \sum_{i,j} c_{ij} \, e_{ij} , \qquad \text{with } c_{ij} = \langle e_{ij} \, , F \rangle_2 . \tag{II.40}$$

Then for two such general bosonic two particle functions satisfy

$$\boxed{\langle F, F' \rangle_2 = \int \overline{f(\vec{x}_1, \vec{x}_2)} \, f'(\vec{x}_1, \vec{x}_2) \, d\vec{x}_1 d\vec{x}_2 = \sum_{ij} \overline{c_{ij}} \, c'_{ij}} . \tag{II.41}$$

[3]Note that the physics normalization convention of using $\sqrt{2}$ in (II.34) is really a *definition*. It differs from the normalization in many mathematics books, where one often finds $f_1 \otimes_s f_2 = \frac{1}{2} (f_1 \otimes f_2 + f_2 \otimes f_1)$, yielding the natural relation $f \otimes_s f = f \otimes f$, and also

$$\langle f_1 \otimes_s f_2, f_1' \otimes_s f_2' \rangle = \frac{1}{2} \langle f_1, f_1' \rangle_1 \langle f_2, f_2' \rangle_1 + \frac{1}{2} \langle f_1, f_2' \rangle_1 \langle f_2, f_1' \rangle_1 . \tag{II.35}$$

This differs from (II.36) by a factor 1/2; in the case of n particles a relative factor of $1/n!$ arises.

II.6 Dirac Notation for Two Identical Bosons

We use the following Dirac notation for the symmetrized product state of two bosons,

$$\boxed{|f_1, f_2\rangle\!\rangle \;=\frac{1}{\sqrt{2}}\,|f_1\rangle\!\rangle\;|f_2\rangle\!\rangle\;+\frac{1}{\sqrt{2}}\,|f_2\rangle\!\rangle\;|f_1\rangle\!\rangle}\;. \tag{II.42}$$

When there may be ambiguity about whether the state is bosonic, we write $|f_1, f_2\rangle\!\rangle^b$ instead of $|f_1, f_2\rangle\!\rangle$. Similarly, define the continuum, configuration-space states

$$|\vec{x}_1, \vec{x}_2\rangle\!\rangle \;=\frac{1}{\sqrt{2}}\,|\vec{x}_1\rangle\!\rangle\;|\vec{x}_2\rangle\!\rangle\;+\frac{1}{\sqrt{2}}\,|\vec{x}_2\rangle\!\rangle\;|\vec{x}_1\rangle\!\rangle\;\;, \tag{II.43}$$

so

$$\langle\!\langle\vec{x}_1, \vec{x}_2|\vec{x}\,'_1, \vec{x}\,'_2\rangle\!\rangle_2 = \delta\left(\vec{x}_1-\vec{x}\,'_1\right)\,\delta\left(\vec{x}_2-\vec{x}\,'_2\right)+\delta\left(\vec{x}_1-\vec{x}\,'_2\right)\,\delta\left(\vec{x}_2-\vec{x}\,'_1\right)\;. \tag{II.44}$$

A short calculation shows that

$$\langle\!\langle\vec{x}_1, \vec{x}_2|f_1, f_2\rangle\!\rangle_2 = f_1(\vec{x}_1)f_2(\vec{x}_2)+f_1(\vec{x}_2)f_2(\vec{x}_1) = \sqrt{2}f_1\otimes_s f_2\;. \tag{II.45}$$

Note the factor $\sqrt{2}$. This relation extends by linearity to linear combinations of vectors of the form $|f_1, f_2\rangle\!\rangle$, and therefore for any vector $|F\rangle\!\rangle_2 \in \mathcal{H}_2^b$ one has,

$$\boxed{F(\vec{x}_1, \vec{x}_2) = \frac{1}{\sqrt{2}}\;\langle\!\langle\vec{x}_1, \vec{x}_2|F\rangle\!\rangle_2}\;. \tag{II.46}$$

Comparing this identity with (II.41), we infer that on the two-particle space $\mathcal{H}_2^b$,

$$\boxed{E_2^b = \frac{1}{2}\int |\vec{x}_1, \vec{x}_2\rangle\!\rangle\,\langle\!\langle\vec{x}_1, \vec{x}_2|\;d\vec{x}_1\,d\vec{x}_2 = I}\;. \tag{II.47}$$

The operator E_2^b also acts on all of $\mathcal{H}_2$, and when applied to vectors in $\mathcal{H}_2$ that are orthogonal to $\mathcal{H}_2^b$, it gives 0.[4]

In terms of these wave functions for two symmetric tensor products,

$$\tilde{f}(\vec{k}_1, \vec{k}_2) = \frac{1}{\sqrt{2}}f_1(\vec{k}_1)\,f_2(\vec{k}_2)+\frac{1}{\sqrt{2}}f_1(\vec{k}_2)\,f_2(\vec{k}_1)\;, \tag{II.50}$$

one has

$$\left\langle\tilde{f}, \tilde{f}'\right\rangle_2 = \int \tilde{f}(\vec{k}_1, \vec{k}_2)^*\,\tilde{f}'(\vec{k}_1, \vec{k}_2)\,d\vec{k}_1 d\vec{k}_2\;. \tag{II.51}$$

[4]A short calculation shows that any anti-symmetric (fermionic) state of the form

$$|f_1', f_2'\rangle\!\rangle^f = \frac{1}{\sqrt{2}}|f_1'\rangle|f_2'\rangle - \frac{1}{\sqrt{2}}|f_2'\rangle|f_1'\rangle = -|f_2', f_1'\rangle\!\rangle^f\;, \tag{II.48}$$

is orthogonal has scalar product 0 with every vector in $\mathcal{H}_2^b$. Also $|\vec{x}_1, \vec{x}_2\rangle\!\rangle \;= |\vec{x}_1, \vec{x}_2\rangle\!\rangle^b$ satisfies

$${}^b\langle\!\langle\vec{x}_1, \vec{x}_2|f_1', f_2'\rangle\!\rangle^f = 0\;, \qquad \text{and}\;\; E_2^b|f_1', f_2'\rangle\!\rangle^f = 0\;. \tag{II.49}$$

$$|f_1, f_2\rangle = \frac{1}{\sqrt{2}}\,|f_1\rangle|f_2\rangle + \frac{1}{\sqrt{2}}\,|f_2\rangle|f_1\rangle\ , \tag{II.52}$$

and

$$|\vec{k}_1, \vec{k}_2\rangle = \frac{1}{\sqrt{2}}\,|\vec{k}_1\rangle|\vec{k}_2\rangle + \frac{1}{\sqrt{2}}\,|\vec{k}_2\rangle|\vec{k}_1\rangle\ , \tag{II.53}$$

so

$$\langle\vec{k}_1, \vec{k}_2|\vec{k}'_1, \vec{k}'_2\rangle = \delta\left(\vec{k}_1 - \vec{k}'_1\right)\,\delta\left(\vec{k}_2 - \vec{k}'_2\right) + \delta\left(\vec{k}_1 - \vec{k}'_2\right)\,\delta\left(\vec{k}_2 - \vec{k}'_1\right)\ . \tag{II.54}$$

Furthermore for an arbitrary two-particle bosonic ket $|F\rangle$, the momentum representative is,

$$\boxed{F(\vec{k}_1, \vec{k}_2) = \frac{1}{\sqrt{2}}\,\langle\vec{k}_1, \vec{k}_2|F\rangle_2}\ , \tag{II.55}$$

from which we infer that on the two-particle space $\mathcal{H}_2^b$, also

$$\boxed{\frac{1}{2}\int|\vec{k}_1, \vec{k}_2\rangle\langle\vec{k}_1, \vec{k}_2|\;d\vec{k}_1\,d\vec{k}_2 = I}\ . \tag{II.56}$$

II.7 Operators Acting on States of Two Identical Bosons

The tensor product operator $T_1 \otimes T_2$ acting on the symmetric tensor product state $f_1 \otimes_s f_2$ gives

$$(T_1 \otimes T_2)\,(f_1 \otimes_s f_2) = \frac{1}{\sqrt{2}}\,(T_1 f_1 \otimes T_2 f_2 + T_1 f_2 \otimes T_2 f_1)\ \ , \tag{II.57}$$

which in general is not symmetric. However two operators that always transform bosonic (symmetric) two-particle states to other bosonic (symmetric) states are the two particle operators $T_2 = T \otimes T$ and $S_2 = S \otimes I + I \otimes S$ of §II.4. In these cases it is easy to check that

$$\boxed{T_2\,(f_1 \otimes_s f_2) = Tf_1 \otimes_s Tf_2}\ , \tag{II.58}$$

and

$$\boxed{S_2\,(f_1 \otimes_s f_2) = (Sf_1 \otimes_s f_2) + (f_1 \otimes_s Sf_2)}\ . \tag{II.59}$$

In Dirac notation,

$$\boxed{T_2|f_1, f_2\rangle = |Tf_1, Tf_2\rangle}\ , \qquad \text{and}\ \boxed{S_2|f_1, f_2\rangle = |Sf_1, f_2\rangle + |f_1, Sf_2\rangle}\ . \tag{II.60}$$

Special cases of these identities are,

$$\mathfrak{F}_2|f_1, f_2\rangle = |\mathfrak{F}f_1, \mathfrak{F}f_2\rangle\ , \qquad \text{and}\ U(a,\Lambda)_2|f_1, f_2\rangle = |U(a,\Lambda)f_1, U(a,\Lambda)f_2\rangle\ , \tag{II.61}$$

as well as

$$\vec{P}_2|\vec{k}_1, \vec{k}_2\rangle = \left(\vec{k}_1 + \vec{k}_2\right)|\vec{k}_1, \vec{k}_2\rangle\ , \qquad \text{and}\ \vec{H}_0|\vec{k}_1, \vec{k}_2\rangle = \left(\omega(\vec{k}_1) + \omega(\vec{k}_2)\right)|\vec{k}_1, \vec{k}_2\rangle\ . \tag{II.62}$$

Here H_0 is the energy for freely moving, relativistic particles of mass m.

III n-Particles

Having understood two particles, it is easy to generalize the concepts to the case of an arbitrary number of particles, denoted by n.

III.1 Wave Functions for n-Identical Bosons

One can consider n-particle states that are tensor products of n one-particle states. For instance, we take $\mathcal{H}_n$ to be spanned by states $f^{(n)}$ that are finite linear combinations of states of the form

$$\underbrace{f_1 \otimes_s \cdots \otimes_s f_n}_{n \text{ factors}} = \frac{1}{\sqrt{n!}} \sum_\pi f_{\pi_1} \otimes \cdots \otimes f_{\pi_n} \ , \tag{III.1}$$

where the sum extends over permutations π of $1, \ldots, n$. This defines a wave function for the multiple symmetric tensor product equal to

$$f^{(n)}(\vec{x}_1, \ldots, \vec{x}_n) = (f_1 \otimes_s \cdots \otimes_s f_n)\,(\vec{x}_1, \ldots, \vec{x}_n) = \frac{1}{\sqrt{n!}} \sum_\pi f_{\pi_1}(\vec{x}_1) \cdots f_{\pi_n}(\vec{x}_n) \ . \tag{III.2}$$

In Dirac notation we write

$$f_1 \otimes_s \cdots \otimes_s f_n \quad \longrightarrow \quad |f_1, \ldots, f_n\rangle\!\rangle \ , \tag{III.3}$$

or in momentum space

$$\tilde{f}_1 \otimes_s \cdots \otimes_s \tilde{f}_n \quad \longrightarrow \quad |\tilde{f}_1, \ldots, \tilde{f}_n\rangle \ . \tag{III.4}$$

The scalar product of two such states is

$$\begin{aligned}
\left\langle f^{(n)}, f'^{\,(n)} \right\rangle_{\mathcal{H}_n} &= \int f^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)^* f^{(n)\prime}(\vec{x}_1, \ldots, \vec{x}_n)\, d\vec{x}_1 \cdots d\vec{x}_n \\
&= \langle f_1 \otimes_s \cdots \otimes_s f_n, f_1' \otimes_s \cdots \otimes_s f_n' \rangle_{\mathcal{H}_n} \\
&= \frac{1}{n!} \sum_{\pi,\pi'} \left(f_{\pi_1} \otimes \cdots \otimes f_{\pi_n}, f_{\pi_1'} \otimes \cdots \otimes f_{\pi_n'} \right)_{\mathcal{H}_n} \\
&= \sum_\pi \left\langle f_1, f_{\pi_1}' \right\rangle_{\mathcal{H}_1} \cdots \left\langle f_n, f_{\pi_n}' \right\rangle_{\mathcal{H}_1} \ .
\end{aligned} \tag{III.5}$$

In particular the state that we call an *n-fold symmetric tensor power*

$$f^{\otimes_s n} = \underbrace{f \otimes_s \cdots \otimes_s f}_{n \text{ factors}} \ , \tag{III.6}$$

has length squared

$$\left\langle f^{\otimes_s n}, f^{\otimes_s n} \right\rangle_{\mathcal{H}_n} = n!\, \langle f, f \rangle_{\mathcal{H}_1}^n \ . \tag{III.7}$$

III.2 Permanents, Scalar Products, and Recursion Relations

The scalar product of two n-particle boson tensor product states is related to the $n \times n$ Gram matrix M. This is the matrix whose entries are the scalar products $M_{ij} = \left\langle f_i, f_j' \right\rangle_{\mathcal{H}_1}$. Sometimes M is written in terms of the $2n$ single-particle wave-functions in the form

$$M = \begin{pmatrix} f_1, f_2, \ldots, f_n \\ f_1', f_2', \ldots, f_n' \end{pmatrix} , \tag{III.8}$$

where the first row of vectors $f_1, f_2, \ldots,$ labels the rows of the matrix M, while the second row of vectors $f_1', f_2', \ldots,$ indexes the columns.

The expression for the scalar product (III.5)is called the *permanent* of M, namely

$$\mathrm{Perm}_n M = \sum_\pi M_{1\,\pi_i} M_{2\,\pi_2} \cdots M_{n\,\pi_n} , \tag{III.9}$$

or

$$\boxed{\langle\!\langle f_1, \ldots, f_n | f_1', \ldots, f_n' \rangle\!\rangle = \mathrm{Perm}_n \begin{pmatrix} f_1, f_2, \ldots, f_n \\ f_1', f_2', \ldots, f_n' \end{pmatrix}} . \tag{III.10}$$

The combinatorial expression is similar to the expression for the determinant of M, but it is totally symmetric, rather than alternating under an exchange of neighboring columns or rows. Although the permanent does not have a geometric interpretation like the determinant, it does satisfy similar recursion relations. For an $(n+1) \times (n+1)$-matrix M, one has an $n \times n$ minor $\hat{M}_{ij}$ obtained by omitting the i^{th} row and the j^{th} column of M,

$$\hat{M}_{ij} = \begin{pmatrix} f_1, \ldots, \not{f}_i, \ldots, f_n \\ f_1', \ldots, \not{f}_j', \ldots, f_n' \end{pmatrix} . \tag{III.11}$$

Then the permanent satisfies

$$\mathrm{Perm}_{n+1} M \;=\; \sum_{j=1}^{n+1} M_{ij} \, \mathrm{Perm}_n \hat{M}_{ij} , \tag{III.12}$$

which one can also write as

$$\mathrm{Perm}_{n+1} \begin{pmatrix} f_1, f_2, \ldots, f_{n+1} \\ f_1',_2', \ldots, f_{n+1}' \end{pmatrix} = \sum_{j=1}^{n+1} \mathrm{Perm}_1 \begin{pmatrix} f_i \\ f_j' \end{pmatrix} \mathrm{Perm}_n \begin{pmatrix} f_1, f_2, \ldots, \not{f}_i \ldots, f_{n+1} \\ f_1', f_2', \ldots, \not{f}_j', \ldots, f_{n+1}' \end{pmatrix} . \tag{III.13}$$

In Dirac notation, for any i,

$$\boxed{\langle\!\langle f_1, \ldots, f_{n+1} | f_1', \ldots, f_{n+1}' \rangle\!\rangle = \sum_{j=1}^{n+1} \langle\!\langle f_i | f_j' \rangle\!\rangle \; \langle\!\langle f_1, \ldots, \not{f}_i \ldots, f_{n+1} | f_1', \ldots, \not{f}_j', \ldots, f_{n+1}' \rangle\!\rangle} . \tag{III.14}$$

Continuum Bosonic Eigenkets: We have a corresponding relation for continuum eigenkets. We give the expressions in momentum space, although similar formulas hold in the configuration-space representation. In the n-particle space, take the continuum basis set $|\vec{k}_1, \ldots, \vec{k}_n\rangle$. Then

$$\boxed{\langle \vec{k}_1, \ldots, \vec{k}_n | \vec{k}'_1, \ldots, \vec{k}'_n \rangle = \operatorname*{Perm}_n \begin{pmatrix} \vec{k}_1, \ldots, \vec{k}_n \\ \vec{k}'_1, \ldots, \vec{k}'_n \end{pmatrix} = \sum_\pi \delta\left(\vec{k}_1 - \vec{k}'_{\pi_1}\right) \cdots \delta\left(\vec{k}_n - \vec{k}'_{\pi_n}\right)} \,. \tag{III.15}$$

The corresponding recursion relation is (for any chosen $i = 1, \ldots, n+1$),

$$\boxed{\langle \vec{k}_1, \ldots, \vec{k}_{n+1} | \vec{k}'_1, \ldots, \vec{k}'_{n+1} \rangle = \sum_{j=1}^{n+1} \langle \vec{k}_i | \vec{k}'_j \rangle \langle \vec{k}_1, \ldots, \not{\vec{k}}_i \ldots, \vec{k}_{n+1} | \vec{k}'_1, \ldots, \not{\vec{k}}'_j, \ldots, \vec{k}'_{n+1} \rangle} \,. \tag{III.16}$$

One also has n-particle position space eigenkets $|\vec{x}_1, \ldots, \vec{x}_n\rangle\!\rangle$, or the corresponding bras, obtained by taking $f_j(\vec{x}) = \delta(\vec{x} - \vec{x}_j)$, for each $1 \le j \le n$. One also has the momentum space eigenkets $|\vec{k}_1, \ldots, \vec{k}_n\rangle$ arising from taking the wave functions $\tilde{f}_j(\vec{k}) = \delta(\vec{k} - \vec{k}_j)$, for each $1 \le j \le n$. The position space vectors satisfy

$$\boxed{\langle\!\langle \vec{x}_1, \ldots, \vec{x}_n | \vec{x}'_1, \ldots, \vec{x}'_n \rangle\!\rangle = \operatorname*{Perm}_n \begin{pmatrix} \vec{x}_1, \ldots, \vec{x}_n \\ \vec{x}'_1, \ldots, \vec{x}'_n \end{pmatrix} = \sum_\pi \prod_{j=1}^n \delta(\vec{x}_j - \vec{x}_{\pi_j})} \,. \tag{III.17}$$

Corresponding to (I.44), the scalar product between the two sorts of states is

$$\boxed{\langle\!\langle \vec{x}_1, \ldots, \vec{x}_n | \vec{k}_1, \ldots, \vec{k}_n \rangle = \frac{1}{(2\pi)^{3n/2}} \sum_\pi \prod_{j=1}^n e^{i\vec{x}_j \cdot \vec{k}_{\pi_j}}} \,. \tag{III.18}$$

As in the one-particle case, and as in the Homework 2 for the two-particle case, one has on $\mathcal{H}_n$ the completeness relation,

$$\begin{aligned} \frac{1}{n!} \int |\vec{x}_1, \ldots, \vec{x}_n\rangle\!\rangle \langle\!\langle \vec{x}_1, \ldots, \vec{x}_n| \, d\vec{x}_1 \cdots d\vec{x}_n &= \frac{1}{n!} \int |\vec{k}_1, \ldots, \vec{k}_n\rangle \langle \vec{k}_1, \ldots, \vec{k}_n| \, d\vec{k}_1 \cdots d\vec{k}_n \\ &= I \,. \end{aligned} \tag{III.19}$$

n-Particle Bases: If e_j for $j = 0, 1, \ldots$ is an ortho-normal basis for $\mathcal{H}_1$, then the vectors $\Omega_{\mathbf{n}}$ that we now define are an ortho-normal basis for $\mathcal{H}_n$. Choose a sequence of non-negative integers

$$\mathbf{n} = \{n_0, n_1, \ldots\} \,, \qquad \text{and with} \quad |\mathbf{n}| = \sum_j n_j = n \,. \tag{III.20}$$

Then the set of

$$\Omega_{\mathbf{n}} = \otimes_{j=0}^{\infty} \left(\frac{1}{\sqrt{n_j!}} \, e_j^{\otimes_s n_j} \right) \,, \tag{III.21}$$

is our basis for $\mathcal{H}_n$.

Polarization: We can express the symmetric tensor product of n distinct one-particle states $f_1 \otimes_s \cdots \otimes_s f_n$ as a sum of n-fold tensor powers. This is a generalization of the polarization identity (I.4) for a bilinear, symmetric function, to an identity for an n-linear function. We find

$$f_1 \otimes_s \cdots \otimes_s f_n = \frac{1}{2^n n!} \sum_{\epsilon_j^2=1} \epsilon_1 \cdots \epsilon_n \left(\epsilon_1 f_1 + \cdots + \epsilon_n f_n\right)^{\otimes_s n} , \qquad \text{(III.22)}$$

from which one can derive (I.4) in case $n = 2$.

IV Fock Space

Fock space $\mathcal{F}$ is the appropriate Hilbert space for a free field theory. In the case of one type of identical scalar boson particle, we call sometimes indicate the space as $\mathcal{F}^b$. Here we consider the states in such a space, and also some operators that act on this space.

IV.1 States in Fock Space

Fock space $\mathcal{F}$ is the space with vectors that have the possibility to contain an arbitrary number of particles. A state $\mathbf{f} \in \mathcal{F}$ is given by a sequence of wave functiosn $f^{(n)}$, each with n particles. Thus it has the form

$$\mathbf{f} = \{f^{(0)}, f^{(1)}, f^{(2)}, \ldots, f^{(n)}, \ldots\} , \qquad \text{where } f^{(n)} \in \mathcal{F}_n , \qquad \text{(IV.1)}$$

The zero-particle space $\mathcal{F}_0 = \mathbb{C}$ is just the space of complex numbers $\mathbb{C}$. The scalar product in $\mathcal{F}$ is just

$$\langle \mathbf{f}, \mathbf{f}' \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, f'^{(n)} \right\rangle_{\mathcal{H}_n} . \qquad \text{(IV.2)}$$

with

$$\left\langle f^{(n)}, f'^{(n)} \right\rangle_{\mathcal{H}_n} = \int \overline{f^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)}\, f'^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)\, d\vec{x}_1 \cdots d\vec{x}_n . \qquad \text{(IV.3)}$$

If we normalize a vector $f \in \mathcal{F}$ so that $\langle f, f \rangle_{\mathcal{F}} = 1$, then we can interpret $\left\langle f^n, f^{(n)} \right\rangle_{\mathcal{H}_n}$ as the probability p_n of f containing n particles, with $\sum_{n=0}^{\infty} p_n = 1$.

In the case this is a bosonic Fock space $\mathcal{F}^b$, then we require

$$\mathcal{F}_n = \mathcal{H}_n^b = \otimes_s^n \mathcal{H}_1 = \underbrace{\mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1}_{\text{n factors}} , \qquad \text{(IV.4)}$$

to be the n^{th} symmetric tensor power of the one-particle space $\mathcal{H}_1$. Then the function $f^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)$ is totally symmetric in the interchange of the vectors $\vec{x}_j$, namely

$$f^{(n)}(\vec{x}_1, \ldots, \vec{x}_n) = f^{(n)}(\vec{x}_{\pi_1}, \ldots, \vec{x}_{\pi_n}) , \qquad \text{(IV.5)}$$

where π is a permutation $(\pi_1, \ldots, \pi_n)$ of $(1, \ldots, n)$.

IV.2 Operators Preserving Particle Number

Given an operator T on the one-particle space $\mathcal{H}_1$, there are two simple ways to obtain an operator on Fock space $\mathcal{F}$. In both of these simple cases the resulting operator maps $\mathcal{H}_n$ into $\mathcal{H}_n$ for each n. One can say that such operators are *diagonal in the particles number.* Our first three examples illustrate these methods. These examples also map the bosonic n-particle states $\mathcal{F}_n^b$ into bosonic states. We call these two methods the multiplicative or additive "second quantization" of T.

Example 1. Multiplicative "Second Quantization" Given T a linear transformation on $\mathcal{H}_1$, define its multiplicative quantization $\Gamma(T) : \mathcal{H}_n \mapsto \mathcal{H}_n$ as:

$$\Gamma(T)\lceil\mathcal{H}_0 = I \,, \qquad \text{and } \Gamma(T)\lceil\mathcal{H}_n = \underbrace{T\otimes\cdots\otimes T}_{n \text{ times}} \,, \qquad \text{for } n \geq 1 \,. \tag{IV.6}$$

Then for arbitrary vectors in $\mathcal{F}_n$ one has

$$\Gamma(T)\,(f_1\otimes\cdots\otimes f_n) = Tf_1\otimes\cdots\otimes Tf_n \,, \tag{IV.7}$$

and also on vectors in $\mathcal{F}_n^b$,

$$\Gamma(T)\,(f_1\otimes_s\cdots\otimes_s f_n) = Tf_1\otimes_s\cdots\otimes_s Tf_n \,. \tag{IV.8}$$

Some examples of operators on $\mathcal{F}$ obtained in this way from operators on the one-particle space $\mathcal{F}_1 = \mathcal{H}_1$ are

$$\begin{aligned} \Gamma(I) &= I \,, \\ \Gamma(U(a,\Lambda)_1) &= U(a,\Lambda) \,, \\ \Gamma(e^{-t}) &= e^{-tN} \,, \text{ defining the number of particles operator } N \,, \\ \Gamma(e^{it\omega}) &= e^{itH_0} \text{ defining the free Hamiltonian } \; H_0 \,, \\ \Gamma(e^{-i\vec{a}\cdot\vec{P}}) &= e^{-i\vec{a}\cdot\vec{P}} \text{ defining the momentum operator } \vec{P} \,. \end{aligned} \tag{IV.9}$$

In particular

$$U(a,\Lambda)\Omega = \Omega \,, \tag{IV.10}$$

where Ω is the Fock vacuum vector $\Omega = \{1, 0, 0, \ldots,\}$.

Example 2. Additive "Second Quantization" The exponential representation for the multiplicative second quantization gives rise to the second variant of the construction, namely *additive second* quantization. Given the operator S on the one-particle space $\mathcal{H}_1$ one can define the operator $d\Gamma(S)$ on $\mathcal{H}_n$ as

$$d\Gamma(S)\lceil\mathcal{H}_0 = 0 \,, \qquad \text{and } d\Gamma(S)\lceil\mathcal{H}_n = \underbrace{S\otimes\cdots\otimes I + \cdots + I\otimes\cdots\otimes S}_{n \text{ terms}} \,, \qquad \text{for } n \geq 1 \,. \tag{IV.11}$$

For example,

$$\begin{aligned} d\Gamma(I) &= N \ , \\ d\Gamma(\omega) &= H_0 \ , \\ d\Gamma(\vec{P}) &= \vec{P} \ . \end{aligned} \tag{IV.12}$$

In the last case $\vec{P}$ denotes the total 3-momentum on $\mathcal{H}_n$. This method is appropriate for physical quantities that are additive.

Example 3. The Relation between Examples 1 and 2 Suppose that T can be diagonalized (this is true if T is self-adjoint or unitary) and that 0 is not in the spectrum (e.g. an eigenvalue) of T. Then $S = \ln T$ exists, and one can write

$$T = e^S \ . \tag{IV.13}$$

Also

$$T^t = e^{tS} \ , \qquad \text{for } t \text{ real.} \tag{IV.14}$$

One also says that the operator S is the infinitesimal generator of T^t. In other words,

$$S = \left. \frac{d}{dt} T^t \right|_{t=0} \ . \tag{IV.15}$$

The values in the following table are all cases of Examples 1–2 above:

$T = e^S$	S	$\Gamma(T)$	$d\Gamma(S)$	Comment
I	0	I	0	
e	I	e^N	N	The Number Operator
e^{-s}	$-s$	e^{-sN}	$-sN$	
$e^{is\omega}$	$is\omega$	e^{isH_0}	isH_0	Free Hamiltonian
$e^{-i\vec{a}\cdot\vec{P}}$	$-i\vec{a}\cdot\vec{P}$	$e^{-i\vec{a}\cdot\vec{P}}$	$-i\vec{a}\cdot\vec{P}$	Total Momentum

IV.3 Creation and Annihilation Operators

Of course, one is also interested in operators that act on Fock space between spaces with different numbers of particles. The simplest example of an operator that changes the particle number is the creation operator that maps a state in $\mathcal{H}_n$ with exactly n particles into a state in $\mathcal{H}_{n+1}$. It is natural to call such a state a *creation operator*. One usually denotes the creation operator that creates a particle with the wave-function f by $a^*(f)$. Its adjoint decreases the number of particles and is an *annihilation number*.

Example 4. Creation Operators It is sufficient to define the creation operator on a basis set in Fock space, and conventionally one normalizes this operator so that acting on the tensor product bosonic n-particle state $f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n$ defined in (III.1) it produces a tensor product state with one additional wave function,

$$\boxed{a^*(f)\,(f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n) = f \otimes_s f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n} \ . \tag{IV.16}$$

Thus one could also write

$$\boxed{a^*(f) = f\otimes_s} \; . \tag{IV.17}$$

Translating into the Dirac notation, one finds

$$\boxed{a^*(f)|f_1, \ldots, f_n\rangle\!\rangle \; = |f, f_1, \ldots, f_n\rangle\!\rangle} \; . \tag{IV.18}$$

One can express the creation operator in terms of a density $a^*(\vec{x}\,)$, and one writes

$$a^*(f) = \int a^*(\vec{x}\,)\, f(\vec{x}\,)\, d\vec{x} \; . \tag{IV.19}$$

Similarly one can express the creation operator in terms of a momentum-space wave packet $\tilde{f}(\vec{k}\,)$ and the basis $|\vec{k}\,\rangle$, and one finds that

$$a^*(f) = \tilde{a}^*(\tilde{f}) = \int \tilde{a}^*(\vec{k}\,)\, \tilde{f}(\vec{k}\,)\, d\vec{k} \; , \tag{IV.20}$$

meaning that

$$\tilde{a}^*(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} \int a^*(\vec{x}\,) e^{i\vec{k}\cdot\vec{x}}\, d\vec{x} \; , \qquad \text{and} \;\; a^*(\vec{x}\,) = \frac{1}{(2\pi)^{3/2}} \int a^*(\vec{k}\,) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} \; , \tag{IV.21}$$

as well as the relations for the adjoints

$$\tilde{a}(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} \int a(\vec{x}\,) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} \; , \qquad \text{and} \;\; a(\vec{x}\,) = \frac{1}{(2\pi)^{3/2}} \int a(-\vec{k}\,) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} \; . \tag{IV.22}$$

In a fashion similar to the derivation of (I.46), one can show that the ket vectors $|f_1, \ldots, f_n\rangle\!\rangle$ and $|\tilde{f}_1, \ldots, \tilde{f}_n\rangle$ just amount to expressing one vector in two different bases sets, so

$$\boxed{|f_1, \ldots, f_n\rangle\!\rangle \; = |\tilde{f}_1, \ldots, \tilde{f}_n\rangle} \; . \tag{IV.23}$$

Thus

$$\boxed{a^*(f)\, |\tilde{f}_1, \ldots, \tilde{f}_n\rangle = \tilde{a}^*(\tilde{f})\, |\tilde{f}_1, \ldots, \tilde{f}_n\rangle = |\tilde{f}, \tilde{f}_1, \ldots, \tilde{f}_n\rangle} \; . \tag{IV.24}$$

It is easy to derive the transformation law for the creation operators under the action of the Poincaré group. As this group acts on the one-particle states according to the representation $U(b, \Lambda)$, and on the n-particle states according to a tensor product of this representation that we also denote by $U(b, \Lambda)$. On the one hand, we have

$$\begin{aligned} U(b,\Lambda)\, a^*(f)\, |f_1, \ldots, f_n\rangle\!\rangle &= U(b,\Lambda)\, |f, f_1, \ldots, f_n\rangle\!\rangle \\ &= |U(b,\Lambda)f, U(b,\Lambda)f_1, \ldots, U(b,\Lambda)f_n\rangle\!\rangle \\ &= a^*(U(b,\Lambda)f)\, |U(b,\Lambda)f_1, \ldots, U(b,\Lambda)f_n\rangle\!\rangle \; , \end{aligned} \tag{IV.25}$$

while on the other

$$\begin{aligned} U(b,\Lambda)a^*(f)\, |f_1, \ldots, f_n\rangle\!\rangle &= U(b,\Lambda)a^*(f)U(b,\Lambda)^*\, U(b,\Lambda)\, |f_1, \ldots, f_n\rangle\!\rangle \\ &= U(b,\Lambda)\, a^*(f)\, U(b,\Lambda)^*\, |U(b,\Lambda)f_1, \ldots, U(b,\Lambda)f_n\rangle\!\rangle \; . \end{aligned} \tag{IV.26}$$

Comparing (IV.25)–(IV.26), we conclude that

$$\boxed{U(b,\Lambda)\, a^*(f)\, U(b,\Lambda)^* = a^*(U(b,\Lambda)f)} \; . \tag{IV.27}$$

Example 5. Annihilation Operators The annihilation operator

$$a(f) = (a^*(f))^* = \int a(\vec{x})\overline{f(\vec{x})}d\vec{x} \ , \tag{IV.28}$$

is defined as the adjoint of the creation operator $a^*(f)$. Defined in this way, the annihilation operator anti-linear in f, rather than linear. We claim that for $f^{(n+1)} \in \mathcal{F}^b_{n+1}$, it is the case that $a(f)f^{(n+1)} \in \mathcal{F}^b_n$ and

$$\left(a(f)f^{(n+1)}\right)(\vec{x}_1, \ldots, \vec{x}_n) = \sqrt{n+1}\int \overline{f(\vec{x})}\, f^{(n+1)}(\vec{x}, \vec{x}_1, \ldots, \vec{x}_n)\, d\vec{x} \ . \tag{IV.29}$$

Also $a(f)\Omega = 0$, where $\Omega \in \mathcal{F}_0$ denotes the zero-particle vacuum state $\Omega = \{1, 0, 0, \ldots\}$. This form for the annihilation operator is equivalent to the following action on a bosonic tensor product wave function $f_1 \otimes_s \cdots \otimes_s f_{n+1} = |f_1, \ldots, f_{n+1}\rangle\!\rangle$, namely

$$\boxed{a(f)|f_1, \ldots, f_{n+1}\rangle\!\rangle \ = \sum_{j=1}^{n+1} \langle f, f_j\rangle\, |f_1, \ldots, \not{f}_j, \ldots, f_{n+1}\rangle\!\rangle} \ . \tag{IV.30}$$

Here $\not{f}_j$ denotes the omission of the one-particle wave function f_j.

We compute exactly what the adjoint does on a vector by using the general definition for the adjoint of a linear transformation T. For arbitrary vectors F, G, the adjoint satisfies

$$\langle F, T^*G\rangle = \langle TF, G\rangle \ . \tag{IV.31}$$

It is sufficient to compute the adjoint of $T = a^*(f)$ on $\mathcal{H}$ by choosing G to be of the form $f'_1 \otimes_s \ldots \otimes_s f'_n \in \mathcal{H}_n$, for arbitrary $f'_j \in \mathcal{H}_1$ and for arbitrary n. Since $TG \in \mathcal{H}_{n+1}$, the inner product vanishes (IV.31) vanishes unless $G \in \mathcal{H}_{n+1}$. Thus in particular,

$$\boxed{a(f)\mathcal{H}_0 = 0} \ . \tag{IV.32}$$

In order to calculate $a(f)$ on $\mathcal{H}_{n+1}$, we take $G = f_1 \otimes_s \ldots \otimes_s f_{n+1}$. Then using the definition (III.5) of the scalar product,

$$\begin{aligned} &\langle f'_1 \otimes_s \cdots \otimes_s f'_n,\, a(f)\, f_1 \otimes_s \cdots \otimes_s f_{n+1}\rangle_{\mathcal{F}^b_n} \\ &\qquad = \ \langle f'_1 \otimes_s \cdots \otimes_s f'_n,\, (a^*(f))^*\, f_1 \otimes_s \cdots \otimes_s f_{n+1}\rangle_{\mathcal{F}^b_n} \\ &\qquad = \ \langle a^*(f)\, f'_1 \otimes_s \cdots \otimes_s f'_n, f_1 \otimes_s \cdots \otimes_s f_{n+1}\rangle_{\mathcal{F}^b_n} \\ &\qquad = \ \langle f \otimes_s f'_1 \otimes_s \cdots \otimes_s f'_n, f_1 \otimes_s \cdots \otimes_s f_{n+1}\rangle_{\mathcal{F}^b_n} \ . \end{aligned} \tag{IV.33}$$

Using the recursion relation (III.14) for the inner product,

$$\begin{aligned} &\langle f'_1 \otimes_s \cdots \otimes_s f'_n,\, a(f)\, f_1 \otimes_s \cdots \otimes_s f_{n+1}\rangle_{\mathcal{F}^b_n} \\ &\qquad = \ \sum_{j=1}^{n+1} \langle f, f_j\rangle\, \langle f'_1 \otimes_s \cdots \otimes_s f'_n, f_1 \otimes_s \cdots \not{f}_j \cdots \otimes_s f_{n+1}\rangle \\ &\qquad = \ \left\langle f'_1 \otimes_s \cdots \otimes_s f'_n, \sum_{j=1}^{n+1} \langle f, f_j\rangle\, f_1 \otimes_s \cdots \not{f}_j \cdots \otimes_s f_{n+1}\right\rangle \ . \end{aligned} \tag{IV.34}$$

Here $\not{f}_j$ means that f_j is omitted from the product. This is true for arbitrary $f'_1, \ldots, f'_n$, so

$$\boxed{a(f)\, f_1 \otimes_s \cdots \otimes_s f_{n+1} = \sum_{j=1}^{n+1} \langle f, f_j \rangle\, f_1 \otimes_s \cdots \not{f}_j \cdots \otimes_s f_{n+1}}\,, \tag{IV.35}$$

which is the tensor-product notation for (IV.30).

IV.4 The Canonical Commutation Relations

The canonical commutation relations (CCR) for bosonic creation and annihilation operators can be written,

$$\boxed{[a(f), a^*(f')] = \langle f, f' \rangle}\,. \tag{IV.36}$$

In terms of the densities $a(\vec{x}\,)$, this means that

$$\left[a(\vec{x}\,), a^*(\vec{x}\,')\right] = \delta^3(\vec{x} - \vec{x}\,')\,. \tag{IV.37}$$

In momentum space, the corresponding relations are

$$\left[a(\tilde{f}), a^*(\tilde{f}')\right] = \left\langle \tilde{f}, \tilde{f}' \right\rangle\,, \qquad \text{and} \quad \left[a(\vec{k}\,), a^*(\vec{k}\,')\right] = \delta^3(\vec{k} - \vec{k}\,')\,. \tag{IV.38}$$

Here we have omitted the tilde ˜ from the creation and annihilation operators $\tilde{a}(\vec{k}\,), \tilde{a}^*(\vec{k}\,)$. This shorthand is common, and we do not think it causes confusion.

Before checking these relations, let us remark that if we choose an ortho-normal basis e_i for $\mathcal{H}_1$, then the relations have a simple form in terms of the creation and annihilation operators

$$a_j^* = a^*(e_j)\,, \qquad \text{and their adjoints} \quad a_j = \left(a_j^*\right)^*\,. \tag{IV.39}$$

Then these operators satisfy the canonical commutation relations

$$\boxed{\left[a_i, a_j^*\right] = \langle e_i, e_j \rangle = \delta_{ij}}\,, \tag{IV.40}$$

which are familiar commutation relations in non-relativistic quantum theory. In fact, one can expand f in the basis e_j, namely $f = \sum_j \langle e_j, f \rangle\, e_j$ yielding $[a(f), a^*(f')] = \sum_{j,j'} \langle f, e_j \rangle \langle e_j, f' \rangle = \langle f, f' \rangle$, so (IV.36) holds as a consequence of (IV.40).

We now check that (IV.40) holds. Take the ortho-normal basis for $\mathcal{F}^b$ composed of vectors of the form $\Omega_{\mathbf{n}} \in \mathcal{F}_n^b$ given in (III.21). Here n is arbitrary. Using (IV.16) and (IV.35) one calculates

$$a_i a_j^* \Omega_{\mathbf{n}} = \sqrt{(n_i + \delta_{ij})(n_j + 1)}\, \Omega_{\mathbf{n}'}\,, \ \text{ and } \ a_j^* a_i \Omega_{\mathbf{n}} = \sqrt{(n_j + 1 - \delta_{ij}) n_i}\, \Omega_{\mathbf{n}'}\,, \tag{IV.41}$$

where in both cases

$$\text{where} \ \ n'_\ell = n_\ell + \delta_{j\ell} - \delta_{i\ell}\,. \tag{IV.42}$$

Note that $|\mathbf{n}'| = |\mathbf{n}| = n$, and if also $i = j$, then $\mathbf{n} = \mathbf{n}'$. In either case, $\Omega_{\mathbf{n}'} \in \mathcal{H}_n$. Furthermore

$$\sqrt{(n_i + \delta_{ij})(n_j + 1)} - \sqrt{(n_j + 1 - \delta_{ij}) n_i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}\,. \tag{IV.43}$$

Therefore

$$\left[a_i, a_j^*\right] \Omega_{\mathbf{n}} = \delta_{ij} \Omega_{\mathbf{n}'} = \delta_{ij} \Omega_{\mathbf{n}} \ . \tag{IV.44}$$

As a consequence, the operator identity (IV.40) holds applied to any basis element for $\mathcal{F}$, and hence it holds as claimed.

In order to illustrate this argument completely, we also give a second derivation of the canonical commutation relations, in particular (IV.38). We use the momentum space definition of Fock space, which by Fourier transformation is unitarily equivalent to the configuration space definition. In this case we use Dirac notation. In this case,

$$a^*(\vec{k}\,)|\vec{k}_1, \ldots, \vec{k}_n\rangle = |\vec{k}, \vec{k}_1, \ldots, \vec{k}_n\rangle \ . \tag{IV.45}$$

One calculates the adjoint $a(\vec{k}\,)$ by computing its action on vectors $|\vec{k}_1, \ldots, \vec{k}_{n+1}\rangle$. This is given uniquely by the matrix elements

$$\begin{aligned} \langle \vec{k}'_1, \ldots, \vec{k}'_n | \, a(\vec{k}\,) | \vec{k}_1, \ldots, \vec{k}_{n+1}\rangle &= \langle \vec{k}, \vec{k}'_1, \ldots, \vec{k}'_n | \vec{k}_1, \ldots, \vec{k}_{n+1}\rangle \\ &= \sum_{j=1}^{n+1} \delta\left(\vec{k} - \vec{k}_j\right) \langle \vec{k}'_1, \ldots, \vec{k}'_{n+1} | \vec{k}_1, \ldots, \not{\vec{k}}_j, \ldots, \vec{k}_{n+1}\rangle \ . \end{aligned} \tag{IV.46}$$

To derive the last equality, we use (III.16). Therefore, we infer that the formula for the annihilation operator is

$$\boxed{a(\vec{k}\,)|\vec{k}_1, \ldots, \vec{k}_{n+1}\rangle = \sum_{j=1}^{n+1} \delta\left(\vec{k} - \vec{k}_j\right) |\vec{k}_1, \ldots, \not{\vec{k}}_j, \ldots, \vec{k}_{n+1}\rangle} \ . \tag{IV.47}$$

We can now calculate the commutation relations. Both $a(\vec{k}\,)a^*(\vec{k}'\,)$ and $a^*(\vec{k}'\,)a(\vec{k}\,)$ preserve the total particle number N. Then using (IV.47),

$$\begin{aligned} a(\vec{k}\,)a^*(\vec{k}'\,)|\vec{k}_1, \ldots, \vec{k}_n\rangle &= a(\vec{k}\,)|\vec{k}', \vec{k}_1, \ldots, \vec{k}_n\rangle \\ &= \delta(\vec{k} - \vec{k}'\,)|\vec{k}_1, \ldots, \vec{k}_n\rangle + \sum_{j=1}^{n} \delta(\vec{k} - \vec{k}_j)|\vec{k}', \vec{k}_1, \ldots, \not{\vec{k}}_j, \ldots, \vec{k}_n\rangle \ . \end{aligned} \tag{IV.48}$$

On the other hand,

$$\begin{aligned} a^*(\vec{k}'\,)a(\vec{k}\,)|\vec{k}_1, \ldots, \vec{k}_n\rangle &= \sum_{j=1}^{n} \delta(\vec{k} - \vec{k}_j) a^*(\vec{k}'\,)|\vec{k}', \vec{k}_1, \ldots, \not{\vec{k}}_j, \ldots, \vec{k}_n\rangle \\ &= \sum_{j=1}^{n} \delta(\vec{k} - \vec{k}_j)|\vec{k}', \vec{k}_1, \ldots, \not{\vec{k}}_j, \ldots, \vec{k}_n\rangle \ . \end{aligned} \tag{IV.49}$$

Therefore

$$\left[a(\vec{k}\,), a^*(\vec{k}'\,)\right] |\vec{k}_1, \ldots, \vec{k}_n\rangle = \delta(\vec{k} - \vec{k}'\,)|\vec{k}_1, \ldots, \vec{k}_n\rangle \ . \tag{IV.50}$$

V The Free Quantum Field

One defines the real, time-zero quantum field $\varphi(\vec{x}\,)$, averaged with a configuration space wave function $f(\vec{x}\,)$ as

$$\begin{aligned}\varphi(f) &= \int \varphi(\vec{x}\,)\, f(\vec{x}\,)\, d\vec{x} = \int \tilde{\varphi}(\vec{k}\,)^{*} \tilde{f}(\vec{k}\,)\, d\vec{k} \\ &= a^{*}\left((2\omega)^{-1/2} f\right) + a\left((2\omega)^{-1/2} f\right) .\end{aligned} \tag{V.1}$$

Here ω is the energy operator (I.48) on the space of one-particle wave functions, and the factor $\omega^{-1/2}$ ensures that the field is Lorentz covariant. The field at time t is

$$\begin{aligned}\varphi(f,t) &= e^{itH_0}\varphi(f)e^{-itH_0} \\ &= a^{*}\left((2\omega)^{-1/2}\, e^{it\omega} f\right) + a\left((2\omega)^{-1/2}\, e^{-it\omega} f\right) .\end{aligned} \tag{V.2}$$

Taking $f(\vec{x}\,') = \delta_{\vec{x}}(\vec{x}\,')$ one has $\tilde{f}(\vec{k}\,) = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}}$,

$$\boxed{\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{\sqrt{2\omega(\vec{k}\,)}} \left(\tilde{a}^{*}(\vec{k}\,)e^{it\omega(\vec{k}\,)} + \tilde{a}(-\vec{k}\,)e^{-it\omega(\vec{k}\,)}\right) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k}} \, . \tag{V.3}$$

This field is defined to be a solution to the Klein-Gordon equation

$$\left(\Box + m^2\right)\varphi(x) = 0 . \tag{V.4}$$

As a second-order differential equation, the initial value together with the initial time derivative specifies the solution at all times. The initial value is

$$\varphi(\vec{x}\,) = \varphi(\vec{x}\,,0) = \frac{1}{(2\pi)^3} \int \left(2\omega(\vec{k}\,)\right)^{-1/2} \left(\tilde{a}^{*}(\vec{k}\,) + \tilde{a}(-\vec{k}\,)\right) e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} \ , \tag{V.5}$$

while the time derivative of $\varphi(x)$ is

$$\pi(x) = \frac{\partial\varphi(x)}{\partial t} = \frac{i}{(2\pi)^3} \int \left(\frac{\omega(\vec{k}\,)}{2}\right)^{1/2} \left(\tilde{a}^{*}(\vec{k}\,)e^{it\omega(\vec{k}\,)} - \tilde{a}(-\vec{k}\,)e^{-it\omega(\vec{k}\,)}\right)e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} \ , \tag{V.6}$$

with initial value

$$\pi(\vec{x}\,) = \frac{\partial\varphi(x)}{\partial t} = \frac{i}{(2\pi)^3} \int \left(\frac{\omega(\vec{k}\,)}{2}\right)^{1/2} \left(\tilde{a}^{*}(\vec{k}\,) - \tilde{a}(-\vec{k}\,)\right)e^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} \ , \tag{V.7}$$

When applied to a state with exactly n particles, this field creates a new particle with a wave packet $(2\omega)^{-1/2}\, e^{it\omega} f$, and it annihilates one particle with the wave packet $(2\omega)^{-1/2}\, e^{it\omega}\,\bar{f}$. In case that f is a real wave packet ($f = \bar{f}$), the field operator is self-adjoint. Note that with $\Omega = \{1, 0, \ldots\}$ the zero-particle state (vacuum) in $\mathcal{F}^b$, then

$$\varphi(f,t)\Omega = \{0, (2\omega)^{-1/2}e^{it\omega}f, \ldots, \} , \tag{V.8}$$

is the one-particle state with wave function $(2\omega)^{-1/2}f$, propagated to time t by $U(t) = e^{it\omega}$ acting on the one-particle space.

V.1 Poincaré Covariance

The scalar field transforms under Lorentz transformations Λ and space-time translations according to the scalar transformation law,

$$\boxed{U(a,\Lambda)\,\varphi(x)\,U(a,\Lambda)^* = \varphi(\Lambda x + a)}\ . \tag{V.9}$$

There are other standard ways of writing this: for example, if one denotes the transformed field as φ' and the transformed space-time point as x' one could write $\varphi'(x) = \varphi(x')$. Also, we often average the field $\varphi(x)$ with a *space-time* wave packet $g(x)$ and define

$$\varphi(g) = \int \varphi(x)\, g(x)\, d^4x\ , \tag{V.10}$$

where one takes $g(x)$ to be a well-behaved function on four-dimensional Minkowski space. Then one has

$$\boxed{U(a,\Lambda)\,\varphi(g)\,U(a,\Lambda)^* = \varphi(g')}\ , \qquad \text{where } \boxed{g'(x) = g\left(\Lambda^{-1}\,(x-a)\right)}\ . \tag{V.11}$$

One can regard g as a family of wave-packets $f^{(t)} \in \mathcal{H}_1$ depending on the time parameter, so that $g(\vec{x}\,,t) = f^{(t)}(\vec{x}\,)$. In any case, we obtain the simple transformation law (V.9) by considering the action of the Poincaré group on space-time, rather than on the wave functions that depend only on the spatial variable.

Furthermore, the Poincaré transformation $U(a,\Lambda)$ leaves the zero-particle state Ω invariant, $U(a,\Lambda)\Omega = \Omega$. Therefore we claim that the vacuum-expectation of the product of two fields,

$$W_2(x;x') = W_2(x-x') = \langle \Omega, \varphi(x)\varphi(x')\Omega\rangle\ , \tag{V.12}$$

is a function of the difference vector $x - x'$, and unlike the propagator (I.103), it is a Lorentz scalar! We establish this by studying

$$\begin{aligned}\langle \Omega, \varphi(x)\varphi(x')\Omega\rangle &= \langle U(a,\Lambda)\Omega, U(a,\Lambda)\varphi(x)\varphi(x')\Omega\rangle \\ &= \langle \Omega, U(a,\Lambda)\varphi(x)U(a,\Lambda)^*\,U(a,\Lambda)\varphi(x')U(a,\Lambda)^*\,U(a,\Lambda)\Omega\rangle \\ &= \langle \Omega, \varphi(\Lambda x + a)\varphi(\Lambda x' + a)\Omega\rangle\ .\end{aligned} \tag{V.13}$$

The special case $\Lambda = I$ and $a = -x'$ shows that $\langle \Omega, \varphi(x)\varphi(x')\Omega\rangle = \langle \Omega, \varphi(x-x')\varphi(0)\Omega\rangle$, so taking $W_2(x;x')$ is a function $W_2(x-x')$ of the difference variable. Taking a general Λ shows that

$$\boxed{W_2(\Lambda x) = W_2(x)}\ , \tag{V.14}$$

so furthermore W_2 is a Lorentz-scalar function of the difference 4-vector variable x.

In fact, the vacuum expectation values (VEVs or Wightman functions) of arbitrary products of fields turn out to be a very useful tool. They are defined as

$$W_n(x_1,\ldots,x_n) = \langle \Omega, \varphi(x_1)\cdots\varphi(x_n)\Omega\rangle\ . \tag{V.15}$$

These functions are Lorentz-invariant functions of $n-1$ difference vectors,

$$W_n(\Lambda x_1 + a,\ldots,\Lambda x_n + a) = W_n(x_1,\ldots,x_n)\ . \tag{V.16}$$

Such invariant functions are functions of the various Minkowski scalar products $\xi_i \cdot \xi_j$ of the difference variables $\xi_i = x_i - x_{i+1}$, with $i,j = 1,\ldots,n-1$. In the case of the free scalar field, the odd Wightman functions vanish, $W_{2n+1} = 0$. The special case $n = 2$ is the function $W_2(x-x')$ defined in (V.12) of a single difference variable.

V.2 Canonical Commutation Relations for the Fields

The initial data for the field are independent, in the sense that they satisfy *canonical, equal-time commutation relations* at equal times. The equal-time values of the field $\varphi(f,t)$ commute with one-another. For any real f, f', such that $\omega^{1/2}f$ is a square-integrable wave packet, it is the case that the sharp-time field $\varphi(f,t)$ and its time derivative $\pi(f,t)$ are essentially self-adjoint operators, and that for any two such functions f, f',

$$[\varphi(f,t),\varphi(f',t)] = 0 \,, \qquad \text{and} \quad [\pi(f,t),\pi(f',t)] = 0 \,. \tag{V.17}$$

For real f, with $\omega^{-1/2}f$ square-integrable, such fields $\varphi(f)$ and $\pi(f)$ are self adjoint. All the $\varphi(f)$'s can be simultaneously diagonalized; or all the $\pi(f)$'s can be simultaneously diagonalized.

The field π is canonical with respect to the field φ. At equal times the commutation relations are

$$[\pi(f,t),\varphi(f',t)] = -i\left\langle \bar{f}, f'\right\rangle_{\mathcal{H}_1} \,. \tag{V.18}$$

In terms of the fields at a point, the canoncial commutation relations are

$$[\pi(\vec{x}\,,t),\varphi(\vec{x}\,',t)] = -i\delta(\vec{x}\,-\vec{x}\,')\,, \qquad \text{while} \quad \left[\varphi(\vec{x}\,),\varphi(\vec{x}\,')\right] = 0 = \left[\pi(\vec{x}\,),\pi(\vec{x}\,')\right] \,. \tag{V.19}$$

One says that the values of the time-field are *independent*.

To check these relations, write out the commutators, each of which involves four terms. Let

$$\varphi(x) = e^{itH}\left(A^*(\vec{x}\,) + A(\vec{x}\,)\right)e^{-itH}\,, \qquad \text{where} \;\; A(\vec{x}\,) = \frac{1}{(2\pi)^{3/2}}\int (2\omega(\vec{k}\,))^{-1/2} a(-\vec{k}\,)e^{-i\vec{k}\cdot\vec{x}}\,d\vec{k} \,. \tag{V.20}$$

Then

$$\begin{aligned}
\left[\varphi(\vec{x}\,,t),\varphi(\vec{x}\,',t)\right] &= \left[A^*(\vec{x}\,)+A(\vec{x}\,), A^*(\vec{x}\,')+A(\vec{x}\,')\right] \\
&= \left[A^*(\vec{x}\,), A(\vec{x}\,')\right] + \left[A(\vec{x}\,), A^*(\vec{x}\,')\right] \\
&= \frac{1}{(2\pi)^3}\int \frac{1}{2\omega}e^{-i\vec{k}\,(\vec{x}\,-\vec{x}\,')}d\vec{k} - \frac{1}{(2\pi)^3}\int \frac{1}{2\omega}e^{i\vec{k}\,(\vec{x}\,-\vec{x}\,')}d\vec{k} \\
&= 0\,,
\end{aligned} \tag{V.21}$$

as the integral is rotationally invariant. One finds similarly that $\left[\pi(\vec{x}\,,t),\pi(\vec{x}\,',t)\right] = 0$. However,

$$\begin{aligned}
\left[\pi(\vec{x}\,,t),\varphi(\vec{x}\,',t)\right] &= \left[i\omega A^*(\vec{x}\,) - i\omega A(\vec{x}\,), A^*(\vec{x}\,')+A(\vec{x}\,')\right] \\
&= -i\frac{1}{(2\pi)^3}\int \frac{1}{2}e^{-i\vec{k}\,(\vec{x}\,-\vec{x}\,')}d\vec{k} - i\frac{1}{(2\pi)^3}\int \frac{1}{2}e^{i\vec{k}\,(\vec{x}\,-\vec{x}\,')}d\vec{k} \\
&= -i\delta(\vec{x}\,-\vec{x}\,')\,.
\end{aligned} \tag{V.22}$$

V.3 Consequences of Positive Energy

As a consequence of positive energy, $\omega \geq m$, the Wightman function $W_n(x_1, \ldots, x_n)$ has an analytic continuation as a function of the difference variables $\xi_j = x_j - x_{j+1}$. It is analytic for

$$\xi_j = \alpha_j - i\beta_j \; , \tag{V.23}$$

where α_j is a real Minkowski vector, and where β_j is a real, time-like Minkowski vector with a strictly positive time component, such as $\beta_j = (\vec{0}, s_j)$ with $s_j > 0$.

In particular, if we choose the time components of each $\alpha_j = 0$, and the space-components of each β_j to vanish, then the scalar products $\xi_j \cdot \xi_{j'}$ are all negative; they are $i^2 = -1$ times the scalar product of two positive-time-like vectors. One says that such points are "Euclidean," and that W_n analytically continues to Euclidean points. At these Euclidean points of purely imaginary time, one says that the analytic continuation of the Wightman functions are *Schwinger functions.*

V.4 Local Commutation Relations

Define the commutation for the field $\varphi(x)$ as the function Δ, namely

$$\Delta(x; x') = [\varphi(x), \varphi(x')] = -\Delta(x'; x) \; . \tag{V.24}$$

A bosonic field is said to be *local* if $\Delta(x; x')$ vanishes whenever $x - x'$ is space-like (i.e. when x and x' cannot communicate through sending light signals to each other). In terms of φ,

$$\boxed{[\varphi(x), \varphi(x')] = 0} \; , \qquad \text{when } \boxed{(x - x')^2 < 0} \; . \tag{V.25}$$

One can average these commutation relations with a space-time wave packet $g(x)$ to give

$$\varphi(g) = \int \varphi(x)\, g(x)\, d^4x \; . \tag{V.26}$$

One can express locality in terms of the space-time averaged fields. The commutator is

$$[\varphi(g), \varphi(g')] = \Delta(g; g') = \int \Delta(x - x')\, g(x)\, g'(x')\, dx\, dx' \; . \tag{V.27}$$

One says that two domains $\mathcal{D}, \mathcal{D}'$ are space-like separated, if every point in $x \in \mathcal{D}$ is space-like separated from every point $x' \in \mathcal{D}'$. In case that g, g' vanish outside domains $\mathcal{D}, \mathcal{D}'$ that are space-like separated, then the commutator $\Delta(g; g') = 0$ for a local field φ.

We have seen that the free scalar field is local. One also requires locality for scalar fields that are not free (i.e. fields that satisfy non-linear wave equations, possibly coupling them to other fields).

The equal-time commutators of a scalar field vanishes if

$$\boxed{[\varphi(x), \varphi(x')]|_{x_0 = x_0'} = 0} \; . \tag{V.28}$$

According to the canonical commutation relations (IV.36), the equal-time commutator of the free scalar field vanishes. We have the following criteria for locality of a scalar field φ, which holds for the free scalar field.

Proposition V.1. *Any covariant scalar field whose equal-time commutators vanish is* local.

Proof. We use only two facts to study :

- The field $\varphi(x)$ is a Lorentz scalar, $U(b,\Lambda)\varphi(x)U(b,\Lambda)^* = \varphi(\Lambda x + b)$, see (V.9).
- If $(x-x')^2 < 0$, then there is a Lorentz transformation Λ that interchanges x, x'; see Proposition I.2.

From Lorentz covariance one infers that for any Lorentz transformation Λ,

$$\Delta(x;x') = U(0,\Lambda)^*\Delta(\Lambda x;\Lambda x')U(0,\Lambda) \; . \tag{V.29}$$

Assume that $x - x'$ is space-like. We know from Proposition I.2 that exists a Lorentz transformation Λ_B that brings the two points to the same time, $(\Lambda_B(x-x'))_0 = 0$. Denote $\Lambda_B x$ by y and $\Lambda_B x'$ by y'. Then from (V.28) we infer

$$\Delta(\Lambda_B x;\Lambda_B x') = \Delta(y;y') = [\varphi(\vec{y}, y_0), \varphi(\vec{y}\,', y_0)] = 0 \; . \tag{V.30}$$

From (V.29) we infer that $\Delta(x;x') = U^*\Delta(\Lambda_B x;\Lambda_B x')U = U^* 0 U = 0$, for $(x-x')^2 < 0$. □

For any local field, the fields averaged in space-like separated regions can be simultaneously diagonalized, and measured simultaneously. If g and g' are space-time wave packets that vanish outside space-time domains $\mathcal{D}$ and $\mathcal{D}'$ respectively, and if every point $x \in \mathcal{D}$ is space-like separated from every point $x' \in \mathcal{D}'$, then the space-time averaged fields (V.10) commute,

$$[\varphi(g), \varphi(g')] = 0 \; . \tag{V.31}$$

V.5 Properties of the Free-Field Two Point Wightman Function

The two-point functions are the various vacuum expectation values of the product of two fields. The simplest two-point function is the Wightman function $W_2(x-x') = \langle \Omega, \varphi(x)\varphi(x')\Omega\rangle$, which has the explicit integral representation

$$\boxed{W_2(x;x') = W_2(x-x') = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega(\vec{k}\,)} e^{-i\omega(\vec{k}\,)(x_0-x_0')+i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)} d\vec{k}} \; . \tag{V.32}$$

There are two related integral representation that make the Lorentz covariance of the two-point function explicit. The first is

$$\boxed{W_2(x-x') = \frac{1}{(2\pi)^3}\int_{k_0>0} \delta(k_M^2 - m^2)\, e^{-ik\cdot(x-x')}\, d^4k} \; , \tag{V.33}$$

where $k\cdot(x-x') = k\cdot_M(x-x')$ denotes the Minkowski scalar product. That (V.33) reduces to (V.32) can be seen from the property of the one-dimensional delta function and a real constant a,

$$\int \delta(ak_0)\, dk_0 = \frac{1}{|a|} \; . \tag{V.34}$$

In terms of a delta function whose argument vanishes at isolated zeros $k^{(i)}$ of a function $F(k)$,

$$\int \delta(F(k))\, dk_0 = \sum_i \frac{1}{|F'(k^{(i)})|} \; . \tag{V.35}$$

Here $F'(k^{(i)})$ denotes the $\partial F(k)/\partial k_0$ and $\left|F'(k^{(i)})\right|^{-1}$ arises as the Jacobian of the change of variable of integration from dk_0 to $dF(k)$. The summation ranges over the isolated zeros $k^{(i)}$ of $F(k)$. In the case $F(k) = k_M^2 - m^2 = k_0^2 - \omega(\vec{k}\,)^2$, the function $F(k)$ has two isolated zeros at $k_0 = \pm\omega(\vec{k}\,)$, and at these points $F'(k^{(i)}) = 2k_0|_{k^{(i)}}$. Thus $k^{(i)} = (\pm k_0, \omega(\vec{k}\,))$. The restriction of the integral (V.33) to $k_0 > 0$ then ensures that it equals (V.32).

A second integral representation for W_2 allows one to express it as a contour integral by extending the function of the energy variable k_0 to the complex k_0 plane. The function

$$e^{-ik\cdot(x-x')} \frac{1}{k_M^2 - m^2} = e^{-ik\cdot(x-x')} \frac{1}{(k_0 - \omega(\vec{k}\,))(k_0 + \omega(\vec{k}\,))} \; , \tag{V.36}$$

extends to an analytic function with the exception of two simple poles at $k_0 = \pm\omega(\vec{k}\,)$. Let C_+ denote a closed, circular contour around the pole at $k_0 = \omega(\vec{k}\,)$ that circles the pole in a *clockwise* fashion, but does not enclose the pole at $k_0 = -\omega(\vec{k}\,)$. Then by Cauchy's integral formula, one sees that (V.32) agrees with

$$\boxed{W_2(x - x') = \frac{i}{(2\pi)^4} \int d\vec{k} \int_{C_+} dk_0 \, \frac{1}{k_M^2 - m^2} \, e^{-ik\cdot(x-x')}} \; . \tag{V.37}$$

The solution $W_2(x - x')$ to the Klein-Gordon equation has the initial values (at $x_0 = x_0'$) given by

$$\left\langle \Omega, \varphi(\vec{x}\,)\varphi(\vec{x}\,')\Omega \right\rangle = W_2(\vec{x} - \vec{x}\,', 0) = G(\vec{x} - \vec{x}\,') \; , \tag{V.38}$$

and

$$\left\langle \Omega, \pi(\vec{x}\,)\varphi(\vec{x}\,')\Omega \right\rangle = \left(\frac{\partial W_2}{\partial t}\right)(\vec{x} - \vec{x}\,', 0) = -\frac{i}{2}\delta(\vec{x} - \vec{x}\,') \; . \tag{V.39}$$

Analyticity: From the condition that the energy $\omega(\vec{k}\,)$ is positive, we infer that the Wightman function $W_2(x - x')$ has an analytic continuation to a complex half plane in the time variable. In the Fourier representation (V.32) or (V.33) this is reflected in the fact that $k_0 \geq 0$. (This is a general property that is a consequence of the assumption that the energy H is a positive operator in quantum theory, and does not depend on the fact that the field $\varphi(x)$ we are considering is "free.")

More generally the energy-momentum vector k in the Fourier transform lies in the forward cone: it is a time-like vector with positive energy,

$$k_M^2 \geq 0 \; , \qquad \text{with } \; k_0 \geq 0 \; . \tag{V.40}$$

As a consequence, $W_2(x)$ has an analytic continuation into the region $z = x - i\xi$ of complex Minkowski space, where the real part x of the four-vector z is arbitrary and the imaginary part ξ of z is a vector in the forward cone. (This includes the previous case that $\xi = (\vec{0}\,, \xi_0)$ with $\xi_0 > 0$.)

V.6 Properties of the Free-Field Commutator Function

Another two point function for the free field is the commutator Green's function. Since the field is a linear function of creation and annihilation operators, the commutator (V.24) of two fields is a function and not an operator, namely

$$\Delta(x-y) = \langle\Omega, [\varphi(x), \varphi(x')]\,\Omega\rangle = W_2(x-x') - W_2(x'-x)\ . \tag{V.41}$$

Both $\Delta(x)$ and $W(x)$ are Lorentz-invariant solutions to the massive wave equation,

$$\left(\Box_x + m^2\right)\Delta(x-x') = 0\ . \tag{V.42}$$

The solution $\Delta(x-x')$ has the initial values at $(x-x')_0 = 0$ given by

$$\Delta(\vec{x}-\vec{x}\,',0) = 0\ , \qquad \text{and} \qquad \left(\frac{\partial\Delta}{\partial x_0}\right)(\vec{x}-\vec{x}\,',0) = -i\delta(\vec{x}-\vec{x}\,')\ . \tag{V.43}$$

As the initial data vanish for x strictly space-like, so does the entire solution $\Delta(x)$. Thus we have an independent argument that for the free field

$$\boxed{[\varphi(x), \varphi(x')] = \Delta(x-x') = 0}\ , \qquad \text{for} \ \boxed{(x-x')^2 < 0}\ . \tag{V.44}$$

One can give an integral representation for the free-field commutator function $\Delta(x-x')$ analogous to (V.37), namely

$$\boxed{\Delta(x-x') = \frac{i}{(2\pi)^4}\int d\vec{k}\int_C dk_0\,\frac{1}{k_M^2 - m^2}\,e^{-ik\cdot(x-x')}}\ . \tag{V.45}$$

Here C denotes a contour in the complex k_0-plane that encloses both the poles at $\pm\omega(\vec{k}\,)$ and circles them in a *clockwise direction.* The contribution from the pole at $k_0 = \omega$ was analyzed in (V.37). The corresponding contribution from a circular contour C_- that surrounds only the second pole at $k_0 = -\omega$ is

$$\begin{aligned}
\frac{i}{(2\pi)^4}\int d\vec{k}\int_{C_-} dk_0\,e^{-ik\cdot(x-x')}\frac{1}{k_M^2-m^2} &= -\frac{1}{(2\pi)^3}\int \frac{1}{2\omega(\vec{k}\,)} e^{i\omega(\vec{k}\,)(x_0-x_0')+i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}d\vec{k} \\
&= -\frac{1}{(2\pi)^3}\int \frac{1}{2\omega(\vec{k}\,)} e^{i\omega(\vec{k}\,)(x_0-x_0')-i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}d\vec{k} \\
&= -W_2(x'-x)\ .
\end{aligned} \tag{V.46}$$

The second to last equality follows from making the substitution of variables $\vec{k}\,' = -\vec{k}$ and using $\omega(\vec{k}\,) = \omega(-\vec{k}\,)$. The last equality follows by (V.32). Thus (V.45) is the sum $W_2(x-x') - W_2(x'-x)$ and equals the commutator function $\Delta(x-x')$.

The function $\Delta(x-x') = W_2(x-x') - W_2(x'-x) = \langle\Omega, [\varphi(x), \varphi(x')]\,\Omega\rangle$ satisfies the homogeneous Klein-Gordon equation; as a consequence of $(\Box + m^2)\,\varphi(x) = 0$, so

$$\left(\Box + m^2\right)\Delta(x-x') = 0\ , \tag{V.47}$$

with the initial values (V.43) above. Such a Green's function provides a solution $F(x)$ to the Klein-Gordon equation

$$\left(\Box + m^2\right) F(x) = 0 \,, \tag{V.48}$$

with initial values

$$F(\vec{x}\,,0) = \alpha(\vec{x}\,)\;, \;\text{ and }\; \partial F(\vec{x}\,,0)/\partial t = \beta(\vec{x}\,)\;. \tag{V.49}$$

Namely, one can check that

$$F(x) = i\int\left(\frac{\partial\Delta(\vec{x}\,-\vec{y}\,,t)}{\partial t}\,\alpha(\vec{y}) + \Delta(\vec{x}\,-\vec{y}\,,t)\,\beta(\vec{y})\right)d\vec{y}\;, \tag{V.50}$$

does solve this equation. Furthermore the solution is uniquely determined by the initial conditions, so this is the only such solution.

V.7 The Time-Ordered Product

The time ordered product of two fields is defined as $\varphi(x)\varphi(x')$ in case $x_0 > x_0'$, and $\varphi(x')\varphi(x)$ in case $x_0' > x_0$. Thus the time in the product is defined to increase from *right to left*. One often denotes time-ordering by the symbol T, so

$$T\varphi(x)\varphi(x') = \theta(x_0' - x_0)\,\varphi(x')\varphi(x) + \theta(x_0 - x_0')\,\varphi(x)\varphi(x')\;. \tag{V.51}$$

Note that we do not define the time-ordered product of fields for equal times.[5] Note that the time-ordered product of two fields is a symmetric function,

$$T\varphi(x)\varphi(x') = T\varphi(x')\varphi(x)\;. \tag{V.53}$$

One does not see this symmetry so clearly when expressing the time dependence of the time-ordered product, namely

$$\begin{aligned} T\varphi(x)\varphi(x') &= \theta(t'-t)\,e^{itH}\,\varphi(\vec{x}\,)\,e^{-i(t-t')H}\,\varphi(\vec{x}\,')\,e^{-it'H} \\ &\quad + \theta(t-t')\,e^{it'H}\,\varphi(\vec{x}\,')\,e^{-i(t'-t)H}\,\varphi(\vec{x}\,)\,e^{-itH}\;. \end{aligned} \tag{V.54}$$

The Feynman Propagator: The vacuum expectation value of the time-ordered product of two free fields is another important function Δ_F, sometimes called the Feynman propagator. In particular,

$$\begin{aligned} \Delta_F(x;x') &= \Delta_F(x';x) = \Delta_F(x-x') = \langle\Omega, T\varphi(x)\varphi(x')\Omega\rangle \\ &= \theta(x_0'-x_0)\,W_2(x'-x) + \theta(x_0-x_0')\,W_2(x-x')\;. \end{aligned} \tag{V.55}$$

[5] In fact, one cannot define the time-ordering in case $x = x'$, because at that point the product of fields $\varphi(x)^2$ is singular. In fact the vacuum expectation value $W_2(x-x')$ of $\varphi(x)\varphi(x')$ has no limit as $x' \to x$, as it would be

$$W_2(0) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega(\vec{k}\,)}d\vec{k} = \frac{1}{(2\pi)^2}\int_0^\infty\frac{k^2\,dk}{\omega(k)}\;, \tag{V.52}$$

which is quadratically divergent in the momentum. In general, for space-time dimension $d > 2$, this divergence is of order κ^{d-2}, where κ is the magnitude of the largest momentum.

Note that for $x_0 \neq x'_0$, the first term continues analytically to the lower-half $x_0 - x'_0$ plane in the time-difference variable. The same is true for the second term. Thus the Feynman propagator at unequal times continues to an analytic function in the lower-half time-difference plane.

One can write a complex integral representation for Δ_F, similar to the representations for $W_2(x-x')$ and $\Delta(x-x')$. In particular, let C_F denote a contour extending from $-\infty$ to $+\infty$ along the real k_0 axis; this contour avoids the pole at $k_0 = -\omega$ by making a small semicircle around the pole in the lower half plane, and it avoids the pole at $k_0 = +\omega$ by making a small semicircle around the pole in the upper half plane. We claim that

$$\boxed{\Delta_F(x-x') = \frac{i}{(2\pi)^4} \int d\vec{k} \int_{C_F} dk_0 \, \frac{1}{k_M^2 - m^2} \, e^{-ik\cdot(x-x')}} \,. \tag{V.56}$$

Here

The effect of the two small excursions into the complex k_0 plane near $k_0 = \pm\omega(\vec{k}\,)$ is unchanged if we displace the two singularities slightly off the real axis into the complex k_0 plane, and then take the limit of the answers calculated with the displaced singularities. Replacing the momentum space expression

$$\frac{1}{k_M^2 - m^2} \,, \qquad \text{by} \quad \frac{1}{k_M^2 - m^2 + i\epsilon} \,, \quad \text{with small } \epsilon > 0 \,, \tag{V.57}$$

shifts the pole at $k_0 = +\omega(\vec{k}\,)$ to $\sqrt{\omega(\vec{k}\,)^2 - i\epsilon}$, which lies close to $\omega(\vec{k}\,)$ but is in the lower half k_0-plane. Likewise it shifts the pole at $k_0 = -\omega(\vec{k}\,)$ to $-\sqrt{\omega(\vec{k}\,)^2 - i\epsilon}$ which is nearby and in the upper half k_0-plane.

The modified integral clearly converges to the original Green's function as $\epsilon \to 0$. But after introducing $\epsilon > 0$, Cauchy's theorem says that we can shift the contour C_F to integrate in the k_0-plane along the real axis, from $-\infty$ to ∞. This does not change the value of the integral. Therefore

$$\boxed{\boxed{\Delta_F(x-x') = \langle \Omega, T\varphi(x)\varphi(x')\Omega \rangle = \lim_{\epsilon\to 0+} \frac{i}{(2\pi)^4} \int \frac{1}{k_M^2 - m^2 + i\epsilon} \, e^{-ik\cdot(x-x')} \, d^4k}} \,. \tag{V.58}$$

The Feynman propagator satisfies the differential equation

$$\left(\Box + m^2\right) \Delta_F(x-x') = -i\delta^4(x-x') \,. \tag{V.59}$$

One says that the energy-momentum-space representation of the Feynman propagator is

$$\boxed{\widetilde{\Delta}_F(k) = \lim_{\epsilon\to 0+} \frac{i}{k_M^2 - m^2 + i\epsilon}} \,. \tag{V.60}$$

Here the limit $\epsilon \to 0+$ is only meant to be taken after evaluating the Fourier transform. Also by convention, we assign the entire $(2\pi)^{-4}$ to the Fourier transform of the momentum-space propagator, rather than $(2\pi)^{-2}$ to the propagator and $(2\pi)^{-2}$ to the Fourier transform in four dimensions. When we use this expression later in perturbation theory, we will reintroduce the missing powers of 2π in an appropriate way.

V.8 The Retarded Commutator

The *retarded commutator* of fields at x and x' is the commutator $[\varphi(x), \varphi(x')]$ restricted to the domain $t > t'$ where the field at x can be influenced by an event at x'. In particular, define

$$R\varphi(x)\varphi(x') = \theta(t-t')\,[\varphi(x), \varphi(x')] \ . \tag{V.61}$$

Define also the expectation of the retarded commutator as the function

$$\Delta_R(x;x') = \langle \Omega, R\varphi(x)\varphi(x')\Omega\rangle \ . \tag{V.62}$$

For the free field, $\Delta_R(x;x') = \Delta_R(x-x') = R\varphi(x)\varphi(x')$.

The retarded commutator is related to the time-ordered product,

$$R\varphi(x)\varphi(x') = T\varphi(x)\varphi(x') - \varphi(x')\varphi(x) \ . \tag{V.63}$$

Taking the vacuum expectation value of (V.63) gives for the free-field case the relation,

$$\Delta_R(x-x') = \Delta_F(x-x') - W_2(x'-x) \ . \tag{V.64}$$

The equation of motion (V.59) for Δ_F shows that

$$\boxed{\left(\Box + m^2\right)\Delta_R(x-x') = -i\delta^4(x-x')}\ , \qquad \text{and } \boxed{\Delta_R(x-x') = 0\ , \ \text{ for } t < t'} \ . \tag{V.65}$$

In other words

$$\boxed{G_R(x-x') = i\Delta_R(x-x')}\ , \tag{V.66}$$

is the ordinary retarded Green's function for the Klein-Gordon equation.

The relation (V.64) gives a complex integral representation for Δ_R. The contour $C_F \cup C_-$ in the k_0-plane can be deformed into the contour C_R along the real axis from $-\infty$ to ∞ and avoiding both poles at $\pm\omega$ with small semi-circles in the upper-half plane. Thus

$$\Delta_R(x-x') = \frac{i}{(2\pi)^4}\int d\vec{k} \int_{C_R} dk_0\, \frac{1}{k_M^2 - m^2}\, e^{-ik\cdot(x-x')} \ . \tag{V.67}$$

One can also write

$$\Delta_R(x-x') = \lim_{\epsilon\to 0+} \frac{i}{(2\pi)^4}\int \frac{1}{(k_0+i\epsilon)^2 - \vec{k}^{\,2} - m^2}\, e^{-ik\cdot(x-x')}\, d^4k \ . \tag{V.68}$$

In the second representation, both poles are displaced slightly into the lower-half k_0-plane. We can verify that this representation ensures the retarded property for Δ_R. In fact if $t < t'$, then the exponential $e^{-ik_0(t-t')}$ decays in the upper-half k_0-plane, which encloses no singularity. Closing the k_0 contour in the upper-half plane at infinity then shows that $\Delta_R(t-t')$ vanishes.

Likewise

$$\Delta_A(x-x') = \lim_{\epsilon\to 0+} \frac{i}{(2\pi)^4}\int \frac{1}{(k_0-i\epsilon)^2 - \vec{k}^{\,2} - m^2}\, e^{-ik\cdot(x-x')}\, d^4k \ , \tag{V.69}$$

is the advanced commutator function with

$$\left(\Box + m^2\right)\Delta_A(x-x') = -i\delta(x-x')\ , \qquad \text{and } \Delta_A(x-x') = 0\ , \ \text{ for } t > t' \ . \tag{V.70}$$

Also the commutator is related to the advanced and retarded functions through

$$\boxed{\Delta(x-x') = \Delta_R(x-x') - \Delta_A(x-x')} \ . \tag{V.71}$$

V.9 The Minkowski-Euclidean Correspondence

In §V.5 we saw that the Wightman function $W_2(x-x')$ analytically continues into the lower-half complex plane in the time-difference variable, namely to a negative imaginary part for $t-t'$. The time-ordered function $\Delta_F(x;x')$ also analytically continues for $t \neq t'$ and defines a symmetric function of complex x, x'. These indications suggest that the time coordinate of the field itself can be analytically continued to be purely imaginary.

As the time dependence of the field is $\varphi(\vec{x}\,,t) = e^{itH}\varphi(\vec{x}\,)e^{-itH}$, from positivity of the energy we infer that e^{itH} is continues analytically to a bounded function in the *upper*-half complex t-plane. Furthermore take any state vector in Fock space of the form $\mathbf{g} = e^{-TH}\mathbf{f}$, with $\mathbf{f} \in \mathcal{F}^b$ and $t < T$. Then the transformation e^{+tH} can be applied to such a vector $\mathbf{g}$, and $\varphi(x)\mathbf{g} = \varphi(\vec{x}\,,t)\mathbf{g}$ has an analytic continuation to imaginary time.[6]

Under this change to purely imaginary time, a real space-time vector $x_M = (\vec{x}\,,t)$ in Minkowski space continues to the complex space-time vector $z_M = (\vec{x}, it)$ with the property that the Minkowski square $x_M^2 = t^2 - \vec{x}^{\,2}$ continues to the negative of the Euclidean square,

$$x^2 = t^2 + \vec{x}^{\,2}\ . \tag{V.72}$$

We now consider the transformation $x_M \to x$, from a real point in Minkowski space to the corresponding real point in Euclidean 4-space. Under this transformation, the Lorentz transformations Λ on Minkowski space map to rotations of Euclidean 4-space, $\Lambda \to R$.

The invariance of the Minkowski square x_M^2 under Lorentz transformations in space-time continues to the invariance of the Euclidean square x^2 under rotations in 4-space. Correspondingly the invariance of the Minkowski square k_M^2 of the energy-momentum vector under Lorentz transformations continues to invariance of the Euclidean square k^2 under rotations in a real energy-momentum 4-space with Euclidean geometry.

Let us denote complex 4-vectors in Minkowski space by $z = x + iy$ and $z' = x' + iy'$. The analytic continuation for W_2 or Δ_F in question take place in the region where the difference 4-vector variable $z - z' = (x-x') + i(y-y')$ has the property that the imaginary part $y - y'$ is time-like and $(y-y')_0 < 0$. Included in this set of points of analyticity are those z, z' for which the spatial components $\vec{z}, \vec{z}\,'$ are real, and the time components iy_0, iy_0' are purely imaginary with negative time differences, namely $y_0' > y_0$. These imaginary-time points are called *Euclidean points* as $z^2 < 0$ and $z'^{\,2} < 0$.

Imaginary Time Fields: We are especially interested in the analytic continuation of the Minkowski space field to purely imaginary time. Define the *imaginary-time field*

$$\boxed{\varphi_I(x) = \varphi(\vec{x}\,,it) = e^{-tH}\varphi(\vec{x}\,)e^{tH}}\ . \tag{V.73}$$

Note that the imaginary-time fields are *not* hermitian. In terms of matrix elements,

$$\boxed{\varphi_I(\vec{x}\,,t)^* = \varphi_I(\vec{x}\,,-t)}\ , \tag{V.74}$$

meaning that $\langle \mathbf{g}, \varphi_I(\vec{x}\,,t)^*\,\mathbf{g}'\rangle_{\mathcal{F}} = \langle \varphi_I(\vec{x}\,,-t)\mathbf{g}, \mathbf{g}'\rangle_{\mathcal{F}}$, for $\mathbf{g} = e^{-TH}\mathbf{f}$, $\mathbf{g}' = e^{-TH}\mathbf{f}'$ of the form

[6] For any $\epsilon > 0$, the expression $e^{-\epsilon H}\varphi(\vec{x}\,,t)\mathbf{g}$ is a *vector* in $\mathcal{F}$ that analytically continues to the strip in the upper-half t-plane with the imaginary part of t bounded by $\Im(t) < T$.

It is natural to ask when the product of two imaginary time fields $\varphi(x)\varphi(x')$ has an analytic continuation to a product $\varphi_I(x)\varphi_I(x')$ at purely imaginary times it, it'. Writing out the product, one sees that

$$\varphi(x)\varphi(x') = e^{itH}\,\varphi(\vec{x}\,)\,e^{i(t'-t)H}\,\varphi(\vec{x}\,')e^{t'H}\ . \tag{V.75}$$

One can repeat the argument for the single field above and analytically continue this product to an analytic function

$$\varphi_I(x)\varphi_I(x') \tag{V.76}$$

in the time-ordered imaginary-time region $0 < t < t' < T$.

Therefore it is natural to also define a time-order for imaginary time fields. But in this case, it is natural to take the time-order to increase from left to right, as in our example (V.76). In order to distinguish this from we real-time ordering, we use the notation "$_+$" to designate the imaginary-time-ordered product,

$$\boxed{\left(\varphi_I(x)\varphi_I(x')\right)_+ = \theta(t'-t)\varphi_I(x)\varphi_I(x') + \theta(t-t')\varphi_I(x')\varphi_I(x)}\ . \tag{V.77}$$

The vacuum expectation value of the imaginary-time-ordered-product of two fields is called the two-point Schwinger function S_2,

$$\boxed{S_2(x-x') = \left\langle \Omega, \left(\varphi_I(x)\varphi_I(x')\right)_+ \Omega\right\rangle}\ . \tag{V.78}$$

This Schwinger function is Euclidean invariant, it is symmetric under the interchange of x, x', and in fact it equals

$$\begin{aligned} S_2(x,x') &= S_2(x-x') \\ &= \left\langle \Omega, \varphi(\vec{x}\,')e^{-|t-t'|H}\varphi(\vec{x}\,)\Omega\right\rangle \\ &= \frac{1}{(2\pi)^3}\int \frac{1}{2\omega(\vec{k}\,)} e^{-\omega(\vec{k}\,)|t-t'|+i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)} d\vec{k}\ . \end{aligned} \tag{V.79}$$

As S_2 is rotationally invariant, we can evaluate $S_2(x-x')$ in a frame where $\vec{x}-\vec{x}\,'=0$. Then the representation (V.79) shows that

$$S_2(x-x') > 0\ . \tag{V.80}$$

This positivity is a fundamental property that we later see also holds for interacting fields, and also for n-point Schwinger functions!

One can give make the Euclidean-invariance of the Schwinger function manifest, by using a representation with this property. This representation is closely tied to the 4-dimensional integral representations of the various real-time Green's functions studied above. With $k\cdot x$ the Euclidean scalar product in 4-space $\sum_{i=0}^{3} k_i x_i$, one can also write the *free*-two-point Schwinger function as

$$\boxed{\boxed{S_2(x-x') = \frac{1}{(2\pi)^4}\int \frac{1}{k^2+m^2}\, e^{ik\cdot(x-x')}d^4k}}\ . \tag{V.81}$$

In fact the *free* Schwinger function $S_2(x - x')$ is a Green's function for the Helmnolz operator

$$\left(-\nabla_x^2 + m^2\right) = \left(-\Delta + m^2\right) , \tag{V.82}$$

where $\Delta = \Delta_x$ is the 4-dimensional Laplacian[7]

$$\Delta_x = \nabla_x^2 = \sum_{i=0}^{4} \frac{\partial^2}{\partial x_i^2} . \tag{V.83}$$

By differentiating (V.79) we see that S_2 satisfies the (elliptic) Helmholz equation for a Green's function,

$$\left(-\Delta_x + m^2\right) S_2(x - x') = \delta^4(x - x') . \tag{V.84}$$

Time-Ordered Products and Schwinger Functions: One can generalize this to consider the "Euclidean" analytic continuations of the time-ordered product of n time-ordered fields to n time-ordered imaginary-time fields. (By considering the time-ordered product, we are certain to obtain a symmetric function of the Euclidean vectors $x_1, \ldots, x_n$.) Consider n Minkowski-space points $x_j = (\vec{x}_j, t_j)$ at unequal times, and let π be the permutation of $1, \ldots, n$ such that $t_{\pi_1} > \cdots > t_{\pi_n}$. Define the time-ordered product

$$\boxed{T\varphi(x_1) \cdots \varphi(x_n) = \varphi(x_{\pi_1}) \cdots \varphi(x_{\pi_n})} , \qquad \text{where in } \textit{real} \text{ time } \boxed{t_{\pi_1} > \cdots > t_{\pi_n}} . \tag{V.85}$$

These time ordered products, just as in the case $n = 2$ considered above, analytically continue to purely imaginary time for unequal times. The analytic continuation is the (imaginary)-time-ordered product that generalizes (V.77),

$$\begin{aligned} (\varphi_I(x_1)\varphi_I(x_2) \cdots \varphi_I(x_n))_+ &= \varphi_I(x_{\pi_1})\varphi_I(x_{\pi_2}) \cdots \varphi_I(x_{\pi_n}) \\ &= e^{-t_{\pi_1}H}\varphi(\vec{x}_{\pi_1})e^{-(t_{\pi_2}-t_{\pi_1})H}\varphi(\vec{x}_{\pi_2}) \cdots \varphi(\vec{x}_{\pi_n})e^{t_{\pi_n}H} . \end{aligned} \tag{V.86}$$

Unlike the imaginary parts of the real time differences that are negative, the individual imaginary times $t_1, \ldots, t_n$ are all taken to be positive. Here π is not the permutation that enters (V.85); rather in *imaginary* time π is chosen as the permutation that ensures

$$\boxed{0 < t_{\pi_1} < \cdots < t_{\pi_n}} . \tag{V.87}$$

Therefore in the imaginary-time-ordered product, it is natural for the times to increase from left to right, which is the reverse of the situation in the real-time-ordered product.

The vacuum expectation values of the time-ordered products generalize Δ_F, so we define

$$\Delta_{F,n}(x_1, \ldots, x_n) = \langle \Omega, T\varphi(x_1) \cdots \varphi(x_n)\Omega \rangle . \tag{V.88}$$

These expectations analytically also continue to purely imaginary time, giving the n-point Schwinger functions defined as

$$\boxed{S_n(x_1, \ldots, x_n) = \left\langle \Omega, (\varphi_I(x_1) \cdots \varphi_I(x_n))_+ \Omega \right\rangle = S_n(x_{\pi_1}, \ldots, x_{\pi_n})} , \tag{V.89}$$

totally symmetric under permutation of the Euclidean space-time points $x_1, \ldots, x_n$.

[7] It is standard notation to use the symbol Δ both to denote the commutator function (V.24) or the Feynman propagator Δ_F, as well as the Laplacian Δ_x in (V.83). Hopefully this will not cause confusion.