Reflection Positivity for Parafermions

Arthur Jaffe[1,2]**, Fabio L. Pedrocchi**[3,2]

[1] Harvard University, Cambridge, Massachusetts 02138, USA
    E-mail: arthur_jaffe@harvard.edu
[2] Department of Physics, University of Basel, Basel, Switzerland
[3] JARA Institute for Quantum Information, RWTH Aachen University, Aachen, Germany
    E-mail: fabio.pedrocchi@rwth-aachen.de

Dedicated to the memory of Ursula Eva Holliger-Hänggi.

Abstract: We establish reflection positivity for Gibbs trace states for a class of gauge-invariant, reflection-invariant Hamiltonians describing parafermion interactions on a lattice. We relate these results to recent work in the condensed-matter physics literature.

I. Introduction

In the early 1960's, Keijiro Yamazaki introduced a family of algebras generalizing a Clifford algebra.[1] These algebras are characterized by a primitive n^{th} root of unity $\omega = e^{2\pi i/n}$, and generators c_j, where $j = 1, 2, \ldots, L$, with each generator of order n. Alun Morris studied these algebras and showed that for even L when considered over the complex field they have an irreducible representation on a Hilbert space $\mathcal{H}$ of dimension $N = n^{L/2}$, and this is unique up to unitary equivalence [33]. Here we consider L even and c_j unitary. In the physics literature, one calls the operators c_j *a set of parafermion generators of order n* (or simply "parafermions") if they satisfy Yamazaki's relations:

$$c_j^n = I \;, \quad \text{and} \quad c_j c_{j'} = \omega \, c_{j'} c_j \;, \quad \text{for} \quad j < j' \;. \tag{I.1}$$

Consequently $c_j^* = c_j^{n-1}$, and also $c_j c_{j'} = \omega^{-1} c_{j'} c_j$ for $j > j'$. The choice $n = 2$ reduces to a self-adjoint representation of a Clifford algebra; it describes Majoranas, namely fermionic coordinates. For $n \geqslant 3$ one obtains a generic algebra of parafermionic coordinates, whose generators are not self-adjoint. Note that if $\{c_j\}$ are a set of L parafermion generators of order n, then $\{c_j^*\}$ is another set of L parafermion generators of order n.

Parafermion commutation relations appeared in both the mathematics and the physics literature, long before the definitions of the algebras cited above.

---

[1] See 1) and 2) in the middle of page 193 in §7.5 of [43].

J. J. Sylvester introduced matrices satisfying parafermion commutation relations in 1882, see [39,40]. In 1953, Herbert S. Green proposed such commutators for fields [20]. More recent examples occur in [22,16].

The relations (I.1) arise from studying representations of the braid group; a new discussion appears in [12]. Generally, representations of the braid group lead to a variety of statistics and have been the focus of intense research over the last decades, see for example [19].

Paul Fendley [14,15] gave a parafermion representation for Rodney Baxter's clock Hamiltonian and for some related spin chains [4,5,6], and discovered matrices similar to those in [39]; see our remarks in §VIII. Some further examples occur in [7,1]. Recently there has been a great deal of interest in the possibility to obtain parafermion states in one and two-dimensional model systems, see [2,10,30,42,3,32,27,28]. Two sets of authors have proposed a classification of topological and non-topological phases in parafermionic chains [34,9].

I.1. Reflection Positivity (RP). Konrad Osterwalder and Robert Schrader discovered RP for bosons and fermion fields [37], after which RP became the standard way to relate statistical physics to quantum theory, especially quantum field theory, justifying inverse Wick rotation. In condensed-matter physics RP leads to a self-adjoint transfer matrix acting on a Hilbert space. Variations of this property have been central in hundreds of subsequent papers on quantum theory and also on condensed-matter physics, especially in the study of ground states and phase transitions [18,13,8]. So RP is fundamental, and it is important to know when it holds.

Let $A \in \mathfrak{A}_-$ belong to an algebra of observables localized on one side of a reflection plane; let $\vartheta(A)$ denote the reflected observable localized on the other side of the plane. The reflection ϑ is said to have the RP-property on $\mathfrak{A}_-$ with respect to the expectation $\langle \, \cdot \, \rangle$, if always $\langle A \, \vartheta(A) \rangle \geqslant 0$.

In this paper we show that RP applies in lattice statistical mechanical systems generated by parafermions. The expectation that we study here is a trace defined with the Boltzmann weight e^{-H} for a class of Hamiltonians specified in §VI. Our Hamiltonians are not necessarily hermitian. However in case the Hamiltonian is reflection symmetric, then the partition function is automatically real and positive,

$$\mathfrak{Z} = \mathrm{Tr}(e^{-H}) > 0 \, . \tag{I.2}$$

We give our main result in Theorem 6 of §VI, where we show that the corresponding expectations of the form

$$\langle \, \cdot \, \rangle = \mathrm{Tr}(\, \cdot \, e^{-H}) \tag{I.3}$$

are RP with respect to an *algebra of observables* $\mathfrak{A}_-^n$ generated by monomials in parafermions of degree n. This paper generalizes our earlier results on the algebra of fermionic coordinates [25].

I.2. Non-Hermitian Hamiltonians. We remark that non-hermitian Hamiltonians describe the dynamics of physical systems which are not conservative. They may arise in different systems, such as those with a singular potential, or when considering conditional expectations. Although such models appear throughout

statistical physics, there has been a proliferation of recent papers that focus on non-hermitian Hamiltonians.

Interesting examples with non-hermitian Hamiltonians occur both in statistical physics; see for example [4,15], as well as in different areas of science. We mention the study of flux lines in superconducting materials [21], and the analysis of population density in biological processes [36]. Trefethen and Embree have a book on some general properties of non-normal operators, that arise from non-hermitian Hamiltonians [41].

In the complimentary direction of Euclidean quantum field theory, the action $\mathfrak{S}$ plays the role of the Hamiltonian multiplied by time or inverse temperature. In this context, expectation values defined with a reflection-positive Boltzmann weight $e^{-\mathfrak{S}}$ are the analytic continuation of physical expectation values in a quantum theory with a self-adjoint Hamiltonian. A non-hermitian action $\mathfrak{S}$ occurs not only for action functions involving fermion fields, see [38], but it can also arise for purely bosonic interactions that are not time-reflection invariant, see [23,24].

II. Basic Properties of Monomials in Parafermions

Parafermions c_j yield ordered monomials with exponents taken mod n,

$$C_{\mathfrak{J}} = c_1^{n_1} c_2^{n_2} \cdots c_L^{n_L}, \quad \text{where} \quad 0 \leqslant n_j \leqslant n - 1 . \tag{II.1}$$

Define the sequence of exponents, $\mathfrak{J} = \{n_1, \ldots, n_L\}$, and denote the total degree as

$$|\mathfrak{J}| = \sum_{j=1}^{L} n_j . \tag{II.2}$$

II.1. Algebras of Parafermions. The parafermion monomials $C_{\mathfrak{J}}$ generate an algebra that we denote $\mathfrak{A}$. Divide the L parafermions c_i into two subsets, according to whether or not $i \leqslant \frac{1}{2}L$. Define $\mathfrak{A}_-$ as the algebra generated by monomials $C_{\mathfrak{J}}$, for which $n_j = 0$ for all $j > \frac{1}{2}L$. We use the short-hand notation $\mathfrak{J} \subset \Lambda_-$ to mean that the sequence $\mathfrak{J}$ determines a monomial $C_{\mathfrak{J}} \in \mathfrak{A}_-$.

Correspondingly let $\mathfrak{A}_+$ denote the algebra generated by monomials $C_{\mathfrak{J}}$, for which $n_j = 0$ for all $j \leqslant \frac{1}{2}L$, etc. In addition, define the "order k"-parafermion subalgebras $\mathfrak{A}_\pm^k \subset \mathfrak{A}_\pm$ as follows:

$$\mathfrak{A}_\pm^k \text{ is the algebra generated by } C_{\mathfrak{J}} \in \mathfrak{A}_\pm , \text{ with } |\mathfrak{J}| = k . \tag{II.3}$$

One can add the sets indexing parafermions by setting

$$\mathfrak{J} + \mathfrak{J}' = \{n_1 + n_1', \ldots, n_L + n_L'\} . \tag{II.4}$$

Clearly there is no loss in generality to require that one takes each sum $n_j + n_j'$ mod n. Define the numbers

$$\mathfrak{J} \circ \mathfrak{J}' = \sum_{1 \leqslant j < j' \leqslant L} n_j n_{j'}' , \quad \text{and} \quad \mathfrak{J} \wedge \mathfrak{J}' = \mathfrak{J} \circ \mathfrak{J}' - \mathfrak{J}' \circ \mathfrak{J} . \tag{II.5}$$

With these definitions

$$C_{\mathfrak{I}} C_{\mathfrak{I}'} = \omega^{-\mathfrak{I} \circ \mathfrak{I}'} C_{\mathfrak{I}+\mathfrak{I}'} = \omega^{-\mathfrak{I} \wedge \mathfrak{I}'} C_{\mathfrak{I}'} C_{\mathfrak{I}} . \qquad \text{(II.6)}$$

Denote the complement of $\mathfrak{I}$ by

$$\mathfrak{I}^c = \{n - n_1, \ldots, n - n_L\} . \qquad \text{(II.7)}$$

One has

$$C_{\mathfrak{I}}^* = \omega^{-\mathfrak{I} \circ \mathfrak{I}} C_{\mathfrak{I}^c} , \quad \text{and} \quad C_{\mathfrak{I}}^* C_{\mathfrak{I}} = I = C_{\mathfrak{I}} C_{\mathfrak{I}}^* . \qquad \text{(II.8)}$$

II.2. Reflection. Define the reflection ϑ as the map

$$i \mapsto \vartheta i = L - i + 1 . \qquad \text{(II.9)}$$

Represent ϑ as an anti-linear *-automorphism of $\mathfrak{A}$, whose action on elements we denote $\vartheta(A)$. Thus

$$\vartheta(c_i) = c_{\vartheta i}^* = c_{\vartheta i}^{n-1} , \quad \text{and} \quad \vartheta(c_j c_k) = \vartheta(c_j) \vartheta(c_k) . \qquad \text{(II.10)}$$

Set $\vartheta \mathfrak{I} = \{n_L, \ldots, n_1\}$, and note that $(\vartheta \mathfrak{I})^c = \vartheta(\mathfrak{I}^c) = \vartheta \mathfrak{I}^c$. Using (II.5), one sees that

$$\vartheta(C_{\mathfrak{I}}) = \omega^{-\mathfrak{I} \circ \mathfrak{I}} C_{\vartheta \mathfrak{I}^c} . \qquad \text{(II.11)}$$

Take $\Lambda_- = \{1, 2, \ldots, L/2\}$ and $\Lambda_+ = \{L/2 + 1, \ldots, L\}$ to divide the points $\Lambda = \Lambda_- \cup \Lambda_+$ into two sets $\Lambda_\pm$ exchanged by reflection. To simplify notation, we relabel the sites in order to put sites 1 to $L/2$ on one side of the reflection plane and sites $L/2+1$ to L on the other side. Throughout the rest of this paper we use the property that parafermions c_{i_-} with $i_- \in \Lambda_-$ and c_{i_+} with $i_+ \in \Lambda_+$ satisfy $c_{i_-} c_{i_+} = \omega c_{i_+} c_{i_-}$, which follows from our assumption $i_- < i_+$. For $\mathfrak{I} \subset \Lambda_+$ and $\mathfrak{I}' \subset \Lambda_-$, one has $\mathfrak{I} \circ \mathfrak{I}' = 0$. So in this case

$$\mathfrak{I} \wedge \mathfrak{I}' = -\mathfrak{I}' \circ \mathfrak{I} = -\sum_{j,j'} n_j n_{j'}' = -|\mathfrak{I}| |\mathfrak{I}'| . \qquad \text{(II.12)}$$

II.3. Gauge Transformations. We introduce the family of local gauge automorphisms U_j defined by

$$c_j \mapsto U_{j'}(c_j) = \omega^{\delta_{jj'}} c_j , \quad \text{for} \quad j = 1, \ldots, L . \qquad \text{(II.13)}$$

Here $\delta_{jj'}$ is the Kronecker delta function. As shown in [12], this transformation can be implemented on the Hilbert space of parafermions by the unitary transformation $V_j = e^{-2\pi i N_j / n}$, where N_j is a parafermionic number operator, and $U_{j'}(c_j) = V_{j'} c_j V_{j'}^*$. The different V_j commute.

Global gauge transformations are defined by $U = \prod_{j=1}^L U_j$ and transform all parafermions by the same phase ω. Special significance is attached to the parafermion monomials that are invariant under global gauge transformations. In fact we say that the globally-gauge-invariant parafermion monomials are *observables*. We call the gauge-invariant algebra $\mathfrak{A}^n$ the *algebra of observables*.

III. Reflection Symmetry and Gauge Invariance

Here we show that certain multiples of the monomials (II.1) are both reflection-symmetric and gauge invariant. These monomials may not be hermitian. We also discuss the general form of reflection-symmetric, gauge-invariant, polynomial Hamiltonians.

Lemma 1 (Elementary Rearrangement) *For* $\mathfrak{I}_\pm \subset \Lambda_\pm$,

$$C_{\mathfrak{I}_+} \, C_{\mathfrak{I}_-} = \omega^{-|\mathfrak{I}_+||\mathfrak{I}_-|} \, C_{\mathfrak{I}_-} \, C_{\mathfrak{I}_+} \; . \tag{III.1}$$

Also for $\mathfrak{I}, \mathfrak{I}' \subset \Lambda_-$,

$$\vartheta(C_{\mathfrak{I}}) \, C_{\mathfrak{I}'} = \omega^{|\mathfrak{I}||\mathfrak{I}'|} \, C_{\mathfrak{I}'} \, \vartheta(C_{\mathfrak{I}}) \; . \tag{III.2}$$

Proof. For $\mathfrak{I}_\pm \subset \Lambda_\pm$, one has $\mathfrak{I}_+ \circ \mathfrak{I}_- = 0$. Hence

$$\mathfrak{I}_+ \wedge \mathfrak{I}_- = -\mathfrak{I}_- \circ \mathfrak{I}_+ = -|\mathfrak{I}_-| \, |\mathfrak{I}_+| \; . \tag{III.3}$$

Therefore (II.6) can be written in this case as (III.1). Also $\vartheta\mathfrak{I}^c \in \Lambda_+$, so (II.11) and (III.3) ensures

$$\vartheta(C_{\mathfrak{I}}) \, C_{\mathfrak{I}'} = \omega^{-\vartheta\mathfrak{I}^c \circ \mathfrak{I}'} \, C_{\mathfrak{I}'} \, \vartheta(C_{\mathfrak{I}}) \; . \tag{III.4}$$

But $|\vartheta\mathfrak{I}^c| = nL - |\mathfrak{I}|$, so (III.2) holds.   □

Proposition 2 *Let* $C_{\mathfrak{I}} \in \mathfrak{A}_-$ *have the form (II.1), and let*

$$X_{\mathfrak{I}} = \omega^{\frac{1}{2}|\mathfrak{I}|^2} \, C_{\mathfrak{I}} \, \vartheta(C_{\mathfrak{I}}) \; , \quad where \quad \omega = e^{\frac{2\pi i}{n}} \; . \tag{III.5}$$

Then $X_{\mathfrak{I}}$ *is both reflection invariant and globally gauge invariant. More generally for* $X_{\mathfrak{I}} = e^{i\theta} \, C_{\mathfrak{I}} \, \vartheta(C_{\mathfrak{I}})$, *the reflection-invariant combination* $X_{\mathfrak{I}} + \vartheta(X_{\mathfrak{I}})$ *is a real multiple of (III.5).*

Proof. One has

$$\vartheta(X_{\mathfrak{I}}) = \vartheta(\omega^{\frac{1}{2}|\mathfrak{I}|^2} \, C_{\mathfrak{I}} \, \vartheta(C_{\mathfrak{I}})) = \omega^{-\frac{1}{2}|\mathfrak{I}|^2} \, \vartheta(C_{\mathfrak{I}}) \, C_{\mathfrak{I}} \; . \tag{III.6}$$

Substitute the elementary rearrangement of Lemma 1 with $\mathfrak{I} = \mathfrak{I}'$ into (III.6). This entails $\vartheta(X_{\mathfrak{I}}) = X_{\mathfrak{I}}$ as claimed.

Furthermore $X_{\mathfrak{I}}$ is a globally-gauge-invariant monomial, for

$$U C_{\mathfrak{I}} U^* = \omega^{|\mathfrak{I}|} \, C_{\mathfrak{I}} \; , \quad while \quad U\vartheta(C_{\mathfrak{I}})U^* = \omega^{-|\mathfrak{I}|} \, \vartheta(C_{\mathfrak{I}}) \; . \tag{III.7}$$

As U is linear, we infer $U X_{\mathfrak{I}} U^* = X_{\mathfrak{I}}$.

The second assertion also follows, by noting that the multiple in question is $2\cos\left(\theta - \frac{1}{2}|\mathfrak{I}|^2\right)$.   □

Corollary 3 *Reflection-invariant, globally-gauge-invariant polynomials that are linear combinations of monomials (III.5) can be written as*

$$\sum_{\substack{\mathfrak{I} \subset \Lambda_- \\ |\mathfrak{I}| > 0}} (-1)^{1+|\mathfrak{I}|} \, \omega^{\frac{1}{2}|\mathfrak{I}|^2} \, J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \, C_{\mathfrak{I}} \, \vartheta(C_{\mathfrak{I}}) \; , \quad with \ real \ couplings \ J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \; . \tag{III.8}$$

III.1. Hermitian Hamiltonians. A general monomial entering the sum (III.8) is not hermitian, namely

$$Y_{\mathfrak{I}} = (-1)^{1+|\mathfrak{I}|}\, \omega^{\frac{1}{2}|\mathfrak{I}|^2}\, J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, C_{\mathfrak{I}^c}\,\vartheta(C_{\mathfrak{I}^c})\,, \tag{III.9}$$

for

$$Y_{\mathfrak{I}}^* = (-1)^{1+|\mathfrak{I}|}\, \omega^{\frac{1}{2}|\mathfrak{I}|^2}\, J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, C_{\mathfrak{I}^c}\,\vartheta(C_{\mathfrak{I}^c})\,. \tag{III.10}$$

In this equality we use (II.8) and (III.1), so the monomial $Y_{\mathfrak{I}}$ is hermitian only if $\mathfrak{I}^c = \mathfrak{I}$. This entails $n_i = \frac{1}{2}n$, for every i. A necessary condition for $Y_{\mathfrak{I}}$ to be hermitian is that n is even.

For example if $n = 2$ and $L = 2$ one can label the two sites by 1,2, with $\mathfrak{I} = \{1\}$ and $\vartheta\mathfrak{I} = \{2\}$. Then $|\mathfrak{I}| = 1$ and $\omega = -1$. The specific monomial

$$Y_{\mathfrak{I}} = iJ_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, c_1\vartheta(c_1) \tag{III.11}$$

of the form (III.9) is both reflection-symmetric and hermitian. On the other hand, a general $Y_{\mathfrak{I}}$ of the form (III.9), which may not be hermitian, always yields the polynomial $Y_{\mathfrak{I}} + Y_{\mathfrak{I}}^*$, that is both reflection symmetric and hermitian. For example, with $n = 3$ and $L = 2$, $\omega = e^{\frac{2\pi i}{3}}$, the monomial

$$Y_{\mathfrak{I}} = \omega^{\frac{1}{2}} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, c_1\vartheta(c_1) = \omega^{\frac{1}{2}} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, c_1 c_2^* = \omega^{\frac{1}{2}} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, c_1 c_2^2\,, \tag{III.12}$$

yields the reflection-symmetric, hermitian polynomial

$$Y_{\mathfrak{I}} + Y_{\mathfrak{I}}^* = \omega^{\frac{1}{2}} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, (c_1 c_2^2 + c_1^2 c_2)\,. \tag{III.13}$$

IV. A Basis for Parafermions

Let $C_{\mathfrak{I}} = c_1^{n_1}\cdots c_L^{n_L}$ be one of n^L monomials of the form (II.1), with L even. As a consequence of the results of Morris [33], one can take $C_{\mathfrak{I}}$ to act on a Hilbert space $\mathcal{H}$ of $\dim(\mathcal{H}) = n^{L/2}$.

Proposition 4 *The monomials $C_{\mathfrak{I}}$ are linearly independent, and provide a basis for the n^L linear transformations on $\mathcal{H}$. Furthermore $\mathrm{Tr}\,(C_{\mathfrak{I}}) = 0$, unless $|\mathfrak{I}| = 0$. Any linear transformation A on $\mathcal{H}$ has the decomposition*

$$A = \sum_{\mathfrak{I}} a_{\mathfrak{I}}\, C_{\mathfrak{I}}\,, \quad \text{where} \quad a_{\mathfrak{I}} = \frac{1}{n^{L/2}}\,\mathrm{Tr}\,(C_{\mathfrak{I}}^* A)\,. \tag{IV.1}$$

Proof. If $C_{\mathfrak{I}} = I$, then $\mathrm{Tr}\,(C_{\mathfrak{I}}) = \dim \mathcal{H} = n^{L/2}$. So we need only analyze $|\mathfrak{I}| > 0$. We consider two cases.

Case I: A particular c_j does not occur in $C_{\mathfrak{J}}$. Distinguish between two subcases, according to whether or not $\sum_{i<j} n_i - \sum_{i>j} n_i = 0$ mod n. If this quantity does not vanish, then cyclicity of the trace and the parafermion relations (I.1) ensure that

$$\mathrm{Tr}\,(C_{\mathfrak{J}}) = \mathrm{Tr}\left(C_{\mathfrak{J}} c_j^n\right) = \mathrm{Tr}\left(c_j C_{\mathfrak{J}} c_j^{n-1}\right)$$
$$= \omega^{\sum_{i<j} n_i - \sum_{i>j} n_i}\,\mathrm{Tr}\,(C_{\mathfrak{J}})\;.$$

The last equality is a consequence of (I.1), allowing one to move c_j to the right through $C_{\mathfrak{J}}$. As $\omega^{\sum_{i<j} n_i - \sum_{i>j} n_i} \neq 1$, we infer that $\mathrm{Tr}(C_{\mathfrak{J}}) = 0$.

On the other hand, when $\sum_{i<j} n_i - \sum_{i>j} n_i = 0 \mod n$, there exists $j' \neq j$ with $n_{j'} \neq 0 \mod n$, and also $|j - j'|$ is minimized. If $j' < j$, then

$$\mathrm{Tr}\,(C_{\mathfrak{J}}) = \mathrm{Tr}\left(C_{\mathfrak{J}} c_{j'}^n\right) = \mathrm{Tr}\left(c_{j'} C_{\mathfrak{J}} c_{j'}^{n-1}\right)$$
$$= \omega^{-n_{j'} + \left(\sum_{i<j} n_i - \sum_{i>j} n_i\right)}\,\mathrm{Tr}\,(C_{\mathfrak{J}})$$
$$= \omega^{-n_{j'}}\,\mathrm{Tr}\,(C_{\mathfrak{J}}) = 0\;.$$

In the last equality we use that $\omega^{n_{j'}} \neq 1$. If $j' > j$ the same reasoning can be followed, except $\omega^{n_{j'}}$ replaces $\omega^{-n_{j'}}$.

Case II: Every c_j occurs in $C_{\mathfrak{J}}$.. Here we have

$$n_j \in \{1, 2, \ldots, n-1\}\;, \tag{IV.2}$$

for each j. Move one of the c_j's cyclically through the trace, and back to its original position. For $j = 1$, this shows that

$$\mathrm{Tr}\,(C_{\mathfrak{J}}) = \omega^{\sum_{j=2}^{L} n_j}\,\mathrm{Tr}\,(C_{\mathfrak{J}})\;. \tag{IV.3}$$

Hence either $\mathrm{Tr}\,(C_{\mathfrak{J}}) = 0$, or else

$$\sum_{j=2}^{L} n_j = 0 \mod n\;. \tag{IV.4}$$

Likewise for $2 \leqslant j \leqslant L$, either $\mathrm{Tr}(C_{\mathfrak{J}}) = 0$, or

$$-\sum_{j=1}^{k-1} n_j + \sum_{j=k+1}^{L} n_j = 0 \mod n\;, \quad \text{for} \quad k = 2, \ldots, L-1\;, \tag{IV.5}$$

and for $k = L$,

$$\sum_{j=1}^{L-1} n_j = 0 \mod n\;. \tag{IV.6}$$

The conditions (IV.4) and (IV.5) for the case $k = 2$, show that $n_1 + n_2 = 0$ mod n. Condition (IV.2) ensures that both n_1 and n_2 are strictly greater than 0 and strictly less than n, so $n_1 + n_2 = n$.

Next subtract the condition (IV.5) for $k = 3$ from the same condition for $k = 2$. This shows that $n_2 + n_3 = 0 \mod n$, and the restriction (IV.2) ensures

that $n_2 + n_3 = n$. Continue in this fashion for $k = j + 1$ and $k = j$, in order to infer that $n_j + n_{j+1} = n$ for $j = 3, \ldots, L - 2$. Finally consider the condition (IV.6). As we have seen that $n_j + n_{j+1} = n$ for $j = 1, 3, 5, \ldots, L - 3$, we infer that $n_{L-1} = 0 \mod n$. But this is incompatible with $1 < n_{L-1} < n$ required by (IV.2). So we conclude that $\mathrm{Tr}(C_{\mathfrak{I}}) = 0$ in all cases for which $\mathfrak{I} \neq 0$.

Note that $C_{\mathfrak{I}}^* C_{\mathfrak{I}} = I$ for each $\mathfrak{I}$. Assuming that $\mathfrak{I} \neq \mathfrak{I}'$, it follows from the form (II.1) for $C_{\mathfrak{I}}$, that $C_{\mathfrak{I}'}^* C_{\mathfrak{I}} = \pm C_\gamma$ for some $\gamma \neq 0$. Suppose that there are coefficients $a_{\mathfrak{I}} \in \mathbb{C}$ such that $\sum_{\mathfrak{I}} a_{\mathfrak{I}} C_{\mathfrak{I}} = 0$. Then for any $\mathfrak{I}'$, one has $C_{\mathfrak{I}'}^* \sum_{\mathfrak{I}} a_{\mathfrak{I}} C_{\mathfrak{I}} = \sum_{\mathfrak{I}} a_{\mathfrak{I}} C_{\mathfrak{I}'}^* C_{\mathfrak{I}} = 0$. Taking the trace shows that $a_{\mathfrak{I}'} = 0$, so the $C_{\mathfrak{I}}$ are actually linearly independent. As there are n^L linearly independent matrices $C_{\mathfrak{I}}$, namely the square of the dimension of the representation space $n^{L/2}$ of parafermions, these monomials are a basis set for all matrices. Expanding an arbitrary matrix A in this basis, we calculate the coefficients in (IV.1) using $\mathrm{Tr}\, I = n^{L/2}$. $\qquad\square$

V. Primitive Reflection-Positivity

Proposition 5 *Consider an operator $A \in \mathfrak{A}_\pm$, then*

$$\mathrm{Tr}(A\,\vartheta(A)) \geqslant 0\,. \tag{V.1}$$

Proof. The operator $A \in \mathfrak{A}_\pm$ can be expanded as a polynomial in the basis $C_{\mathfrak{I}}$ of Proposition 4. One can restrict to $\mathfrak{I} \in \Lambda_\pm$, so the monomials that appear in the expansion all belong to $\mathfrak{A}_\pm$. Write

$$A = \sum_{\mathfrak{I}} a_{\mathfrak{I}}\, C_{\mathfrak{I}}\,, \qquad \text{and} \quad \vartheta(A) = \sum_{\mathfrak{I}} \overline{a_{\mathfrak{I}}}\, \vartheta(C_{\mathfrak{I}})\,. \tag{V.2}$$

With $A \in \mathfrak{A}_-$, one can take $C_{\mathfrak{I}} = c_1^{n_1} \cdots c_{L/2}^{n_{L/2}}$, so

$$\mathrm{Tr}\,(A\,\vartheta(A)) = \sum_{\mathfrak{I},\mathfrak{I}'} a_{\mathfrak{I}}\, \overline{a_{\mathfrak{I}'}}\, \mathrm{Tr}\,(C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}'}))\,. \tag{V.3}$$

Since $C_{\mathfrak{I}} \in \mathfrak{A}_-$ and $\vartheta(C_{\mathfrak{I}'}) \in \mathfrak{A}_+$, they are products of different parafermions. We infer from Proposition 4 that the trace vanishes unless $|\mathfrak{I}| = |\vartheta\mathfrak{I}'| = 0$. Then

$$\mathrm{Tr}\,(A\,\vartheta(A)) = n^{L/2}\, |a_0|^2 \geqslant 0\,, \tag{V.4}$$

as claimed. $\qquad\square$

VI. The Main Results

Fix the order n of parafermions, and consider positive-temperature states determined by a Hamiltonian H that is reflection invariant $\vartheta(H) = H$, and globally gauge invariant $UHU^* = H$. But H is not necessarily hermitian.

Assume that H has the form

$$H = H_- + H_0 + H_+\,, \tag{VI.1}$$

with $H_\pm \in \mathfrak{A}_\pm^n$ and $H_+ = \vartheta(H_-)$. Here H_0 is a sum of interactions (III.8) across the reflection plane, namely

$$H_0 = \sum_{\substack{\mathfrak{I} \subset \Lambda_- \\ |\mathfrak{I}| > 0}} (-1)^{|\mathfrak{I}|+1} \omega^{\frac{1}{2}|\mathfrak{I}|^2}\, J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, C_{\mathfrak{I}}\, \vartheta(C_{\mathfrak{I}})\,. \tag{VI.2}$$

VI.1. Assumptions on the coupling constants. For any n, our results hold if the coupling constants in (VI.2) satisfy

$$J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \geqslant 0\,, \quad \text{for all } \mathfrak{I}\,. \tag{VI.3}$$

Alternatively, for even n, our results hold if the coupling constants satisfy

$$(-1)^{|\mathfrak{I}|} J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \geqslant 0\,, \quad \text{for all } \mathfrak{I}\,. \tag{VI.4}$$

Note that we only restrict the signs of the coupling constants for those interactions that cross the reflection plane.[2] The functional

$$\mathrm{Tr}(A\,\vartheta(B)\,e^{-H})\,, \qquad \text{for} \quad A, B \in \mathfrak{A}^n_\pm\,, \tag{VI.5}$$

which is linear in A and anti-linear in B.

VI.2. Reflection Positivity on the Algebra of Observables. Here we show that a reflection-symmetric, globally-gauge-invariant Hamiltonian H has the reflection-positivity property on the algebra $\mathfrak{A}^n_\pm$ of gauge-invariant observables.

Theorem 6 *Let $A \in \mathfrak{A}^n_\pm$ and H of the form (VI.1)–(VI.4). Then the functional (VI.5) is positive on the diagonal,*

$$\mathrm{Tr}(A\,\vartheta(A)\,e^{-H}) = \mathrm{Tr}(\vartheta(A)\,A\,e^{-H}) \geqslant 0\,. \tag{VI.6}$$

In particular, the partition function $\mathrm{Tr}(e^{-H}) \geqslant 0$ is real and non-negative.

Proof. Use the Lie product formula for matrices α_1, α_2, and α_3 in the form

$$e^{\alpha_1 + \alpha_2 + \alpha_3} = \lim_{k \to \infty} \left((1 + \alpha_1/k) e^{\alpha_2/k} e^{\alpha_3/k} \right)^k\,, \tag{VI.7}$$

with $\alpha_1 = -H_0$, $\alpha_2 = -H_-$, and $\alpha_3 = -H_+$. (Such an approximation was also used in equation (2.6) of [17].) Using (VI.7), one has $e^{-H} = \lim_{k \to \infty} \left(e^{-H}\right)_k$, where

$$\left(e^{-H}\right)_k$$

$$= \left((I - \frac{1}{k} \sum_{\substack{\mathfrak{I} \subset \Lambda_- \\ |\mathfrak{I}| > 0}} (-1)^{1+|\mathfrak{I}|} \omega^{\frac{1}{2}|\mathfrak{I}|^2} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, C_\mathfrak{I}\, \vartheta(C_\mathfrak{I}))\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k} \right)^k$$

$$= \left((I + \frac{1}{k} \sum_{\substack{\mathfrak{I} \subset \Lambda_- \\ |\mathfrak{I}| > 0}} (-1)^{|\mathfrak{I}|} \omega^{\frac{1}{2}|\mathfrak{I}|^2} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\, C_\mathfrak{I}\, \vartheta(C_\mathfrak{I}))\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k} \right)^k\,. \tag{VI.8}$$

---

[2]   The conditions (VI.3)–(VI.4), taken together with our definition (VI.2) for the phase of the couplings, reduce to the conditions in our earlier work on Majoranas [25], for which $n = 2$ and $\omega = -1$. The phase in (VI.2) is $i^{2|\mathfrak{I}|+2+|\mathfrak{I}|^2} = -1, i$, corresponding to $|\mathfrak{I}|$ being even or odd, respectively. In [25] the corresponding phases were $i^{(|\mathfrak{I}| \mod 2)} = 1, i$. Thus the couplings $J_{\mathfrak{I}\,\vartheta\mathfrak{I}}$ in the present paper have the opposite sign from those in [25] for even $|\mathfrak{I}|$; they have the same sign for odd $|\mathfrak{I}|$. Bearing this in mind, the allowed interactions in the two papers agree for $n = 2$. For the case of general n, our new choice of signs simplifies the formulation of conditions (VI.3)–(VI.4).

One can include the term I in the sums in (VI.8) by defining $J_{\varnothing \, \vartheta \varnothing} = k$, and including $|\mathfrak{I}| = 0$ in the sum. Then

$$
\begin{aligned}
\left(e^{-H}\right)_k &= \frac{1}{k^k} \left(\sum_{\mathfrak{I} \subset \Lambda_-} (-1)^{|\mathfrak{I}|} \omega^{\frac{1}{2}|\mathfrak{I}|^2} J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \, C_{\mathfrak{I}} \, \vartheta(C_{\mathfrak{I}}) \, e^{-H_-/k} \, e^{-\vartheta(H_-)/k} \right)^k \\
&= \sum_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)} \subset \Lambda_-} (-1)^{\sum_{j=1}^k |\mathfrak{I}^{(j)}|} \omega^{\sum_{j=1}^k \frac{1}{2}|\mathfrak{I}^{(j)}|^2} \\
&\qquad\qquad\qquad \times \mathfrak{c}_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} \, Y_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} \; .
\end{aligned}
\tag{VI.9}
$$

In the second equality we have expanded the expression into a linear combination of terms with coefficients

$$
\mathfrak{c}_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} = \frac{1}{k^k} \prod_{j=1}^k J_{\mathfrak{I}^{(j)} \, \vartheta\mathfrak{I}^{(j)}} \; ,
\tag{VI.10}
$$

and with

$$
\begin{aligned}
Y_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} &= C_{\mathfrak{I}^{(1)}} \vartheta(C_{\mathfrak{I}^{(1)}}) \, e^{-H_-/k} \, e^{-\vartheta(H_-)/k} \cdots \\
&\qquad \times \cdots C_{\mathfrak{I}^{(k)}} \vartheta(C_{\mathfrak{I}^{(k)}}) \, e^{-H_-/k} \, e^{-\vartheta(H_-)/k} \; .
\end{aligned}
\tag{VI.11}
$$

We assume in (VI.1) that $H_- \in \mathfrak{A}_-^n$. Thus $Y_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}}$ has the form in (VI.13) with $B_j = e^{-H_-/k}$ for all j. Let

$$
D_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} = C_{\mathfrak{I}^{(1)}} \, e^{-H_-/k} \, C_{\mathfrak{I}^{(2)}} \, e^{-H_-/k} \cdots C_{\mathfrak{I}^{(k)}} \, e^{-H_-/k} \in \mathfrak{A}_- \; .
\tag{VI.12}
$$

Lemma 7 (General Rearrangement) *Let $C_{\mathfrak{I}^{(j)}} \in \mathfrak{A}_-$, and let $A, B_j \in \mathfrak{A}_-^n$, for $j = 1, \ldots, k$. Also let $D_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} = C_{\mathfrak{I}^{(1)}} B_1 \, C_{\mathfrak{I}^{(2)}} B_2 \cdots C_{\mathfrak{I}^{(k)}} B_k \in \mathfrak{A}_-$. Then*

$$
\begin{aligned}
&A\vartheta(A) C_{\mathfrak{I}^{(1)}} \vartheta(C_{\mathfrak{I}^{(1)}}) \, B_1 \vartheta(B_1) \, C_{\mathfrak{I}^{(2)}} \vartheta(C_{\mathfrak{I}^{(2)}}) \, B_2 \vartheta(B_2) \cdots C_{\mathfrak{I}^{(k)}} \vartheta(C_{\mathfrak{I}^{(k)}}) \, B_k \vartheta(B_k) \\
&\quad = \omega^{\sum_{1 \leqslant j < j' \leqslant k} |\mathfrak{I}^{(j)}| |\mathfrak{I}^{(j')}|} \, A D_{\mathfrak{I}_1, \ldots, \mathfrak{I}_k} \, \vartheta(A D_{\mathfrak{I}_1, \ldots, \mathfrak{I}_k}) \; .
\end{aligned}
\tag{VI.13}
$$

Proof. In order to establish (VI.13), rearrange the order of the factors on the left side of the identity. In doing this, one retains the relative order of A, of the various $C_{\mathfrak{I}^{(j)}}$, and of the various $B_{j'}$ that are elements of $\mathfrak{A}_-$. Likewise one retains the relative order of $\vartheta(A)$, of the various $\vartheta(C_{\mathfrak{I}^{(j)}})$ and of the various $\vartheta(B_{j'})$ that are elements of $\mathfrak{A}_+$. In this manner one obtains $A D_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}} \vartheta(A D_{\mathfrak{I}^{(1)}, \ldots, \mathfrak{I}^{(k)}})$ multiplied by some phase.

The resulting rearrangement only requires that one commutes operators in $\mathfrak{A}_+$ with operators in $\mathfrak{A}_-$. As $\vartheta(A) \in \mathfrak{A}_+^n$ and $\vartheta(B_{j'}) \in \mathfrak{A}_+^n$, each such factor commutes with every operator in $\mathfrak{A}_-$, and in particular with each $C_{\mathfrak{I}^{(j)}}$. Likewise $B_{j'} \in \mathfrak{A}_-^n$ commutes with each operator $\vartheta(C_{\mathfrak{I}^{(j)}})$. Thus one acquires a phase not equal to 1, only by moving one of the operators $\vartheta(C_{\mathfrak{I}^{(j)}}) \in \mathfrak{A}_+$ to the right, past one of the operators $C_{\mathfrak{I}^{(j')}} \in \mathfrak{A}_-$. And this is only required in case $j < j'$. Use the rearrangement identity (III.1) to perform this exchange. This phase is given by the resulting product of phases arising in the elementary moves, and it yields the phase in (VI.13). $\qquad\square$

Lemma 8 (Conservation Law) *Under the hypotheses of Lemma 7, the trace of $AD_{\mathfrak{I}_1,\ldots,\mathfrak{I}_k}\,\vartheta(AD_{\mathfrak{I}_1,\ldots,\mathfrak{I}_k})$ vanishes unless*

$$\sum_{j=1}^{k}|\mathfrak{I}^{(j)}| = 0 \mod n\,. \qquad (VI.14)$$

If (VI.14) holds, then the constants $\mathfrak{c}_{\mathfrak{I}^{(1)},\ldots,\,\mathfrak{I}^{(k)}}$ defined in (VI.10) satisfy

$$0 \leqslant \mathfrak{c}_{\mathfrak{I}^{(1)},\ldots,\,\mathfrak{I}^{(k)}}\,. \qquad (VI.15)$$

Proof. Expand $T = AD_{\mathfrak{I}_1,\ldots,\mathfrak{I}_k}$ and its reflection as a sum of monomials (IV.1),

$$T = \sum_{\widetilde{\mathfrak{I}}\subset \Lambda_-} a_{\widetilde{\mathfrak{I}}}C_{\widetilde{\mathfrak{I}}}\,, \quad \text{and} \quad \vartheta(T) = \sum_{\widetilde{\mathfrak{I}}'\subset \Lambda_-} \overline{a_{\widetilde{\mathfrak{I}}'}}\,\vartheta(C_{\widetilde{\mathfrak{I}}'})\,. \qquad (VI.16)$$

Here we distinguish $\widetilde{\mathfrak{I}} = \{\widetilde{n}_1,\ldots,\widetilde{n}_{L/2},0,\ldots,0\}$ from $\mathfrak{I}^{(j)}$ in the definition of $C_{\mathfrak{I}^{(j)}}$. Proposition 4 ensures that the trace of $C_{\widetilde{\mathfrak{I}}}\,\vartheta(C_{\widetilde{\mathfrak{I}}'})$ vanishes unless each $\widetilde{n}_i = 0 = \widetilde{n}_i'$. The trace of $T\vartheta(T)$ is given by the constant term in the expansion in the monomial basis of parafermions.

Consider first the case in which A and all the B_j are constants. Then the relation (II.6) ensures that

$$T = C_{\mathfrak{I}^{(1)}}\cdots C_{\mathfrak{I}^{(k)}} = \alpha\,C_{\mathfrak{I}^{(1)}+\ldots+\mathfrak{I}^{(k)}} = \alpha\,C_{\widetilde{\mathfrak{I}}}\,, \quad \text{with} \quad \alpha \in \mathbb{C}\,, \qquad (VI.17)$$

namely there is only one term $C_{\widetilde{\mathfrak{I}}}$ in the expansion of T. Thus we have the local conservation law

$$\widetilde{n}_i = \sum_{j=1}^{k} n_i^{(j)} \mod n\,, \qquad (VI.18)$$

for each $i = 1,\ldots,L$, and in fact $\widetilde{n}_i = 0$ for $i > L/2$.

Proposition 4 ensures that the trace of $T\vartheta(T)$ vanishes unless each parafermion c_i appears in $C_{\widetilde{\mathfrak{I}}}$ with an exponent equal to 0 mod n. In other words $\widetilde{n}_i = 0$. Summing this relation over i gives the desired global conservation law (VI.14).

In the general case, the matrices A and B_j are elements of $\mathfrak{A}_-^n$. One obtains T from the previous case by replacing each $C_{\mathfrak{I}^{(j)}}$ by the product $C_{\mathfrak{I}^{(j)}}B_j$, and multiplying $D_{\mathfrak{I}_1,\ldots,\mathfrak{I}_k}$ by A. One can expand A and each B_j using the basis of parafermion monomials, and the total degree of each non-zero term in each of these expansions is an integer multiple of n. In the general case, the multiplications may introduce new parafermion factors, so it may be the case that $\widetilde{n}_i \neq \sum_{j=1}^{k} n_i^{(j)} \mod n$, and the local conservation law (VI.18) may not hold for T. However the relation (II.6) ensures that each multiplication by A or by B_j changes the *total* degree of any monomial in the expansion of T by an integer multiple of n. Thus

$$\sum_{i=1}^{L}\widetilde{n}_i = \sum_{i=1}^{L}\sum_{j=1}^{k} n_i^{(j)} \mod n = \sum_{j=1}^{k}|\mathfrak{I}^{(j)}| \mod n\,, \qquad (VI.19)$$

remains true. Since the trace of $T\vartheta(T)$ vanishes unless $\widetilde{n}_i = 0$ for all i, we infer the global conservation law (VI.14). Hence (VI.14) holds in the general case.

The positivity of $\mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}$ follows in case each of the coupling constants $J_{\mathfrak{J}^{(j)}\,\vartheta\mathfrak{J}^{(j)}}$ are non-negative. In case of even n, we also allow a factor

$$(-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|} = (-1)^{\alpha n} \qquad\qquad (\text{VI}.20)$$

for integer α. But as we are assuming that n is even, this also equals $+1$. $\qquad\square$

Completion of the Proof of Theorem 6. Using (VI.9) and Lemma 7, we infer that

$$
\begin{aligned}
A\,&\vartheta(A)\left(e^{-H}\right)_k \\
&= \sum_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} (-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|}\,\omega^{\sum_{j=1}^{k}\frac{1}{2}|\mathfrak{J}^{(j)}|^2+\sum_{1\leqslant j<j'\leqslant k}|\mathfrak{J}^{(j)}||\mathfrak{J}^{(j')}|} \\
&\qquad\qquad\times \mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,\vartheta(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}) \\
&= \sum_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} (-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|}\,\omega^{\sum_{j=1}^{k}\frac{1}{2}|\mathfrak{J}^{(j)}|^2+\frac{1}{2}\left(\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|\right)^2-\frac{1}{2}\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|^2} \\
&\qquad\qquad\times \mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,\vartheta(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}) \\
&= \sum_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} (-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|}\,\omega^{\frac{1}{2}\left(\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|\right)^2} \\
&\qquad\qquad\times \mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,\vartheta(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}})\,.
\end{aligned}
$$

Taking the trace, we have the approximation

$$
\begin{aligned}
\operatorname{Tr}&\left(A\vartheta(A)\left(e^{-H}\right)_k\right) \\
&= \sum_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} (-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|}\,\omega^{\frac{1}{2}\left(\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|\right)^2}\,\mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} \\
&\qquad\qquad\times \operatorname{Tr}\left(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\,\vartheta(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}})\right)\,. \qquad (\text{VI}.21)
\end{aligned}
$$

From Lemma 8 we infer that the trace vanishes unless $\sum_{j=1}^{k}|\mathfrak{J}^{(j)}| = \alpha n$ for some non-negative integer α. Also in this case $\mathfrak{c}_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}} \geqslant 0$. The phase in (VI.21) is

$$(-1)^{\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|}\omega^{\frac{1}{2}\left(\sum_{j=1}^{k}|\mathfrak{J}^{(j)}|\right)^2} = (-1)^{\alpha n}\omega^{\frac{1}{2}\alpha^2 n^2} = e^{2\pi i n\frac{(1+\alpha)\alpha}{2}} = 1\,.$$

In the final equality we use the fact that $(1+\alpha)\alpha$ is even. Proposition 5 ensures $\operatorname{Tr}(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}\vartheta(AD_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}})) \geqslant 0$. So each term in the sum (VI.21) is non-negative. Therefore the $k\to\infty$ limit of (VI.21) is also non-negative. $\qquad\square$

In §IX, we require a generalization of Lemma 7 and Lemma 8, which reduce to the previous statements in case $A = B$ and $B_j^- = F_j^-$ for all j. The proof of the generalizations follow the prior proofs step by step.

Lemma 9 (General Rearrangement II) *Let* $C_{\mathfrak{J}^{(j)}} \in \mathfrak{A}_-$, *and let* A, B, B_j^-, $F_j^- \in \mathfrak{A}_-^n$, *for* $j = 1,\ldots,k$. *Also let* $D_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}^- = C_{\mathfrak{J}^{(1)}}B_1^-\,C_{\mathfrak{J}^{(2)}}B_2^-\cdots C_{\mathfrak{J}^{(k)}}B_k^- \in \mathfrak{A}_-$, *and correspondingly let* $E_{\mathfrak{J}^{(1)},\ldots,\mathfrak{J}^{(k)}}^- = C_{\mathfrak{J}^{(1)}}F_1^-\,C_{\mathfrak{J}_2}F_2^-\cdots C_{\mathfrak{J}^{(k)}}F_k^- \in \mathfrak{A}_-$. *Then,*

$$
\begin{aligned}
A\vartheta(B)&C_{\mathfrak{J}^{(1)}}\vartheta(C_{\mathfrak{J}^{(1)}})\,B_1^-\vartheta(F_1^-)\cdots C_{\mathfrak{J}^{(k)}}\vartheta(C_{\mathfrak{J}^{(k)}})\,B_k^-\vartheta(F_k^-) \\
&= \omega^{\sum_{1\leqslant j<j'\leqslant k}|\mathfrak{J}^{(j)}||\mathfrak{J}^{(j')}|}\,AD_{\mathfrak{J}_1,\ldots,\mathfrak{J}_k}^-\,\vartheta(BE_{\mathfrak{J}_1,\ldots,\mathfrak{J}_k}^-)\,. \qquad (\text{VI}.22)
\end{aligned}
$$

Lemma 10 (Conservation Law II) *Under the hypotheses of Lemma 9, the trace of $AD^-_{\mathfrak{J}_1,\ldots,\mathfrak{J}_k}\vartheta(BE^-_{\mathfrak{J}_1,\ldots,\mathfrak{J}_k})$ vanishes unless*

$$\sum_{j=1}^{k}|\mathfrak{J}^{(j)}| = 0 \qquad \mathrm{mod}\ \ n\,. \tag{VI.23}$$

If (VI.23) holds, then

$$c_{\mathfrak{J}_1,\ldots,\mathfrak{J}_k} = \frac{1}{k^k}\prod_{j=1}^{k} J_{\mathfrak{J}^{(j)}\,\vartheta\mathfrak{J}^{(j)}} \geqslant 0\,. \tag{VI.24}$$

VII. RP Does Not Hold on $\mathfrak{A}_-$

We have proved that the functional $f(A) = \mathrm{Tr}(A\,\vartheta(A)\,e^{-H})$ is positive for $A \in \mathfrak{A}^n_- \subset \mathfrak{A}_-$. This is what we defined as the algebra of observables after (II.3). Here we remark that $f(A)$ is *not* positive on the full algebra $\mathfrak{A}_-$.

Consider $L = 2$ with the parafermion generators, $c = c_1 \in \mathfrak{A}^1_-$ and $c_2 = \vartheta(c)^* \in \mathfrak{A}^1_+$. Let $A = c$ and take $H = H_0 = \omega^{\frac{1}{2}}c\vartheta(c)$, which has the form (VI.1)–(VI.2), with $H_- = H_+ = 0$. We now show that $f(c)$ is not positive, so ϑ is not RP on $\mathfrak{A}^1_-$. In fact

$$\begin{aligned}
f(c) &= \sum_{k=0}^{\infty}\mathrm{Tr}(c\vartheta(c)(c\vartheta(c))^k)\frac{(-1)^k\omega^{\frac{k}{2}}}{k!}\\
&= \sum_{k=0}^{\infty}\omega^{\left(k+\frac{k(k-1)}{2}\right)}\mathrm{Tr}\left(c^{1+k}\vartheta(c)^{1+k}\right)\frac{(-1)^k\omega^{\frac{k}{2}}}{k!}\\
&= \sum_{k=0}^{\infty}\frac{\omega^{\left(k+\frac{k^2}{2}\right)}(-1)^k}{k!}\mathrm{Tr}\left(c^{1+k}\vartheta(c)^{1+k}\right)\,.
\end{aligned}$$

Use the fact that the trace vanishes unless $1 + k = \ell n$ for $\ell = 1, 2, \ldots$. Then

$$\begin{aligned}
f(c) &= \mathrm{Tr}(I)\sum_{\ell=1}^{\infty}\frac{\omega^{\left((\ell n-1)+\frac{(\ell n-1)^2}{2}\right)}(-1)^{\ell n-1}}{(\ell n-1)!}\\
&= -\omega^{-\frac{1}{2}}\,\mathrm{Tr}(I)\sum_{\ell=1}^{\infty}\frac{\omega^{\frac{1}{2}\ell^2 n^2}(-1)^{\ell n}}{(\ell n-1)!}\,. \tag{VII.1}
\end{aligned}$$

For integer ℓ, the product $\ell(\ell+1)$ is an even, positive integer. Thus the phase inside the sum equals

$$\omega^{\frac{1}{2}\ell^2 n^2}(-1)^{\ell n} = e^{\left(\frac{2\pi i}{n}\right)\left(\frac{1}{2}\ell^2 n^2\right)+\pi i\ell n} = e^{\pi i n\ell(\ell+1)} = 1\,. \tag{VII.2}$$

Therefore one finds that

$$f(c) = -\omega^{-\frac{1}{2}}\,\mathrm{Tr}(I)\sum_{\ell=1}^{\infty}\frac{1}{(\ell n-1)!} \notin \mathbb{R}_+\,. \tag{VII.3}$$

One can also calculate $f(c^j)$ for the same Hamiltonian, noting that $c^j \in \mathfrak{A}_-^j$. In this case there are certain pairs (n, j), with $j < n$, for which $f(c^j)$ is positive. Three such families of pairs are:

1. $n = k^3$, $j = k^2$, with $k \in \mathbb{Z}_+$,
2. $n = 2k^2$, $j = 2kj'$, with $1 \leqslant j' < k$,
3. $n = k^2$, $j = j'k$ with k odd and $1 \leqslant j' < k$.

We do not pursue the question of finding on exactly which subalgebras of $\mathfrak{A}_-$ the functional $f(c^j)$ is positive.

VIII. The Baxter Clock Hamiltonian

As an example of a familiar parafermion interaction, Fendley has shown that the Baxter clock Hamiltonian (originally formulated as interacting spins [4,5]) can be expressed in terms of parafermions. Near the end of §3.1 of [15], he finds that for parafermion generators c_j of degree n,

$$H = \omega^{\frac{n-1}{2}} \sum_{j=1}^{L-1} t_j \, c_{j+1} c_j^* \,, \qquad (\text{VIII.1})$$

where the t_j are real coupling constants. As $c_j^* = c_j^{n-1}$, each term in the Hamiltonian is an element of the algebra $\mathfrak{A}^n$.

In §I we remarked that if $\{c_j\}$ are parafermion generators, then $\{c_j^*\}$ are also parafermion generators. So using this alternative set of parafermions, one can also write the Baxter clock Hamiltonian as

$$H = \omega^{\frac{n-1}{2}} \sum_{j=1}^{L-1} t_j \, c_{j+1}^* c_j = -\omega^{\frac{1}{2}} \sum_{j=1}^{L-1} t_j \, c_j \, c_{j+1}^* \,. \qquad (\text{VIII.2})$$

One can split this sum into three parts,

$$H = H_- + H_0 + H_+ \,, \qquad (\text{VIII.3})$$

where

$$H_- = -\omega^{\frac{1}{2}} \sum_{j=1}^{\frac{1}{2}L-1} t_j \, c_j c_{j+1}^* \,, \quad H_+ = -\omega^{\frac{1}{2}} \sum_{j=\frac{1}{2}L+1}^{L-1} t_j \, c_j c_{j+1}^* \,,$$

and

$$H_0 = -\omega^{\frac{1}{2}} \, t_{\frac{1}{2}L} \, c_{\frac{1}{2}L} \, c_{\frac{1}{2}L+1}^* = -\omega^{\frac{1}{2}} \, t_{\frac{1}{2}L} \, c_{\frac{1}{2}L} \, \vartheta(c_{\frac{1}{2}L}) \,. \qquad (\text{VIII.4})$$

Note that $\vartheta(H_0) = H_0$. Also

$$\vartheta(H_-) = -\omega^{-\frac{1}{2}} \sum_{j=1}^{\frac{1}{2}L-1} t_j \, \vartheta(c_j)\vartheta(c_{j+1}^*) = -\omega^{-\frac{1}{2}} \sum_{j=1}^{\frac{1}{2}L-1} t_j \, c_{L-j+1}^* c_{L-j}$$

$$= -\omega^{-\frac{1}{2}-(n-1)} \sum_{j=1}^{\frac{1}{2}L-1} t_j \, c_{L-j} c_{L-j+1}^* = -\omega^{\frac{1}{2}} \sum_{j=\frac{1}{2}L+1}^{L-1} t_{L-j} \, c_j c_{j+1}^* \,.$$

On the other hand, the parafermion Hamiltonians that we study in (VI.1) include those with $|\mathfrak{I}| = 1$ of the form

$$H = H_- + H_0 + H_+ , \quad \text{with} \quad H_+ = \vartheta(H_-) , \qquad \text{(VIII.5)}$$

and

$$H_0 = \omega^{\frac{1}{2}} J_{\frac{1}{2}L} \, c_{\frac{1}{2}L} \, \vartheta(c_{\frac{1}{2}L}) = \omega^{\frac{1}{2}} J_{\frac{1}{2}L} \, c_{\frac{1}{2}L} \, c^*_{\frac{1}{2}L+1} . \qquad \text{(VIII.6)}$$

Thus Fendley's representation of the Baxter Hamiltonian has the required general form (VIII.5)–(VIII.6) if $J_j = -t_j$ for all j, and also

$$t_{L-j} = t_j , \quad \text{for} \quad j = 1, 2, \ldots, \frac{1}{2}L - 1 . \qquad \text{(VIII.7)}$$

Such a Hamiltonian is reflection invariant, $\vartheta(H) = H$, and it is gauge invariant $UHU^* = H$. It satisfies our RP hypotheses in §VI.1 in case:

$$\text{For odd } n\text{: } t_{\frac{1}{2}L} \leqslant 0 .$$
$$\text{For even } n\text{: } t_{\frac{1}{2}L} \in \mathbb{R} . \qquad \text{(VIII.8)}$$

IX. Reflection Bounds

Reflection positivity allows one to define a pre-inner product on $\mathfrak{A}_\pm$ given by

$$\langle A, B \rangle = \text{Tr}(A \, \vartheta(B)) . \qquad \text{(IX.1)}$$

This pre-inner product satisfies the Schwarz inequality

$$|\langle A, B \rangle|^2 \leqslant \langle A, A \rangle \, \langle B, B \rangle . \qquad \text{(IX.2)}$$

In the standard way, one obtains an inner product $\langle \widehat{A}, \widehat{B} \rangle$ and norm $\|\widehat{A}\|$ by defining the inner product on equivalence classes $\widehat{A} = \{A + n\}$ of A's, modulo elements n of the null space of the functional (IX.1) on the diagonal. In order to simplify notation, we ignore this distinction.

Let us introduce two pre-inner products $\langle \, \cdot \, , \cdot \, \rangle_\pm$ on the algebras $\mathfrak{A}_\pm^n$, corresponding to two reflection-symmetric Hamiltonians. Let

$$\langle A, B \rangle_- = \text{Tr}(A \, \vartheta(B) \, e^{-H_{-,\vartheta_-}}), \quad \text{for} \quad H_{-,\vartheta_-} = H_- + H_0 + \vartheta(H_-) . \quad \text{(IX.3)}$$

Similarly define

$$\langle A, B \rangle_+ = \text{Tr}(A \, \vartheta(B) \, e^{-H_{\vartheta+,+}}), \quad \text{for} \quad H_{\vartheta+,+} = \vartheta(H_+) + H_0 + H_+ . \quad \text{(IX.4)}$$

As in the first paragraph of this section, use the forms (IX.3) and (IX.4) to define inner products and norms $\| \, \cdot \, \|_\pm$.

We now state the reflection bound for a Hamiltonian H for which we do not assume that $H_- = \vartheta(H_+)$. Rather we bound the absolute value of expectations defined by H, in terms of the norms $\| \, \cdot \, \|_\pm$.

Proposition 11 (RP-Bounds) *Let $H = H_- + H_0 + H_+$ with $H_\pm \in \mathfrak{A}_\pm^n$ and H_0 of the form (VI.2), with couplings $J_{\mathfrak{J}\,\vartheta\mathfrak{J}}$ that satisfy (VI.3) or (VI.4). Then for $A, B \in \mathfrak{A}_+^n$,*

$$\left| \mathrm{Tr}(A\,\vartheta(B)\,e^{-H}) \right| \leqslant \|A\|_- \, \|B\|_+ \, . \tag{IX.5}$$

Also

$$\left| \mathrm{Tr}(A\,\vartheta(B)\,e^{-H}) \right| \leqslant \|A\|_+ \, \|B\|_- \, . \tag{IX.6}$$

In particular for $A = B = I$,

$$\left| \mathrm{Tr}(e^{-H}) \right| \leqslant \mathrm{Tr}(e^{-(H_- + H_0 + \vartheta(H_-))})^{1/2} \; \mathrm{Tr}(e^{-(\vartheta(H_+) + H_0 + H_+)})^{1/2} \, . \tag{IX.7}$$

Proof. The proof is an elaboration of the proof of Theorem 6, that yields an upper bound rather than positivity. Use the expression (VI.21), along with the discussion following that identity, to write $A\,\vartheta(B)\left(e^{-H}\right)_k$, which converges to $A\,\vartheta(B)\,e^{-H}$ as $k \to \infty$ in the form

$$\begin{aligned} &\mathrm{Tr}\left(A\vartheta(B)\left(e^{-H}\right)_k\right) \\ &\quad = \sum_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} c_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \left\langle AD^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}}, BE^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \right\rangle \, . \end{aligned} \tag{IX.8}$$

The form $\langle \, \cdot \, , \, \cdot \, \rangle$ in (IX.8) is defined in (IX.1). The matrices $D^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \in \mathfrak{A}_-$ are given by (VI.12), and

$$E^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} = C_{\mathfrak{J}_1} e^{-\vartheta(H_+)/k} C_{\mathfrak{J}_2} e^{-\vartheta(H_+)/k} \cdots C_{\mathfrak{J}_k} e^{-\vartheta(H_+)/k} \in \mathfrak{A}_- \, . \tag{IX.9}$$

We infer from Lemma 8 that $\mathfrak{c}_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \geqslant 0$ whenever $\langle AD^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k}, BE^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \rangle \neq 0$. Use the Schwarz inequality for $\langle \, \cdot \, , \cdot \, \rangle$ and the positivity of $\mathfrak{c}_{\mathfrak{J}_1,\dots,\mathfrak{J}_k}$ to obtain

$$\begin{aligned} \left| \mathrm{Tr}\left(A\,\vartheta(B)\left(e^{-H}\right)_k\right) \right| &\leqslant \sum_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \mathfrak{c}^{1/2}_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \left\langle AD^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k}, AD^-_{\ell_1,\dots,\ell_k} \right\rangle^{1/2} \\ &\qquad\qquad \times \, \mathfrak{c}^{1/2}_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \left\langle BE^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k}, BE^-_{\mathfrak{J}_1,\dots,\mathfrak{J}_k} \right\rangle^{1/2} \\ &\leqslant \left(\sum_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \mathfrak{c}_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \left\langle AD^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}}, AD^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \right\rangle \right)^{1/2} \\ &\qquad \times \left(\sum_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \mathfrak{c}_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \left\langle BE^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}}, BE^-_{\mathfrak{J}^{(1)},\dots,\mathfrak{J}^{(k)}} \right\rangle \right)^{1/2} \, . \end{aligned}$$

Taking the limit $k \to \infty$, one has

$$\left| \mathrm{Tr}\left(A\,\vartheta(B)\,e^{-H}\right) \right| = \langle A, A \rangle_-^{1/2} \, \langle B, B \rangle_+^{1/2} = \|A\|_- \, \|B\|_+ \, . \tag{IX.10}$$

This completes the proof of relation (IX.5).

When $A, B \in \mathfrak{A}_+^n$, substitute in the left-hand side of (IX.6) $A = \vartheta(\widetilde{A})$ and $B = \vartheta(\widetilde{B})$ with $\widetilde{A}, \widetilde{B} \in \mathfrak{A}_-^n$. Since A and B commute with $\vartheta(A)$ and $\vartheta(B)$,

$$\left| \mathrm{Tr}(A\,\vartheta(B)\,e^{-H}) \right| = \left| \mathrm{Tr}(\widetilde{B}\,\vartheta(\widetilde{A})\,e^{-H}) \right| \, . \tag{IX.11}$$

Replacing H_- by $\vartheta(H_+)$ and $\vartheta(H_-)$ by H_+ in the bound (IX.5) completes the proof of (IX.6). $\qquad\square$

Reflection-positivity bounds of the form (IX.5) and (IX.6) turned out to be useful to solve numerous physics problems. As an illustration, we mention here four relevant examples. In [13], the authors applied RP bounds to study phase transitions in anisotropic spin lattice models. RP bounds were also useful to investigate the vortex structure in the ground states of interacting fermions on a lattice [29,31] and of certain spin ladders [11].

X. Topological Order and Reflection Positivity

Define a loop C of length 2ℓ as an ordered sequence of sites $\{i_1, i_2, \ldots, i_{2\ell}\}$. Let $W_A = A\,\vartheta(A) = \mathfrak{B}(C)$ be a product of parafermions around the loop C,

$$\mathfrak{B}(C) = c_{i_1}^{n_{i_1}} c_{i_2}^{n_{i_2}} \cdots c_{i_{2\ell}}^{n_{i_{2\ell}}}, \quad \text{where} \quad i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{2\ell} = i_1 . \qquad (\text{X}.1)$$

Take $A = c_{i_1}^{n_{i_1}} \cdots c_{i_\ell}^{n_{i_\ell}}$ to be the product of parafermions along half of a loop and $\vartheta(A) = \vartheta(c_{i_1}) \cdots \vartheta(c_{i_\ell}) = (c_{\vartheta i_1}^{n_{i_1}})^* \cdots (c_{\vartheta i_\ell}^{n_{i_\ell}})^* = c_{i_{2\ell}}^{n-n_{i_1}} \cdots c_{i_{\ell+1}}^{n-n_{i_\ell}}$ the product of parafermions along the other half of the loop.

Consider a reflection-invariant Hamiltonian H, with a ground-state subspace $\mathcal{P}$. We also use the symbol $\mathcal{P}$ to denote the orthogonal projection onto the ground-state subspace. Define H to have W-order, if the operator W applied to any vector $\Omega \in \mathcal{P}$ has no component in $\mathcal{P}$ that is orthogonal to Ω. In other words, $\mathcal{P}W\mathcal{P}$ is a scalar multiple of $\mathcal{P}$, and W does not cause transitions between different ground states. Topological order involves the additional assumption that W is localized.

In an earlier paper [26], we have the following result for a Hamiltonian describing the interaction of Majoranas. The exact same proof as in [26] applies as well to Hamiltonians describing the interaction of parafermions.

Proposition 12 *Let H be a reflection-positive Hamiltonian that has $W_A = A\vartheta(A)$ topological order, where $A \in \mathfrak{A}_-^n$. Then $0 \leqslant \langle \Omega, W_A \Omega \rangle$ for any $\Omega \in \mathcal{P}$.*

Topologically ordered systems have attracted a lot of attention in the physics community because of their potential use in quantum computation. The idea is to encode the qubit states into the ground-state subspace of such Hamiltonians. Topological order ensures that the encoded qubits are able to tolerate some local noise without being destroyed. In order to change the state of the qubit one must perform a non-local operation. This is the basic premise for topological quantum computation. In this context it is important to understand the ground-state properties of Hamiltonians with topological order.

Related to this line of thinking, we considered the interaction of Majoranas (parafermions of degree two) on a two-dimensional lattice [26]. In this case the operators W_A were conserved and we said that a loop C has a vortex if $\mathfrak{B}(C) = W_A = -1$. We applied Proposition 12 to show that W_A has no vortex in any ground state. One could investigate a similar situation for parafermion interactions.

XI. Acknowledgement

presentation by Jelena Klinovaja. We are grateful to Maissam Barkeshli, Jacques Perk, and Thomas Quella for comments on earlier versions of this manuscript, including Perk's advising us of Sylvester's paper [40], and of an electronic archive where one can download [43]. We thank Barbara Drauschke for suggesting several improvements in style. We are also grateful to the referees of our paper for very constructive suggestions.

References

1. Au-Yang, Helen and Perk, Jacques H. H.: Parafermions in the τ_2 model, arXiv:1402.0061, http://arxiv.org/abs/1402.0061, to appear in J. Phys. A: Math. Theor.
2. Barkeshli, Maissam and Qi, Xiao-Liang, Topological Nematic States and Non-Abelian Lattice Dislocations, Phys. Rev. X **2**, 031013 (2012). doi:10.1103/PhysRevX.2.031013.
3. Barkeshli, Maissam, Jian, Chao-Ming, and Qi, Xiao-Liang: Twist defects and projective non-Abelian braiding statistics, Phys. Rev. B **87**, 045130. doi:10.1103/PhysRevB.87.045130.
4. Baxter, R. J.: A simple solvable Z_N Hamiltonian, Phys. Lett. **A 140**, 155–157 (1989). doi:10.1016/0375-9601(89)90884-0.
5. Baxter, R. J.: Superintegrable Chiral Potts Model: Thermodynamic Properties, an "Inverse" Model, and a Simple Associated Hamiltonian, J. Stat. Phys. **57**, 1–39 (1989). doi:10.1007/BF01023632.
6. Baxter, R. J.: Transfer matrix functional relations for the generalized $\tau_2(t_q)$ model, J. Stat. Phys. **117**, 1–25 (2004). doi:10.1023/B:JOSS.0000044062.64287.b9.
7. Baxter, R. J.: The τ_2 Model and Parafermions, arXiv:1310.7074, http://arxiv.org/abs/1310.7074, to appear in J. Phys. A: Math. Theor.
8. Biskup, Marek, Reflection Positivity and Phase Transitions in Lattice Spin Models, in: R. Kotecky (ed), Methods of Contemporary Mathematical Statistical Physics, Lecture Notes in Mathematics, vol. 1970, Springer-Verlag Berlin Heidelberg, 2009, 1-86. doi:10.1007/978-3-540-92796-9.
9. Bondesan, Roberto and Quella, Thomas, Topological and symmetry broken phases of $\mathbb{Z}_N$ parafermions in one dimension, J. Stat. Mech. (2013) P10024. doi:10.1088/1742-5468/2013/10/P10024.
10. Clarke, David J., Alicea, Jason, and Shtengel, Kirill: Exotic non-Abelian anyons from conventional fractional quantum Hall states, Nat. Commun. **4**, 1348 (2013). doi:10.1038/ncomms2340.
11. Chesi, Stefano, Jaffe, Arthur, Loss, Daniel, and Pedrocchi, Fabio L.: Vortex loops and Majoranas, J. Math. Phys. **54**, 112203 (2013). doi:10.1063/1.4829273.
12. Cobanera, Emilio and Ortiz, Gerardo: Fock Parafermions and Self-Dual Representations of the Braid Group, Phys. Rev. A **89**, 012328 (2014). doi:10.1103/PhysRevA.89.012328.
13. Dyson, Freeman J., Lieb, Elliott H., and Simon, Barry: Phase Transitions in Quantum Spin Systems with Isotropic and Nonisotropic Interactions, J. Stat. Phys. **18**, 335–383 (1978). doi:10.1007/978-3-662-10018-9_12.
14. Fendley, Paul: Parafermionic edge zero modes in $\mathbb{Z}_n$-invariant spin chains, J. Stat. Mech. P11020 (2012). doi:10.1088/1742-5468/2012/11/P11020.
15. Fendley, Paul: Free parafermions, J. Phys. A: Math. Theor. **47**, 075001 (2014). doi:10.1088/1751-8113/47/7/075001.
16. Fradkin, E. and Kadanoff, L.: Disorder variables and parafermions in two-dimensional statistical mechanics, Nucl. Phys. **B170 [FS1]**, 1–15 (1980). doi:10.1016/0550-3213(80)90472-1.
17. Fröhlich, Jürg and Lieb, Elliott: Phase Transitions in Anisotropic Lattice Spin Systems, Commun. Math. Phys. **60**, 233–267 (1978). doi:10.1007/BF01612891.
18. Fröhlich, Jürg, Simon, Barry, and Spencer, Thomas: Infrared Bounds, Phase Transitions, and Continuous Symmetry Breaking, Commun. Math. Phys. **50**, 78–85 (1976). doi:10.1007/BF01608557.
19. Fröhlich, Jürg and Gabbiani, Fabrizio: Braid Statistics in local quantum field theory, Rev. Math. Phys. **2**, 251–353 (1990). doi:10.1142/S0129055X90000107.

20. Green, H. S.: A Generalized Method of Field Quantization, Phys. Rev. **90**, 270–273 (1953). doi:10.1103/PhysRev.90.270.
21. Hatano, Naomichi and Nelson, David R.: Localization Transitions in Non-Hermitian Quantum Mechanics, Phys. Rev. Lett. **77**, 570 (1996). doi:10.1103/PhysRevLett.77.570.
22. 't Hooft, Gerard: On the phase transition towards permanent quark confinement, Nucl. Phys. **B138**, 1–15 (1978). doi:10.1016/0550-3213(78)90153-0.
23. Jaffe, Arthur, Jäkel, Christian, and Martinez II, Roberto E., Complex Classical Fields: A Framework for Reflection Positivity, Commun. Math. Phys. **329**, 1–28 (2014). doi:10.1007/s00220-014-2040-y.
24. Jaffe, Arthur, Jäkel, Christian, and Martinez II, Roberto E., Complex Classical Fields: an Example, Journal of Functional Analysis, **266**, 1833–1881 (2014). doi:10.1016/j.jfa.2013.08.033.
25. Jaffe, Arthur, and Pedrocchi, Fabio L.: Reflection Positivity for Majoranas, Annales Henri Poincaré (2014). doi:10.1007/s00023- 014-0311-y.
26. Jaffe, Arthur, and Pedrocchi, Fabio L.: Topological Order and Reflection Positivity, Europhysics Letters (EPL) **105**, 40002 (2014). doi:10.1209/0295-5075/105/40002.
27. Klinovaja, Jelena and Loss, Daniel: Parafermions in Interacting Nanowire Bundle, Phys. Rev. Lett. **112**, 246403 (2014). doi:10.1103/PhysRevLett.112.246403.
28. Klinovaja, Jelena and Loss, Daniel: arXiv:1312.1998, Time-Reversal Invariant Parafermions in Interacting Rashba Nanowires, http://arxiv.org/abs/1312.1998.
29. Lieb, Elliott H.: Flux Phase of the Half-Filled Band, Phys. Rev. Lett. **73**, 2158 (1994). doi:10.1103/PhysRevLett.73.2158.
30. Lindner, Netanel H., Berg, Erez, Refael, Gil, and Stern, Ady: Fractionalizing Majorana Fermions: Non-Abelian Statistics in the Edges of Abelian Quantum Hall States, Phys. Rev. X **2**, 041002 (2012). doi:10.1103/PhysRevX.2.041002.
31. Macris, Nicolas and Nachtergaele, Bruno: On the Flux Phase Conjecture at Half-Filling: An Improved Proof, J. Stat. Phys. **85**, 745–761 (1996). doi:10.1007/BF02199361.
32. Mong, Roger S. K., Clarke, David J., Alicea, Jason, Lindner, Netanel H., Fendley, Paul, Nayak, Chetan, Oreg, Yuval, Stern, Ady, Berg, Erez, Shtengel, Kirill, Fisher, Matthew P.A.: Universal Topological Quantum Computation from Superconductor-Abelian Quantum Hall Heterostructure, Phys. Rev. X **4**, 011036 (2014). doi:10.1103/PhysRevX.4.011036.
33. Morris, A. O.: On a Generalized Clifford Algebra, Quarterly J. Math., Oxford, **18**, 7–12 (1967). doi:10.1093/qmath/18.1.7.
34. Motruk, Johannes, Berg, Erez, Turner, Ari M., and Pollmann Frank, Topological phases in gapped edges of fractionalized systems, Phys. Rev. B **88**, 085115 (2013). doi:10.1103/PhysRevB.88.085115.
35. Nayak, Chetan, Simon, Steven H., Stern, Ady, Freedman, Michael, and Das Sarma, Shankar: Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. **80**, 1083 (2008). doi:10.1103/RevModPhys.80.1083.
36. Nelson, David R. and Shnerb, Nadav M.: Non-Hermitian localization and population biology, Phys. Rev. E **58**, 1383 (1998). doi:10.1103/PhysRevE.58.1383.
37. Osterwalder, Konrad, and Schrader, Robert: Axioms for Euclidean Green's functions, I and II. Commun. Math. Phys. **31**, 83–112 (1973), and Commun. Math. Phys. **42**, 281–305 (1975). doi:10.1007/BF01645738 and doi:10.1007/BF01608978.
38. Osterwalder, Konrad and Schrader Robert, Euclidean Fermi fields and a Feynman-Kac formula for boson-fermion interactions Helv. Phys. Acta **46**, 227–302 (1973). doi:10.5169/seals-114484.
39. Sylvester, J. J., A Word on Nonions, Johns Hopkins University Circulars **1**, No. 17, 241–242 (1882). See https://jscholarship.library.jhu.edu/bitstream/handle/1774.2/32845/I017.PDF.
40. Sylvester, J. J., On quaternions, nonions, sedenions, etc., Johns Hopkins University Circulars **3**, No. 27, 7–9 (1883). See https://jscholarship.library.jhu.edu/handle/1774.2/32855.
41. Trefethen, Lloyd N. and Embree, Mark, Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators, Princeton University Press 2005.
42. Vaezi, Abolhassan: Fractional topological superconductor with fractionalized Majorana fermions, Phys Rev B **87**, 035132 (2013). doi:10.1103/PhysRevB.87.035132.
43. Yamazaki, K.: On projective representations and ring extensions of finite groups, J. Fac. Sci Univ. Tokyo, Sect I **10** 147–195 (1964). See also http://repository.dl.itc.u-tokyo.ac.jp/dspace/bitstream/2261/6042/1/jfs100205.pdf.