Advances in Mathematics **143**, 1–110 (1999)
Article ID aima.1998.1747, available online at http://www.idealibrary.com on IDEAL®

Quantum Harmonic Analysis and Geometric Invariants*

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Received December 1, 1997; accepted March 3, 1998

We develop two topics in parallel and show their inter-relation. The first centers on the notion of a fractional-differentiable structure on a commutative or a non-commutative space. We call this study *quantum harmonic analysis*. The second concerns homotopy invariants for these spaces and is an aspect of non-commutative geometry.

We study an algebra $\mathfrak{A}$, which will be a Banach algebra with unit, represented as an algebra of operators on a Hilbert space $\mathscr{H}$. In order to obtain a geometric interpretation of $\mathfrak{A}$, we define a derivative on elements of $\mathfrak{A}$. We do this in a Hilbert space context, taking da as a commutator $da = [Q, a]$. Here Q is a basic self-adjoint operator with discrete spectrum, increasing sufficiently rapidly that $\exp(-\beta Q)^2$ has a trace whenever $\beta > 0$.

We can define fractional differentiability of order μ, with $0 < \mu \leqslant 1$, by the boundedness of $(Q^2+I)^{\mu/2}\, a(Q^2+I)^{-\mu/2}$. Alternatively we can require the boundedness of an appropriate smoothing (Bessel potential) of da. We find that it is convenient to assume the boundedness of $(Q^2+I)^{-\beta/2}\, da(Q^2+I)^{-\alpha/2}$, where we choose $\alpha, \beta \geqslant 0$ such that $\alpha+\beta<1$. We show that this also ensures a fractional derivative of order $\mu = 1-\beta$ in the first sense. We define a family of interpolation spaces $\mathfrak{I}_{\beta,\alpha}$. Each such space is a Banach algebra of operator, whose elements have a fractional derivative of order $\mu = 1-\beta > 0$.

We concentrate on subalgebras $\mathfrak{A}$ of $\mathfrak{I}_{\beta,\alpha}$ which have certain additional covariance properties under a group $\mathbb{Z}_2 \times \mathfrak{G}$ acting on $\mathscr{H}$ by a unitary representation $\gamma \times U(g)$. In addition, the derivative Q is assumed to be $\mathfrak{G}$-invariant. The geometric interpretation flows from the assumption that elements of $\mathfrak{A}$ possess an arbitrarily small fractional derivative. We study homotopy invariants of $\mathfrak{A}$ in terms of equivariant, entire cyclic cohomology. In fact, the existence of a fractional derivative on $\mathfrak{A}$ allows the construction of the cochain τ^{JLO}, which plays the role of the integral of differential forms. We give a simple expression for a homotopy invariant $\mathfrak{Z}^Q(a; g)$, determined by pairing τ^{JLO}, with a $\mathfrak{G}$-invariant element $a \in \mathfrak{A}$, such that a is a square root of the identity. This invariant is $\mathfrak{Z}^Q(a; g) = (1/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-t^2}\, \mathrm{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da})\, dt$.

This representation of the pairing is reminiscent of the heat-kernel representation for an index. In fact this quantity is an invariant, in the following sense. We isolate a simple condition on a family $Q(\lambda)$ of differentiations that yields a continuously-differentiable family $\tau^{\mathrm{JLO}}(\lambda)$ of cochains. Since $\mathfrak{Z}^Q(a; g)$ need not be an integer, continuity of $\tau^{\mathrm{JLO}}(\lambda)$ in λ is insufficient to prove the constancy of the pairing. However the existence of the derivative leads to the existence of the homotopy. As

* Supported in part by the Department of Energy under Grant DE-FG02-94ER-25228 and by the National Science Foundation under Grant DMS-94-24344.

0001-8708/99 $30.00

$\langle \tau, a \rangle$ vanishes for τ a coboundary, and as $d\tau^{\mathrm{JLO}}(\lambda)/d\lambda$ is a coboundary, our condition on $Q(\lambda)$ ensures that $\mathfrak{Z}^{Q(\lambda)}(a; g)$ is independent of λ. Hence it is a homotopy invariant.

The theory of $\mathfrak{Z}^{Q}(a; g)$ reduces to the study of the Radon transform of sequences of certain functions. The fractional differentiability properties of elements of $\mathfrak{A}$ translate into properties of the asymptotics of the sequences of Radon transforms. The condition that τ^{JLO} fit into the framework of entire cyclic cohomology translates to the existence of some fractional derivative for functions in the algebra under study, and in particular the assumption $\alpha + \beta < 1$. Thus the study of fractionally-differentiable structures dove-tails naturally with the theory of homotopy invariants.

In our study of quantum harmonic analysis, we introduce spaces $\mathscr{T}(-\beta, \alpha)$ of operator-valued distributions. These spaces are bounded, linear operators between Sobolev spaces. The elements of the interpolation spaces, the Banach algebras $\mathfrak{J}_{\beta, \alpha}$ have derivatives da which belong to the spaces $\mathscr{T}(-\beta, \alpha)$. For a certain range of β and α, we extend the theory of the Radon transform from products of regularized, bounded operators to products of regularized, operator-valued distributions.

We sometimes wish to evaluate such an invariant at the endpoint of an interval such as $\lambda \in (0, 1]$, where $\mathfrak{Z}^{Q(\lambda)}(a; g)$ becomes singular as $\lambda \to 0$. We discuss in brief a procedure to regularize the endpoint, and a method to recover $\mathfrak{Z}^{Q(\lambda)}(a; g)$ fully from certain partial information at the endpoint.

Finally, we generalize this approach to cover the case when Q can be split into the sum of "independent" parts $Q_1 + Q_2$, such that $(Q_1 + Q_2)^2 = (Q_1)^2 + (Q_2)^2$. Here we assume that Q_1 and $(Q_2)^2$ are $\mathfrak{G}$-invariant, but not necessarily Q_2. With further assumptions on a, the most important being that $(Q_1)^2 - (Q_2)^2$ commutes with a, we obtain a modified formula for an invariant, namely $\mathfrak{Z}^{\{Q_j\}}(a; g) = (1/\sqrt{\pi}) \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-(Q_1^2 + Q_2^2)/2 + it\, d_1 a})\, dt$.

Contents

I. INTRODUCTION

Let $\mathscr{H}$ denote a Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear transformations on $\mathscr{H}$. We study certain subalgebras $\mathfrak{A} \subset \mathscr{B}(\mathscr{H})$ with a unit I. Each algebra $\mathfrak{A}$ describes the geometry of a classical manifold or a quantum space, and we obtain certain homotopy invariants of this space. All this fits into Alain Connes' formulation of non-commutative geometry [3], as well as extensions of that theory studied by others. We emphasize here the analytic aspects of this theory, and in doing so we develop the relation between the regularity of the algebra, the regularity of the dependence of a differential acting on the algebra as a function of a parameter, and the existence and the constancy of the invariants as a function of this parameter.

We assume the algebra $\mathfrak{A}$ contains the unit I and has the norm $|||\cdot|||$, that we denote the $\mathfrak{A}$-norm and this norm has two properties: the $\mathfrak{A}$-norm dominates the operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, and $\mathfrak{A}$ is a Banach algebra with respect to the $\mathfrak{A}$-norm. In other words for $a, b \in \mathfrak{A}$,

$$\|a\| \leqslant |||a|||, \qquad \text{and} \qquad |||ab||| \leqslant |||a|||\ |||b|||. \tag{I.1}$$

In particular, the injection $\mathfrak{A} \to \mathscr{B}(\mathscr{H})$ and multiplication $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ are continuous. The norm $|||\cdot|||$ will be a Sobolev-type norm on $\mathfrak{A}$, so we require that a derivative operator d be defined on $\mathfrak{A}$. We specify differentiation in terms of a self-adjoint, linear transformation $Q = Q^*$ on $\mathscr{H}$ with domain $\mathscr{D}$. The first-order derivative da of an element of $\mathfrak{A}$ with respect to Q is given by the commutator

$$da = [Q, a] = Qa - aQ. \tag{I.2}$$

In general da may not be an element of $\mathscr{B}(\mathscr{H})$, or even an unbounded operator, but da is always defined as a sesquilinear form on $\mathscr{H} \times \mathscr{H}$ with domain $\mathscr{D} \times \mathscr{D}$. We eventually make precise the notion that a has a fractional derivative with respect to Q.

In this paper we impose a technical restriction on Q that its spectrum is discrete, and that for $s > 0$ the eigenvalues increase sufficiently rapidly that the heat kernel $\exp(-sQ^2)$ has a trace. In the case that this space is a classical manifold $\mathscr{M}$, this condition relaxes the assumptions that $\mathscr{M}$ be compact or finite dimensional. Some infinite-dimensional examples are given in [19]. Connes calls this condition *Θ-summability* [4].

In Section V, we introduce the family of Sobolev–Hilbert spaces $\mathscr{H}_\mu$ that are the domains of the μth-fractional powers of Q, for $\mu \geqslant 0$. More precisely, let $R = (Q^2 + I)^{-1/2}$. We say that R is a smoothing operator of degree -1 with respect to Q, and we let $\mathscr{H}_\mu$ be the domain of $R^{-\mu}$ with the norm $\|f\|_{\mathscr{H}_\mu} = \|R^{-\mu} f\|$ that determines the inner product. The dual space $\mathscr{H}_{-\mu}$ completes a Gelfand triple of Hilbert spaces

$$\mathscr{H}_\mu \subset \mathscr{H} \subset \mathscr{H}_{-\mu},$$

see [11]. One reasonable definition of an order-μ, differentiable operator $a \in \mathscr{B}(\mathscr{H})$ is that a is a bounded, linear transformation on $\mathscr{H}_\mu$, namely that

$$a\colon \mathscr{H}_\mu \to \mathscr{H}_\mu, \qquad \text{or equivalently} \qquad R^{-\mu} a R^{\mu} \in \mathscr{B}(\mathscr{H}). \tag{I.3}$$

However, because of the fundamental nature of the differential da, we prefer to pose differentiability properties of a directly in terms of properties of da, rather than by using (I.3). The statement that a is differentiable should be characterized by some property of the sesquilinear form (I.2) on $\mathscr{H}$. The simplest formulation of once-differentiability would be to say assume that the form (I.2) is *bounded*, so da uniquely determines a bounded, linear transformation in $\mathscr{B}(\mathscr{H})$. We abbreviate this statement by saying that

$$da\colon \mathscr{H} \to \mathscr{H}, \qquad \text{or} \qquad da \in \mathscr{B}(\mathscr{H}), \qquad \text{or} \qquad d\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H}).$$

In fact this assumption is made in most earlier work.[1]

In this paper we make the weaker requirement that $a \in \mathscr{B}(\mathscr{H})$ and that da defines a bounded, linear transformation between two different Sobolev spaces. Let $\mathscr{T}(-\beta, \alpha)$ denotes the space of bounded, linear maps from $\mathscr{H}_\alpha$ to $\mathscr{H}_{-\beta}$. For $0 < \alpha, \beta$, the elements of $\mathscr{T}(-\beta, \alpha)$ are operator-valued, generalized functions. We write our assumption as

$$da\colon \mathscr{H}_\alpha \to \mathscr{H}_{-\beta}, \qquad \text{or} \qquad R^{\beta}\, da\, R^{\alpha} \in \mathscr{B}(\mathscr{H}), \tag{I.4}$$

where we assume that

$$0 \leqslant \alpha, \beta, \qquad \text{and} \qquad \alpha + \beta < 1. \tag{I.5}$$

The condition (I.5) is crucial to the resulting analysis. We show that (I.4–5) ensures that each a we consider has a fractional derivative of order $\mu = 1 - \beta > 0$. The once-differentiable case is $\beta = 0$. Thus in place of

$$da\colon \mathfrak{A} \to \mathscr{B}(\mathscr{H}), \qquad \text{we now assume} \quad d\colon \mathfrak{A} \to \mathscr{T}(-\beta, \alpha).$$

In Section V we define a family of interpolation spaces $\mathfrak{J}_{\beta,\alpha} \subset \mathscr{B}(\mathscr{H})$. These spaces $\mathfrak{J}_{\beta,\alpha}$ provide us a useful generalization of classical interpolation

[1] This ranges from the differentiable case studied in [6], to the smooth case in [17], where one assumes that for all n, $d^n \mathfrak{A} \in \mathscr{B}(\mathscr{H})$. I am thankful to A. Connes for informing me that he and Moscovici have also considered certain algebras of pseudo-differential operators in [7].

spaces of Hölder-continuous functions, to a context where the functions are replaced by operators acting on a Hilbert space. The space $\mathfrak{J}_{\beta,\alpha}$ is a Banach algebra with unit I, so

$$\|ab\|_{\mathfrak{J}_{\beta,\alpha}} \leqslant \|a\|_{\mathfrak{J}_{\beta,\alpha}} \|b\|_{\mathfrak{J}_{\beta,\alpha}}.$$

Another characteristic property of $\mathfrak{J}_{\beta,\alpha}$ is

$$d\colon \mathfrak{J}_{\beta,\alpha} \to \mathscr{T}(-\beta, \alpha),$$

where $0 \leqslant \alpha, \beta$ and $\alpha + \beta < 1$. Thus it is natural that we assume the algebra $\mathfrak{A}$ under study is a subset of one of these interpolation spaces,

$$\mathfrak{A} \subset \mathfrak{J}_{\beta,\alpha} \subset \mathscr{B}(\mathscr{H}),$$

including the requirement that for all $a \in \mathfrak{A}$,

$$\|a\| \leqslant \|a\|_{\mathfrak{J}_{\beta,\alpha}} \leqslant |||a|||.$$

In addition to our assumption that $d\colon \mathfrak{A} \to \mathscr{T}(-\beta, \alpha)$, we also assume that a group $\mathbb{Z}_2 \times \mathfrak{G}$ acts as an automorphism group of $\mathfrak{A}$. The group $\mathbb{Z}_2$ provides a grading on $\mathscr{H}$. The group $\mathfrak{G}$ generally has a geometrical interpretation, and it may equal the identity. When $\mathfrak{G}$ is not trivial, then it gives rise to a $\mathfrak{G}$-equivariant theory. The group $\mathbb{Z}_2 \times \mathfrak{G}$ acts on $\mathscr{H}$ by a unitary representation $\gamma \times g \to \gamma U(g) = U(g)\,\gamma$. Here γ denotes both that element of $\mathbb{Z}_2$ not equal to the identity and also its representative. The $*$-automorphism of $\mathfrak{A}$ is obtained by the conjugation of the unitary action on $\mathscr{H}$.

The operator representing γ is self-adjoint as well as unitary, since $\gamma^* = \gamma^{-1}$. Denote the action of γ on $\mathfrak{A}$ by

$$a \to a^\gamma = \gamma a \gamma^{-1} = \gamma a \gamma. \tag{I.6}$$

We assume that the algebra $\mathfrak{A}$ is pointwise invariant under this action: $a = a^\gamma$ for all $a \in \mathfrak{A}$. The operator γ also acts on Q, and we assume that

$$\gamma Q \gamma = Q^\gamma = -Q. \tag{I.7}$$

Therefore for all $a \in \mathfrak{A}$,

$$(da)^\gamma = -d(a^\gamma). \tag{I.8}$$

This structure is familiar from ordinary geometry.[2]

[2] In the case that $\mathscr{H}$ is the L^2-Hilbert space of differential forms on a smooth, compact manifold $\mathscr{M}$, the $\mathbb{Z}_2$-grading γ may be taken to equal $(-1)^n$ on the subspace of forms of degree n. One can take the elements of $\mathfrak{A}$ to be smooth forms of degree zero (functions on $\mathscr{M}$). In that case, with d_{ext} the exterior derivative, a possible example of Q is $Q = d_{\text{ext}}^* + d_{\text{ext}}$, see Section V.1.

In Section V.5 we establish the (graded) Leibniz rule for $\mathfrak{J}_{\beta,\alpha}$. In other words, for $a, b \in \mathfrak{J}_{\beta,\alpha}$, we show that

$$d(ab) = (da)\, b + a^{\gamma}(db). \tag{I.9}$$

This statement includes the assertion that both $(da)\, b$ and $a^{\gamma}(db)$ are elements of $\mathcal{T}(-\beta, \alpha)$, so all three terms in (I.9) are elements of the same linear space. If $a \in \mathfrak{A}$, then $a^{\gamma} = a$, and (I.9) reduces to the ordinary Leibniz rule on $\mathfrak{A}$.

Likewise denote the $*$-automorphism group induced by $g \to U(g)$ by

$$a \to a^{g} = U(g)\, aU(g)^{*}. \tag{I.10}$$

In general, $\mathfrak{A}$ is not pointwise invariant under $\mathfrak{G}$, but we let $\mathfrak{A}^{\mathfrak{G}} \subset \mathfrak{A}$ denote the subalgebra of $\mathfrak{A}$ which is pointwise invariant, namely the subalgebra of $\mathfrak{A}$ which commutes with the representation $U(g)$,

$$a^{g} = a \qquad \text{for} \quad a \in \mathfrak{A}^{\mathfrak{G}}. \tag{I.11}$$

Throughout most of this paper we also assume that the group $U(g)$ commutes with Q,

$$Q^{g} = U(g)\, QU(g)^{*} = Q, \tag{I.12}$$

though we relax this assumption in Section IX. These considerations lead us to the notion of a fractionally-differentiable structure $\{\mathscr{H}, Q, \gamma, U(g), \mathfrak{A}\}$ on $\mathfrak{A}$, see Section VI.

As a preliminary to the study of $\mathfrak{A}$, we assume that the norm $|||\cdot|||$ is given, and we consider the theory of continuous, multi-linear functionals[3] on $\mathfrak{A}$. These are the spaces of cochains, introduced in Section II. In particular we focus on the space of entire chains, defined as follows. Consider the space of sequences $f = \{f_n\} \in \mathscr{C}$,

$$\mathscr{C} = \bigoplus_{n=0}^{\infty} \mathscr{C}_n, \tag{I.13}$$

[3] Such functionals often arise in a purely mathematical setting: in analysis, in probability theory, or in geometry. Furthermore, such functionals arise in mathematical physics: in statistical physics, in quantum theory, in quantum field theory, and in string theory. Within each of these fields, it may be true that one can represent a particular functional as a well-defined integral over a function space. When this is the case, the functional integral representation provides a powerful tool in order to prove mathematical properties of the functional in question, as well as a tool to estimate the functional, or possibly to evaluate it in closed form. In particular, constructive quantum field theory provides many examples of the phenomenon.

where the element $f_n \in \mathscr{C}_n$ maps $\mathfrak{A}^{n+1} \times \mathfrak{G}$ to $\mathbb{C}$. Here $f_n(a_0, a_1, ..., a_n; g)$ is an $(n+1)$-linear functional on $\mathfrak{A}$ and a functional on $\mathfrak{G}$. The other basic property of $\mathscr{C}_n$ we assume is that $f_n(a_0, ..., a_n; g)$ vanishes whenever any $a_j = I$ for $j = 1, 2, ..., n$ (but not $j = 0$). This assumption is motivated by the fact that integrals of classical differential forms such as $f_n(a_0, a_1, ..., a_n) = \int a_0 \, da_1 \cdots da_n$ have this property.

Furthermore, differential forms (and their integrals) vanish in finite dimensions if the degree of a form exceeds the dimension. Since a functional $f \in \mathscr{C}$ has an arbitrary number of components f_n, it is reasonable to expect that we need to replace vanishing of f_n for large n, by some estimate on the rate that f_n vanishes as the number of variables n grows to ∞. The *entire* condition species that the norm $|||f_n|||$ of the nth-component of a functional $f = \{f_n\} \in \mathscr{C}$ gets small at a certain rate,

$$n^{1/2} \, |||f_n|||^{1/n} \to 0. \tag{I.14}$$

This condition was originally introduced in [4], in order to prove the existence of a certain "normalization" operator. The space of cochains that we use here does not require that we introduce the concept of normalization, but the entire condition remains a natural analytic assumption, useful for other reasons.

The entire condition (I.14) ensures the existence of a generating function for a cochain evaluated on the diagonal, as well as the Gaussian transform of this generating function. In particular, the entire condition ensures that for $f \in \mathscr{C}$ and for $a \in \mathfrak{A}$, the generating functional given by the power series

$$f(t; a; g) = \sum_{n=0}^{\infty} t^n f_n(a, a, ..., a; g),$$

converges to define an entire function of t. The "extra" factor $n^{1/2}$ in (I.14) ensures that the Gaussian transform $(Rf)(t; a; g)$ of $f(t; a; g)$ also exists. With this hypothesis, whenever $f \in \mathscr{C}$ and $a \in \mathfrak{A}$,

$$(Rf)(s; a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(s-t)^2} f(t; a; g) \, dt$$

exists and extends to an entire function of s. We give details in Section III.2–3.

In Section II we define some continuous, linear operators on the space $\mathscr{C}$ of cochains, including the three fundamental coboundary operators b, B, and $\partial = b + B$. The operator b is Hochschild's coboundary, the operator B is Connes' coboundary, and $\partial = b + B$ is the coboundary operator of entire cyclic cohomology. The latter is defined as

$$\partial : \mathscr{C} \to \mathscr{C}, \qquad \partial^2 = 0. \tag{I.15}$$

In Section III we study a natural pairing $\langle f, a\rangle$ between cochains $f \in \mathscr{C}$ and invariant elements $a \in \mathfrak{A}^{\mathfrak{G}}$, which also are square roots of the identity $a^2 = I$. We show in Section III that the requirements

$$\langle \partial G, a\rangle = 0 \tag{I.16}$$

for all $G \in \mathscr{C}$ and for all $a \in \mathfrak{A}^{\mathfrak{G}}$ for which $a^2 = I$, determines the representation for a pairing to be

$$\langle \tau, a\rangle = (R\mathscr{J})(0; a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \mathscr{J}(t; a; g)\, dt. \tag{I.17}$$

Here $\mathscr{J}(t; a; g) = f_{\mathrm{even}}(it; a; g)$ is the generating functional for the even components of f, defined as

$$\mathscr{J}(t; a; g) = \sum_{n=0}^{\infty} (-t^2)^n \operatorname{tr} f_{2n}(a, a, ..., a; g), \tag{I.18}$$

see Corollary III.5.

In Section IV we introduce a particular even cochain $\tau^{\mathrm{JLO}} \in \mathscr{C}$,

$$\tau^{\mathrm{JLO}} = \{\tau_n^{\mathrm{JLO}}\}.$$

The nth component τ_n^{JLO} of τ^{JLO} is an $(n+1)$-multilinear functional on $\mathfrak{A}$,

$$\tau_n^{\mathrm{JLO}}(a_0, ..., a_n; g) = \langle a_0, da_1, ..., da_n; g\rangle. \tag{I.19}$$

This cochain is even in the sense that,

$$\tau_{2n+1}^{\mathrm{JLO}} = 0,$$

so in the case of the cochain τ^{JLO} the generating function (I.18) equals the full generating function for imaginary t, namely $\mathscr{J}(t; a; g) = \tau^{\mathrm{JLO}}(it; a; g)$. The expectation $\langle\ \rangle$ is defined in Sections IV and V in terms of the Radon transform of the heat-kernel regularized operator

$$X^{\mathrm{JLO}}(s) = a_0 e^{-s_0 Q^2}\, da_1 e^{-s_1 Q^2} \cdots da_n e^{-s_n Q^2}, \tag{I.20}$$

restricted to the sector $s_j > 0$, for $0 \leqslant j \leqslant n$. In this sector, with each $s_j > 0$, we show that $X^{\mathrm{JLO}}(s)$ is trace class. In Sections V–VI, we show that the definition of τ^{JLO} extends to the algebras $\mathfrak{A} \subset \mathfrak{J}_{\beta, \alpha}$ that we introduce here. The trace norm $\|X^{\mathrm{JLO}}(s)\|_1$ of $X^{\mathrm{JLO}}(s)$ may diverge on the hyperplane $s_0 + s_1 + \cdots + s_n = 1$ as any $s_j \to 0$. We will estimate the rate of divergence in terms of $\alpha + \beta$, using the bound on the norm $\|da\|_{\mathscr{T}(-\beta, \alpha)}$ of da in (I.4).

We establish the existence and the properties of the Radon transform, $\tau_n^{\mathrm{JLO}} = \mathrm{Tr}(\gamma U(g)(\int X^{\mathrm{JLO}}(s)\,ds))$, where the integral is taken over the hyperplane $s_0 + s_1 + \cdots + s_n = 1$. In this step, we use the assumption $\alpha + \beta < 1$. Our estimates also allow us to justify the interchange of the integration and the trace in the expression defining τ_n^{JLO},

$$\int (\mathrm{Tr}(\gamma U(g)\, X^{\mathrm{JLO}}(s)))\, ds = \mathrm{Tr}\left(\gamma U(g) \left(\int X^{\mathrm{JLO}}(s)\, ds\right)\right).$$

Thus we simply write

$$\tau_n^{\mathrm{JLO}}(a_0, ..., a_n; g) = \int_{s_j > 0} \mathrm{Tr}(\gamma U(\theta)\, a_0 e^{-s_0 Q^2}\, da_1 e^{-s_1 Q^2} \cdots da_n e^{-s_n Q^2})$$
$$\times \delta(s_0 + \cdots + s_n - 1)\, ds_0\, ds_1 \cdots ds_n. \tag{I.21}$$

In Proposition VI.2 we prove that for $\mathfrak{A} \subset \mathfrak{I}_{\beta, \alpha}$,

$$n^{1/2}\, |||\tau_n^{\mathrm{JLO}}|||^{1/n} \leqslant O(n^{-(1-\alpha-\beta)/2}), \tag{I.22}$$

which yields the required asymptotics (I.14). The behavior of (I.22) for large n is dependent on the analysis in Sections V–VI of the differentiability properties of elements of $\mathfrak{A}$. The importance of the order-μ, fractional-differentiability of elements of $\mathfrak{A}$ emerges once more. We require that the order of fractional derivative $\mu = 1 - \beta$ is greater than 0, which is part of the assumption (I.5). Our methods break down just at the point when elements of $\mathfrak{A}$ have no fractional derivative, or more precisely when $\alpha + \beta = 1$.

Parenthetically, we remark that in Section V we introduce sets $\{x_0, x_1, ..., x_n\}$ of n, operator-valued generalized functions, which we call sets of *vertices*. In our study of quantum harmonic analysis, we define expectations of such sets of operators, as a multilinear functional. Suppose there are $\alpha_j, \beta_j \geqslant 0$, with $\beta_{n+1} = \beta_0$, and such that for $0 \leqslant j \leqslant n$,

$$x_j \in \mathscr{T}(-\beta_j, \alpha_j) \qquad \text{with} \quad \alpha_j + \beta_{j+1} < 2. \tag{I.23}$$

Then we call the set $\{x_0, x_1, ..., x_n\}$ a *regular set* of vertices. The conditions (I.23) require that every $\alpha_j, \beta_j < 2$; however certain configurations of α_j, β_j may allow a particular vertex x_j in a regular set to have $\alpha_j + \beta_j$ close to 4.

We define the heat kernel regularization $X(s)$ of a regular set $\{x_0, x_1, ..., x_n\}$ of vertices as a trace class operator. For parameters $0 < s_j$, let

$$X(s) = (I + Q^2)^{-\beta_0/2}\, x_0 e^{-s_0 Q^2} \cdots x_n e^{-s_n Q^2} (I + Q^2)^{\beta_0/2}. \tag{I.24}$$

While this operator $X(s)$ is trace class, the trace norm on $\mathscr{H}$ may diverge as $s_j \to 0$. The operator $X(s)$ has a trace class Radon transform on the hyperplane $s_0 + \cdots + s_n = 1$, and the operators of taking the Radon transform and the trace commute. We write

$$\langle x_0, x_1, ..., x_n; g \rangle_n = \int_{s_j > 0} \mathrm{Tr}(\gamma U(\theta)\, x_O e^{-s_0 Q^2} x_1 e^{-s_1 Q^2} \cdots x_n e^{-s_n Q^2}) \times \delta\left(1 - \sum_{j=0}^{n} s_j\right) ds_0 \cdots ds_n. \tag{I.25}$$

This functional extends by continuity from its definition on the space of bounded vertices $x_j \in \mathscr{B}(\mathscr{H})$ to a multi-linear functional on the space of vertices $x_j \in \mathscr{T}(-\beta_j, \alpha_j)$ in a regular set. Furthermore we bound this expectation (I.25) in Corollary V.4 by

$$|\langle x_0, x_1, ..., x_n; g \rangle_n| \leqslant \frac{m(\eta_{\mathrm{local}})^{n+1}}{\Gamma(\eta_{\mathrm{tot}})} \mathrm{Tr}(e^{-Q^2/2}) \left(\prod_{j=0}^{n} \|x_j\|_{(-\beta_j, \alpha_j)} \right), \tag{I.26}$$

where $m(\eta_{\mathrm{local}}) < \infty$ is a constant. Here the exponents η that characterize the behavior of the expectations are defined by

$$\eta_j = \tfrac{1}{2}(2 - \alpha_j - \beta_{j+1}), \qquad \eta_{\mathrm{local}} = \min_{0 \leqslant j \leqslant n} \{\eta_j\}, \qquad \text{and} \qquad \eta_{\mathrm{tot}} = \sum_{j=0}^{n} \eta_j.$$

We say that η_{local} characterizes the local regularity of the expectations (I.25), while $\eta_{\mathrm{global}} = \eta_{\mathrm{tot}}/(n+1)$ characterizes the global regularity of sets of such expectations as a function of n.

In Section VI we also return to the fact that the functional τ^{JLO} is a cocycle in $\mathscr{C}$, namely

$$\partial \tau^{\mathrm{JLO}} = 0. \tag{I.27}$$

This was previously known in the differentiable case, for which $\alpha = \beta = 0$. In Section VI we also analyze the generating functional corresponding to τ^{JLO}, namely

$$\mathscr{J}^{JLO}(t; a; g) = \mathrm{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da}), \tag{I.28}$$

and the Gaussian transform $(R\mathscr{J})(0; a; g)$,

$$\mathfrak{Z}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \mathscr{J}(t; a; g)\, dt. \tag{I.29}$$

In the case that $\mathscr{J}(t; a; g) = \mathscr{J}^{\mathrm{JLO}}(t; a; g)$, we indicate the dependence of $\mathfrak{Z}$ on Q. We have

$$\mathfrak{Z}^{Q}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da})\, dt. \tag{I.30}$$

We prove in Sections V–VI that the functional (I.28–30) exist for all $a \in \mathfrak{A}$, when $\mathfrak{A}$ is contained in one of the allowed interpolation spaces $\mathfrak{J}_{\beta, \alpha}$.

Let a be invariant, $a \in \mathfrak{A}^{\mathfrak{G}}$, and also satisfy $a^2 = I$. Then the pairing (I.17) equals

$$\langle \tau^{\mathrm{JLO}}, a \rangle = \mathfrak{Z}^{Q}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da})\, dt. \tag{I.31}$$

In other words, $\mathfrak{Z}^{Q}(a; g)$ can be written as a trace of a heat kernel.[4] For the case $a = I$, we have $da = 0$, and (I.31) reduces to the equivariant index

$$\mathfrak{Z}^{Q}(I; g) = \operatorname{Tr}(\gamma U(g)\, e^{-Q^2}). \tag{I.32}$$

The expression for $\mathfrak{Z}^{Q}(a; g)$ in (I.31) is fundamental, and it provides a generalization of the McKean–Singer heat-kernel representation of the index $\operatorname{Tr}(\gamma e^{-Q^2})$. In this context, the cochain τ^{JLO} is the equivariant Chern character for the fractionally-differentiable structure on $\mathfrak{A}$.

In Section VII we return to the question of the precise sense in which (I.31) is a homotopy invariant. Here we refer to how $\mathfrak{Z}^{Q(\lambda)}(a; g)$ changes as we vary a parameter λ on which Q depends, and which defines a family of cochains $\tau^{\mathrm{JLO}}(\lambda)$. Our basic result can be summarized in the following way:

Main Theorem. *Consider $a \in \mathfrak{A}^{\mathfrak{G}} \subset \mathfrak{J}_{\beta, \alpha}$ for some $0 \leqslant \alpha, \beta$, with $\alpha + \beta < 1$, and also satisfying $a^2 = I$. If $\mathfrak{Z}^{Q(\lambda)}(a; g)$ is continuously differentiable with respect to λ, and if the derivative $\partial \mathfrak{Z}^{Q(\lambda)}(a; g)/\partial\lambda$ equals the expression obtained by exchanging the order of differentiation with summations, integrations, and traces, then $\partial \mathfrak{Z}^{Q(\lambda)}(a; g)/\partial\lambda = 0$.*

[4] In the physics literature, $\mathfrak{Z}^{Q}(a; g)$ is called a *partition function*. Such traces often arise in statistical physics or in quantum theory. The Laplace operator which generates the heat kernel is Q^2, perturbed by $it\, da$. Note that if a is self-adjoint, $a = a^*$, then so is the perturbation $it\, da$. We mentioned earlier that it might be the case that a functional $\operatorname{Tr}(\gamma U(g) \cdot e^{-Q^2})$ could be represented as a functional integral, given by a measure $d\mu_g$, namely $\operatorname{Tr}(\gamma U(g)\, a e^{-Q^2}) = \int a\, d\mu_g$. If this is the case, and if in addition both a and da can be realized as functions on path space, then the representation (I.31) further simplifies. The Gaussian integral can be carried out, giving $\mathfrak{Z}^{Q}(a; g) = \int a e^{-(da)^2/4}\, d\mu_g$. We have presented this formula in [14].

In other words, the pairing $\mathfrak{Z}^{Q(\lambda)}(a; g)$ given in (I.31) is a homotopy invariant under such differentiable deformations of $Q(\lambda)$. In Section VII we give sufficient conditions on $Q(\lambda)$ and on $\mathfrak{A}$ under which these hypotheses hold, and we also establish the invariance of (I.31). These assumptions apply to a wide variety of examples. We suppose that the family $Q(\lambda)$ has the form

$$Q(\lambda) = Q + q(\lambda), \tag{I.33}$$

where $q(\lambda)$ maps between two Sobolev spaces,

$$q(\lambda)\colon \mathscr{H}_1 \to \mathscr{H}_0, \tag{I.34}$$

with (an appropriate) norm less than 1. This condition (I.34) can also be described by saying that $q(\lambda)$ is a perturbation of Q in the sense of T. Kato, see [23]. We also assume that there is a bounded map $\dot{q}(\lambda)$ from $\mathscr{H}_1$ to $\mathscr{H}_0$, that is continuous in λ, and such that the difference quotient is norm convergent to the derivative in the space $\mathscr{T}(-1, 1)$ of bounded, linear maps from $\mathscr{H}_1$ to $\mathscr{H}_0$. This means that

$$\lim_{\lambda' \to \lambda} \left(\frac{q(\lambda) - q(\lambda')}{\lambda - \lambda'} - \dot{q}(\lambda) \right) = 0. \tag{I.35}$$

These assumptions form the basis of our definition in Section VII of a regular family $Q(\lambda)$. We also define a corresponding regular family of fractionally-differentiable structures $\{\mathscr{H}, Q(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ on $\mathfrak{A}$ on an algebra $\mathfrak{A} \subset \mathfrak{J}_{\beta, \alpha}$. Under these hypotheses we prove that

$$\lambda \mapsto \tau^{\mathrm{JLO}}(\lambda) \tag{I.36}$$

is a continuously differentiable function from an interval $\Lambda \subset \mathbb{R}$ to $\mathscr{C}$. Here $\mathscr{C}$ carries the natural topology defined in Section II.

These simple conditions on $Q(\lambda)$ and its derivative allow a complete analysis of the trace-class nature and differentiability in the appropriate Schatten norm of $\lambda \mapsto \exp(-sQ(\lambda)^2)$, for $s > 0$. Our hypotheses cover many applications, and as a consequence, the derivative $d\tau^{\mathrm{JLO}}(\lambda)/d\lambda$ can be computed by differentiating the expression (I.21) under the integral, and under the trace.

Calculation of the derivative shows that there is a cochain $h \in \mathscr{C}$ with coboundary ∂h such that

$$\frac{d\tau^{\mathrm{JLO}}(\lambda)}{d\lambda} = \partial h(\lambda). \tag{I.37}$$

Integrating this relation we obtain

$$\tau^{\mathrm{JLO}}(\lambda) = \tau^{\mathrm{JLO}}(\lambda') + \partial H(\lambda, \lambda'), \tag{I.38}$$

where $H(\lambda, \lambda') \in \mathscr{C}$. Since the pairing (I.16) vanishes on coboundaries, $\langle \partial H, a \rangle = 0$. The linearity of the pairing in τ ensures

$$\mathfrak{Z}^{Q(\lambda)}(a; g) = \mathfrak{Z}^{Q(\lambda')}(a; g). \tag{I.39}$$

In other words, $\mathfrak{Z}^{Q(\lambda)}$ does not depend on λ, and so it is a homotopy invariant. As a special case of this result, we show that the definition of τ_n^{JLO}, where we choose a particular hyperplane $s_0 + s_1 + \cdots + s_n = \beta$ for the Radon transform of (I.19), gives a pairing independent of β. But more generally, $\mathfrak{Z}^Q$ remains unchanged under a regular deformation of a parameter in a potential, of a metric, etc.

We comment in Section VIII on using the homotopy invariance of $\mathfrak{Z}(a; g)$ in various settings, as a tool to study or to compute these quantities. In particular, we study the possibility that the family $\{\mathscr{H}, Q(\lambda), \gamma, U(g), \mathfrak{A}\}$ for $\lambda \in \Lambda$, may have a singularity at one endpoint of an interval Λ. We give a method to study such an endpoint, by introducing a family $\tau^{\mathrm{JLO}}(\lambda, \varepsilon)$ of approximations to $\tau^{\mathrm{JLO}}(\lambda)$.

In Section IX we generalize this approach to cover the case when Q can be split into the sum of "independent" parts $Q = (Q_1 + Q_2)/\sqrt{2}$, such that $(Q_1 + Q_2)^2 = (Q_1)^2 + (Q_2)^2$. We assume here that Q_1 and $(Q_2)^2$, are $\mathfrak{G}$-invariant, but we do not make that assumption about Q_2. With further assumptions on a, the most important being that $(Q_1)^2 - (Q_2)^2$ commutes with a, we obtain a modified formula for a pairing. Namely, with $d_1 a = [Q_1, a]$ and $a^2 = I$, the quantity

$$\mathfrak{Z}^{\{Q_j\}}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, d_1 a})\, dt \tag{I.40}$$

is invariant under regular deformations of $Q_1(\lambda)$ and $Q_2(\lambda)$ which leave $Q_1^2 - Q_2^2$ fixed.

II. COCHAIN AND COBOUNDARY OPERATORS

Functionals on $\mathfrak{A}$ (maps from $\mathfrak{A}$ to $\mathbb{C}$) play a central role in non-commutative geometry. Important examples include integrals or traces of operators in $\mathfrak{A}$, and sometimes they are called *expectations*. We introduce three linear spaces $\mathscr{N} \subset \mathscr{C} \subset \mathscr{D}$ of multilinear functionals on $\mathfrak{A}$. These are *spaces of cochains* on $\mathfrak{A}$.

The space $\mathscr{C}$ plays a fundamental role here, and we call $\mathscr{C}$ the space of entire cochains. The algebraic questions we consider concern the properties of certain linear transformations which map $\mathscr{C}$ into itself. We only use the other spaces $\mathscr{D}$ and $\mathscr{N}$ to simplify the discussion of linear transformations of $\mathscr{C}$ into $\mathscr{C}$. Each of these spaces of functionals has an invariance under the group $\mathfrak{G}$ that acts on $\mathfrak{A}$, so we sometimes denote the spaces by $\mathscr{D} = \mathscr{D}(\mathfrak{A}; \mathfrak{G})$, $\mathscr{C} = \mathscr{C}(\mathfrak{A}; \mathfrak{G})$, etc. However, in order to simplify notation, we generally suppress the dependence of $\mathscr{D}$ on $\mathfrak{A}$ or on $\mathfrak{G}$.

A coboundary operator ∂ is a continuous, linear transformation from $\mathscr{C}$ to $\mathscr{C}$, with the property $\partial^2 = 0$. We study three coboundary operators in $\mathscr{C}$: the Hochschild operator b, the Connes operator B, and their sum $\partial = b + B$ which is the coboundary operator defining entire cyclic cohomology.

II.1. *Spaces of Cochains*

The Space $\mathscr{D}$

Define $\mathscr{D}_n$ as a vector space of functionals on $\mathfrak{A}^{n+1} \times \mathfrak{G}$, where every $f_n \in \mathscr{D}_n$ is an $(n+1)$-continuous, multilinear functional on $\mathfrak{A}$, that is also a continuous function on $\mathfrak{G}$. We denote the values of f_n by $f_n(a_0, ..., a_n; g)$, where $a_j \in \mathfrak{A}$ and $g \in \mathfrak{G}$. We assume that f_n is invariant on the diagonal, in the sense that

$$f_n(a_0^{g^{-1}}, a_1^{g^{-1}}, ..., a_n^{g^{-1}}; g) = f_n(a_0, ..., a_n; g). \tag{II.1}$$

Define the norm $|||f_n|||$ of f_n with respect to the norm $|||\cdot|||$ on $\mathfrak{A}$ and the sup over $\mathfrak{G}$. Specifically,

$$|||f_n||| = \sup_{\substack{|||a_j||| \leqslant 1 \\ g \in \mathfrak{G}}} |f_n(a_0, ..., a_n; g)|. \tag{II.2}$$

An identity $f_n = g_n$ in $\mathscr{D}_n$ means $f_n(a_0, ..., a_n; g) = g_n(a_0, ..., a_n; g)$ for all $a_j \in \mathfrak{A}$, and all $g \in \mathfrak{G}$. We assume that identities, expressed without explicitly stating the $\mathfrak{G}$-dependence, hold pointwise in $\mathfrak{G}$.

Now define the space of sequences $\mathscr{D}$ as the space $\bigoplus_{n=0}^{\infty} \mathscr{D}_n$ of elements

$$f = \{f_n : f_n \in \mathscr{D}_n, n \in \mathbb{Z}_+\},$$

restricted to those sequences that in addition satisfy the *entire* condition:

$$\lim_{n \to \infty} n^{1/2} \, |||f_n|||^{1/n} = 0. \tag{II.3}$$

The Space $\mathscr{C} \subset \mathscr{D}$

The key property of the space of entire cochains $\mathscr{C}$ is that elements of $\mathscr{C}$ vanish when evaluated on I except in the 0th place. More precisely, the elements $f \in \mathscr{C}$ are those elements $f \in \mathscr{D}$ such that for every $n \geqslant 1$,

$$f_n(a_0, ..., a_n; g) = 0$$

when any a_j, other than a_0, satisfies $a_j = I$. (The motivation for this definition is to think of the space of cochains $\mathscr{C}$ as the space of integrals of quantized forms $\int a_0 \, da_1 \cdots da_n$ of forms $a_0 \, da_1 \cdots da_n$. Any (graded) derivation d annihilates the identity I, so $\mathscr{C}$ is a natural space of cochains.)

The Space $\mathscr{N} \subset \mathscr{C}$

The subspace $\mathscr{N}$ of $\mathscr{C}$ is the annihilator of I. In other words, $f \in \mathscr{D}$ is in $\mathscr{N}$ if each $f_n(a_0, ..., a_n; g)$ vanishes whenever *any* $a_j = I$, for $0 \leqslant j \leqslant n$. Thus $f \in \mathscr{C}$ belongs to $\mathscr{N}$ if for every $n \in \mathbb{Z}_+$, $f_n(I, a_1, ..., a_n; g) = 0$.

II.2. *Elementary Linear Transformations*

We define a number of bounded, linear transformations, $\mathscr{S} : \mathscr{D} \to \mathscr{D}$. In other words, all these maps have domain $\mathscr{D}$ and range in $\mathscr{D}$. We remark below specifically which of these operators also map $\mathscr{C}$ into $\mathscr{C}$, map $\mathscr{C}$ into $\mathscr{N}$, etc.

Let $\mathscr{S}$ denote a generic linear transformation $\mathscr{S} : \mathscr{D} \to \mathscr{D}$. Since $\mathscr{D}$ is contained in the direct sum $\bigoplus_n \mathscr{D}_n$ and since $\mathscr{S}$ is linear, it is sufficient to define $\mathscr{S}$ on each $\mathscr{D}_n$, $n \in \mathbb{Z}_+$. We often denote by $\mathscr{S}_n$ the action of $\mathscr{S}$ of $\mathscr{D}_n$. In all the examples here, $\mathscr{S}$ is tridiagonal: the range of $\mathscr{S}_n$ lies in $\mathscr{D}_{n-1} \oplus \mathscr{D}_n \oplus \mathscr{D}_{n+1}$.

Cyclic Transposition: $T: \mathscr{N} \to \mathscr{N}$

$$(Tf_n)(a_0, ..., a_n; g) = (-1)^n \, f_n(a_n^{g^{-1}}, a_0, ..., g_{n-1}; g). \tag{II.4}$$

Note that as a consequence of the invariance (II.1), it is true that

$$T_n^{n+1} = I. \tag{II.5}$$

Cyclic Antisymmetrization: $A: \mathscr{N} \to \mathscr{N}$

The cyclic antisymmetrization A_n on $\mathscr{D}_n$ is defined by $A_n = \sum_{j=0}^n T_n^j$. Then (II.5) ensures that for any $s \in \mathbb{Z}$,

$$A_n = \sum_{j=0}^{n} T_n^{j+s}. \tag{II.6}$$

Annihilation: $U\colon \mathscr{C} \to \mathscr{N}$, $U\colon \mathscr{N} \to 0$

The annihilation transformation U maps $\mathscr{D}_n$ into $\mathscr{D}_{n-1}$. It is defined by

$$U_0 f_0 = 0, \qquad (U_n f_n) = (a_0, ..., a_{n-1}) = f_n(I, a_0, ..., a_{n-1}). \tag{II.7}$$

As U acts on the first variable, $U\colon \mathscr{C} \to \mathscr{N}$ and $U\colon \mathscr{N} \to 0$.

Creation: V

The creation operator maps $\mathscr{D}_n$ to $\mathscr{D}_{n+1}$, according to the rule

$$(V_n f_n)(a_0, ..., a_{n+1}) = f_n(a_0 a_1, a_2, ..., a_{n+1}). \tag{II.8}$$

We also introduce $V(r)\colon \mathscr{D}_n \to \mathscr{D}_{n+1}$, defined by conjugating V by T^r, namely

$$V(r) = T^{-r} V T^r.$$

Then $V(0) = V$, and acting on $\mathscr{D}_n$,

$$(V(r)_n f_n)(a_0, ..., a_{n+1}) = \begin{cases} (-1)^r f_n(a_0, ..., a_r a_{r+1}, ..., a_{n+1}), & 0 \leqslant r \leqslant n, \\ (-1)^{n+1} f_n(a_{n+1}^{g^{-1}} a_0, a_1, ..., a_n), & r = n+1. \end{cases} \tag{II.9}$$

These definitions yield the following elementary identities:

$$U_{n+1} V_n = I_n, \tag{II.10}$$

$$U_{n+1} V(r)_n + V(r-1)_{n-1} U_n = 0, \qquad 1 \leqslant r \leqslant n, \tag{II.11}$$

$$U_{n+1} V(n+1)_n = -T_n, \tag{II.12}$$

and

$$V(r)_{n+1} V(s)_n + V(s+1)_{n+1} V(r)_n = 0, \qquad 0 \leqslant r \leqslant s \leqslant n+1. \tag{II.13}$$

II.3. *Coboundary Operators*

The Hochschild Coboundary: $b\colon \mathscr{C} \to \mathscr{C}$

The Hochshild coboundary operator b maps $\mathscr{D}_n$ to $\mathscr{D}_{n+1}$. It is defined by

$$b_n = \sum_{r=0}^{n+1} V(r)_n. \tag{II.14}$$

Thus

$$(b_n f_n)(a_0, ..., a_{n+1}) = \sum_{j=0}^{n} (-1)^j f_n(a_0, ..., a_j a_{j+1}, ..., a_{n+1}) + (-1)^{n+1} f_n(a_{n+1}^{g^{-1}} a_0, a_1, ..., a_n). \tag{II.15}$$

Note that

$$\begin{aligned} b_{n+1} b_n &= \sum_{r=0}^{n+2} \sum_{s=0}^{n+1} V(r)_{n+1} V(s)_n \\ &= \sum_{0 \leqslant r \leqslant s \leqslant n+1} (V(r)_{n+1} V(s)_n + V(s+1)_{n+1} V(r)_n) = 0, \end{aligned} \tag{II.16}$$

where the last equality follows from (II.13). Hence, on the large space of cochains $\mathscr{D}$, the operator b satisfies

$$b^2 = 0. \tag{II.17}$$

In other words, we have proved that b is a *coboundary operator* on $\mathscr{D}$.

Finally we verify that b acts on $\mathscr{C}$, namely $b: \mathscr{C} \to \mathscr{C}$, from which we conclude that b is also a coboundary operator on $\mathscr{C}$. Assume that $f \in \mathscr{C}$; we need to show that $bf \in \mathscr{C}$. Evaluate $(b_n f_n)(a_0, ..., a_{n+1})$ using (II.15), and also assume that $a_k = I$, for some fixed k with $1 \leqslant k \leqslant n+1$. Then

$$(b_n f_n)(a_0, ..., a_{n+1}) = \begin{cases} ((-1)^{k-1} + (-1)^k) f_n(a_0, ..., a_{k-1}, a_{k+1}, ..., a_{n+1}), & k \leqslant n \\ ((-1)^n + (-1)^{n+1}) f_n(a_0, ..., a_n), & k = n+1. \end{cases} \tag{II.18}$$

In both cases the right-hand side of (II.18) vanishes, so $b: \mathscr{C} \to \mathscr{C}$.

The Connes Coboundary: $B: \mathscr{C} \to \mathscr{N}$

The Connes coboundary operator B is defined on $\mathscr{C}$ by

$$B = AU. \tag{II.19}$$

In particular, $B_n = A_{n-1} U_n$, and on $\mathscr{C}_n$ this can be written as

$$(B_n f_n)(a_0, ..., a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} f_n(I, a_{n-j}^{g^{-1}}, ..., a_{n-1}^{g^{-1}}, a_0, ..., a_{n-j-1}). \tag{II.20}$$

Since $U: \mathscr{C} \to \mathscr{N}$ and $A: \mathscr{N} \to \mathscr{N}$, it follows that $B: \mathscr{C} \to \mathscr{N}$. But $U: \mathscr{N} \to 0$. Thus we have shown that B is a coboundary operator on $\mathscr{C}$, namely[5]

$$B^2: \mathscr{C} \to 0. \tag{II.21}$$

Finally we verify that on $\mathscr{D}$, the two coboundary operators satisfy

$$Bb + bB = 0. \tag{II.22}$$

In fact

$$\begin{aligned}(Bb + bB)_n &= B_{n+1} b_n + b_{n-1} B_n \\ &= A_n U_{n+1} \sum_{r=0}^{n+1} V(r)_n + \sum_{r=0}^{n} V(r)_{n-1} A_{n-1} U_n. \end{aligned} \tag{II.23}$$

Using (II.10, 12), the $r=0$ and $r=n+1$ terms in the first sum in (II.23) are equal to $A_n(I_n - T_n)$. By (II.5–6), this is zero. For the remaining terms in (II.23), we use (II.11) to obtain

$$(Bb + bB)_n = -A_n \sum_{r=0}^{n-1} V(r)_{n-1} U_n + \sum_{r=0}^{n} V(r)_{n-1} A_{n-1} U_n. \tag{II.24}$$

Again by (II.5–6),

$$\begin{aligned}\sum_{r=0}^{n-1} A_n V(r)_{n-1} U_n &= \sum_{j=0}^{n} \sum_{r=0}^{n-1} T_n^j V_{n-1} T_{n-1}^r U_n = A_n V_{n-1} A_{n-1} U_n \\ &= \sum_{r=0}^{n} V(r)_{n-1} A_{n-1} U_n. \end{aligned} \tag{II.25}$$

Hence we have shown that (II.24) vanishes and (II.22) holds.

The Entire Coboundary: $\partial: \mathscr{C} \to \mathscr{C}$

The entire coboundary operator is the sum of b and B. Define

$$\partial = b + B. \tag{II.26}$$

Both b and B act on $\mathscr{C}$ and are coboundaries. By (II.22),

$$\partial^2 = 0. \tag{II.27}$$

[5] Note that on the space $\mathscr{D}$, the Connes operator has an additional term, namely $B = AU(I - T^{-1})$. The term UT^{-1} vanishes on $\mathscr{C}$. It can be checked that $(AU(I - T^{-1}))^2 = 0$ on $\mathscr{D}$, see for example [3].

Alternatively, we could use the coboundary operator $\bar{\partial} = b - B$ which also is nilpotent.

The Cocycle Condition: $\partial\tau = 0$

A cochain $\tau \in \mathscr{C}(\mathfrak{A})$ is a cocycle if

$$\partial\tau = 0. \tag{II.28}$$

Equivalence Classes of Cochains: $[f]$

We define equivalence classes in $\mathscr{C}(\mathfrak{A})$ by considering entire cochains, modulo coboundaries. For $f \in \mathscr{C}(\mathfrak{A})$ the equivalence class $[f]$ is

$$[f] = \{f + \partial G\colon G \in \mathscr{C}(\mathfrak{A})\}. \tag{II.29}$$

II.4. *Cochains as a Countably-Normed Space*

We have already defined linear transformations on $\mathscr{D}$ and on $\mathscr{C}$. We can introduce a topology on these spaces given by a countable set of norms,

$$|||f|||_n = \sup_{m \in \mathbb{Z}_+} n^m m!^{1/2}\, |||f_m|||.$$

Here $n \in \mathbb{Z}_+$. In this topology, the space $\mathscr{D}$ or $\mathscr{C}$ is a countably-normed space. Let us consider a family of cochains indexed by a parameter λ ranging over an index set Λ. (For example, Λ might be $\mathbb{Z}_+$, an interval (λ_1, λ_2) on the real line, a subset of $\mathscr{C}$, etc.)

Let $\mathfrak{F}$ denote a family of cochains $f(\lambda)$,

$$\mathfrak{F} = \{f(\lambda)\colon f(\lambda) \in \mathscr{C} \text{ (or } \mathscr{D}\text{)}, \lambda \in \Lambda\}. \tag{II.30}$$

We say that $\mathfrak{F} \in \mathscr{C}$ (or $\mathscr{D}$) is *bounded*, *convergent*, *or differentiable* if it has this property in the topology of a countably-normed space. We say that the family is *Cauchy* as $\lambda \to \lambda_0$ if $\mathfrak{F}$ is bounded in $\mathscr{C}$ (or $\mathscr{D}$) and for every n,

$$\lim_{\lambda, \lambda' \to \lambda_0} |||f(\lambda) - f(\lambda')|||_n = 0. \tag{II.31}$$

A family $\mathfrak{F}$ that is Cauchy as $\lambda \to \lambda_0$ has a limit f in $\mathscr{C}$ (or $\mathscr{D}$); in this case, for all $n \in \mathbb{Z}_+$,

$$|||f|||_n \leqslant \alpha_n, \qquad \text{and} \qquad \lim_{\lambda \to \lambda_0} |||f(\lambda) - f|||_n = 0. \tag{II.32}$$

We write

$$\lim_{\lambda \to \lambda_0} f(\lambda) = f. \tag{II.33}$$

The standard notions of $\mathfrak{F}$ being closed or compact follow.

If Λ is an open, real interval (λ_1, λ_2), we say that $f(\lambda)$ is differentiable at $\lambda_0 \in \Lambda$ if for $\lambda \in \Lambda$,

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} = g$$

exists. Then $f'(\lambda_0) = g$, etc.

Note that the operators introduced in Section III.2–3 are all bounded operators in the countably-normed topology. For example, the number operator N defined by

$$N: \mathscr{D}_n \to \mathscr{D}_n, \qquad \text{and} \qquad N_n f_n = n f_n \tag{II.34}$$

can be estimated by

$$|||Nf|||_m \leqslant |||f|||_{3m}, \tag{II.35}$$

a consequence of $n \leqslant 3^n$. Similarly, all the operators mentioned are bounded. We state this without explicit proof as

PROPOSITION II.1. *The operators T, A, U, V, b, B, and ∂ are all bounded, linear operators on $\mathscr{D}$ in the countably-normed topology.*

III. PAIRING A COCHAIN

Let us define a pairing $\langle \tau, a \rangle$ between a cochain $\tau \in \mathscr{C}(\mathfrak{A})$, and a root of unity $a \in \mathfrak{A}^{\mathfrak{G}}$, as a non-linear functional $\tau: \mathfrak{A}^{\mathfrak{G}} \to \mathbb{C}$ which depends on τ only through its equivalence class $[\tau]$. Since $\langle \tau, a \rangle$ is linear in τ, it must be true that

$$\langle \partial G, a \rangle = 0 \tag{III.1}$$

for all $G \in \mathscr{C}(\mathfrak{A})$ and all $a \in \mathfrak{A}^{\mathfrak{G}}$. This condition allows us to determine a pairing, at least for even elements $\{\tau_{2n}\} \subset \mathscr{C}(\mathfrak{A})$.

III.1. $\langle \partial G, a \rangle = 0$ *Determines a Pairing*

There is a canonical way to pair a cochain $\tau \in \mathscr{C}(\mathfrak{A})$ with an element $a \in \mathfrak{A}^{\mathfrak{G}}$, such that the result $\langle \tau, a \rangle$ is linear in τ. We suppose that the dependence of $\langle \tau, a \rangle$ on τ_n is a linear function of either

$$\tau_n(a, a, ..., a) \qquad \text{or} \qquad \tau_n(I, a, a, ..., a). \tag{III.2}$$

Under this assumption, the general form of $\langle \tau, a \rangle$ is determined by a sequence of numerical coefficients α_n, β_n independent of τ and a, namely

$$\langle \tau, a \rangle = \sum_{n=0}^{\infty} \alpha_n \tau_n(a, a, ..., a) + \sum_{n=1}^{\infty} \beta_n \tau_n(I, a, a, ..., a). \tag{III.3}$$

We use the requirement $\langle \partial G, a \rangle = 0$, along with the assumption that the odd components of τ vanish, to limit the form of the pairing.

PROPOSITION III.1. *Let $a \in \mathfrak{A}^{\mathfrak{G}}$ satisfy $a^2 = I$. Consider a pairing such that*

$$\langle \tau, a \rangle = \sum_{n=0}^{\infty} \alpha_{2n} \tau_{2n}(a, a, ..., a) + \sum_{n=1}^{\infty} \beta_{2n} \tau_{2n}(I, a, a, ..., a). \tag{III.4}$$

Suppose for all $G \in \mathscr{C}(\mathfrak{A})$, that

$$\langle \partial G, a \rangle = 0.$$

Then

$$\langle \tau, a \rangle = \alpha_0 \sum_{n=0}^{\infty} \left(-\frac{1}{4} \right)^n \frac{(2n)!}{n!} \tau_{2n}(a, a, ..., a). \tag{III.5}$$

Remarks. (1) We remark that it is no loss of generality to normalize the pairing (III.4) so that $\alpha_0 = 1$. With this normalization, $\langle \tau, I \rangle = \tau_0(I)$.

(2) Proposition III.1 replaces Connes "normalization" condition [4], which is unnecessary for this complex. Furthermore, Proposition III.1 is dual to the related result of Getzler and Szenes [12]. These other results concern a pairing of τ with idempotents $p^2 = p \in \mathfrak{A}^{\mathfrak{G}}$, rather than a pairing with operators $a \in \mathfrak{A}^{\mathfrak{G}}$ which are square roots of unity, $a^2 = I$. There is a one-to-one correspondence between square roots of unity a and idempotents p given by $a = 2p - I$. In terms of these variables, the pairing introduced by Connes, which we denote $\langle \tau, p \rangle^C$, equals our pairing for

$a=I$. In general, it is the average of our pairing and its value at $a=I$, namely

$$\langle \tau, p\rangle^{C} = \tfrac{1}{2}\langle \tau, a\rangle + \tfrac{1}{2}\langle \tau, I\rangle. \tag{III.6}$$

(3) In Section IV we introduce a cochain $\tau^{\mathrm{JLO}} \in \mathscr{C}(\mathfrak{A})$ which will be the focus of much of the remainder of this paper. This cochain automatically has the property $\tau^{\mathrm{JLO}}_{2n+1}=0$ of the form (III.4).

Proof. For $a \in \mathfrak{A}^{\mathfrak{G}}$, it is the case that $a=a^{\gamma}=a^{g^{-1}}=a^{g^{-1}\gamma}$. Assume $a^2=I$ and let n be even. Then using (II.15) and (II.20),

$$\begin{aligned} b_{n-1}G_{n-1}(a, a, ..., a) &= G_{n-1}(a^2, a, ..., a) + G_{n-1}(a^{g^{-1}}a, a, ..., a)\\ &= 2G_{n-1}(I, a, ..., a), \end{aligned} \tag{III.7}$$

$$B_{n+1}G_{n+1}(a, ..., a) = (n+1)\, G_{n+1}(I, a, ..., a),$$

and

$$(b_{n-1}G_{n-1})(I, a, ..., a) = 2G_{n-1}(a, ..., a). \tag{III.8}$$

As $B\colon \mathscr{C} \to \mathscr{N}$, we also have $(B_{n+1}G_{n+1})(I, a, ..., a)=0$.

Now let $\tau = \partial G$. Thus for $n \geqslant 1$,

$$\tau_{2n}(a, ..., a) = 2G_{2n-1}(I, a, ..., a) + (2n+1)\, G_{2n+1}(I, a, ..., a), \tag{III.9}$$

and

$$\tau_{2n}(I, a, ..., a) = 2G_{2n-1}(a, ..., a). \tag{III.10}$$

Inserting the identities (III.9–10) into (III.4) yields

$$\begin{aligned} \langle \tau, a\rangle = &\sum_{n=0}^{\infty} (2\alpha_{2n+2} + (2n+1)\,\alpha_{2n})\, G_{2n+1}(I, a, a, ..., a)\\ &+ \sum_{n=1}^{\infty} 2\beta_{2n} G_{2n-1}(a, a, ..., a). \end{aligned} \tag{III.11}$$

The vanishing of (III.11) for all $G \in \mathscr{C}$ ensures vanishing of the coefficients in (III.11). Thus $\beta_{2n}=0$ for $n \geqslant 1$, and $(2n+1)\,\alpha_{2n} + 2\alpha_{2n+2}=0$. This recursion relation is satisfied by $\alpha_{2n} = (-1/4)^n\,(2n)!\,(n!)^{-1}\,\alpha_0$. Substituting the coefficients α_{2n} and β_{2n} into (III.4) yields (III.5).

The space $\mathrm{Mat}_m(\mathfrak{A})$ is the space of $m \times m$ matrices with entries in $\mathfrak{A}$. Elements $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ satisfying $a^2=I$ are matrices $a=\{a_{ij}\}$, for which

$$\sum_{j=1}^{m} a_{ij}a_{jn} = \delta_{in}.$$

The pairing $\langle \tau, a \rangle$ for $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$, $a^2 = I$, is defined by

$$\langle \tau, a \rangle = \sum_{n=0}^{\infty} (-1/4)^n \frac{(2n)!}{n!} \sum_{1 \leqslant i_0, i_1, ..., i_{2n} \leqslant m} \tau_{2n}(a_{i_0 i_1}, a_{i_1 i_2}, ..., a_{i_{2n} i_0}). \quad \text{(III.12)}$$

We use a shorthand notation for (III.12), where $m \times m$ matrices which enter τ are multiplied. Thus we write (III.12) as

$$\langle \tau, a \rangle = \sum_{n=0}^{\infty} (-1/4)^n \frac{(2n)!}{n!} \operatorname{tr} \tau_{2n}(a, a, ..., a). \quad \text{(III.13)}$$

In (III.13), tr denotes the trace in the space of $m \times m$ matrices $\mathrm{Mat}_m(\mathfrak{A})$ with entries in $\mathfrak{A}$. We summarize this discussion by stating

Proposition III.2. *Let $\tau \in \mathscr{C}$ and $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ with $a^2 = I$. Then the pairing* (III.13) *exists. Furthermore, $\langle \tau, a \rangle$ depends on τ only through its class $[\tau]$, so*

$$\langle \tau, a \rangle = \langle \tau + \partial G, a \rangle,$$

where $G \in \mathscr{C}(\mathfrak{A})$, and

$$\langle \partial G, a \rangle = 0. \quad \text{(III.14)}$$

In the case $m = 1$, the pairing reduces to (III.5) normalized so $\alpha_0 = 1$. The proof of (III.13) reduces step by step to the proof of (III.5).

III.2. *A Generating Functional $\mathscr{J}(z; a)$ for τ*

Let us define a generating function $\mathscr{J}(z; a)$ for τ by

$$\mathscr{J}(z; a) = \sum_{n=0}^{\infty} (-z^2)^n \operatorname{tr} \tau_{2n}(a, a, ..., a). \quad \text{(III.15)}$$

As a consequence of the assumption that τ is an entire cochain, and that $|||a||| \leqslant M$, we have

$$n^{1/2} \, |\tau_n(a, ..., a)|^{1/n} \leqslant o(1). \quad \text{(III.16)}$$

Hence the series (III.15) converges to define an entire function of z of order at most two.

Let $h(z)$ denote an entire function of one variable,

$$h(z) = \sum_{n=0}^{\infty} h_n z^n. \quad \text{(III.17)}$$

Consider the class of entire functions $\mathscr{E}_\eta$, for $\eta \geqslant 0$, consisting of $h(z)$ such that

$$n^\eta \, |h_n|^{1/n} \leqslant o(1) \qquad \text{as} \quad n \to \infty. \tag{III.18}$$

We consider the topology on $\mathscr{E}_\eta$ defined as follows: A family of functions $\{h^{(\lambda)}\} \subset \mathscr{E}_\eta$, indexed by λ in an index set Λ, is bounded in $\mathscr{E}_\eta$ if there is a bound $n^\eta \, |h_n^{(\lambda)}|^{1/n} \leqslant o(1)$, where $o(1)$ is independent of λ. The sequence *converges* to $h \in \mathscr{E}_\eta$, if it is bounded and as $\lambda \to \lambda_0$, also $h_n^{(\lambda)} \to h_n$. The space $\mathscr{E}_0$ contains all entire functions.

The operator $D = d/dz$ maps $\mathscr{E}_\eta$ into $\mathscr{E}_\eta$, for all $\eta \geqslant 0$. Likewise, multiplication by z maps $\mathscr{E}_\eta$ to $\mathscr{E}_\eta$. Consider the operator

$$R = \exp(D^2/4) = \sum_{n=0}^{\infty} \left(\frac{D^{2n}}{4^n n!} \right). \tag{II.19}$$

For bounded or L^2 functions, the operator R is given by convolution with a Gaussian kernel, and on those spaces it defines a contraction.

LEMMA III.3. *The transformation R is defined on $\mathscr{E}_\eta$ for $\eta \geqslant 1/2$, and*

$$R\colon \mathscr{E}_\eta \to \mathscr{E}_{\eta-(1/2)}, \qquad \textit{for} \quad \eta \geqslant \tfrac{1}{2}. \tag{III.20}$$

Proof. Let us first establish (III.20) in case $h(z) = h(-z)$. Thus $h(z) = \sum_{n=0}^{\infty} h_{2n} z^{2n}$. Since

$$Rz^{2n} = \sum_{k=0}^{n} \frac{(2n)!}{4^k (2n-2k)! \, k!} \, z^{2n-2k},$$

we write R in the form

$$(Rh)(z) = \sum_{m=0}^{\infty} \beta_{2m} z^{2m}, \tag{III.21}$$

with

$$\beta_{2m} = \sum_{n=m}^{\infty} \frac{(2n)! \, h_{2n}}{4^{n-m} (2m)! \, (n-m)!}. \tag{III.22}$$

Using the hypothesis $(2n)!^\eta \, |h_{2n}| \leqslant o(1)^n$, we obtain

$$\begin{aligned} (2m)!^{(\eta-(1/2))} \, |\beta_{2m}| &\leqslant \sum_{n=m}^{\infty} o(1)^n \, \frac{(2n)!^{1-\eta} \, (2m)!^{\eta-(1/2)}}{(2m)! \, (n-m)!}, \\ &\leqslant \sum_{n=m}^{\infty} o(1)^n \leqslant o(1)^m. \end{aligned} \tag{III.23}$$

Here we used $(2n)!^{1/2}\,(n-m)!^{-1} \leqslant O(1)^n\, m!$, and $\eta \geqslant \frac{1}{2}$. Hence $Rh \in \mathscr{E}_{\eta-(1/2)}$, and we have established (III.20) for even $h(z)$. In general, $h \in \mathscr{E}_\eta$ can be written $h = h_e + zh_o$ where both h_e and h_o are even elements of $\mathscr{E}_\eta$. Since $Rz = (z + 2D)\, R$, the above analysis shows that (III.20) holds in general.

As an entire function of z, $\mathscr{J}(z; a)$ defined in (III.15) is an element of $\mathscr{E}_{1/2}$. This is the consequence of assumption (II.3) for entire cochains. Therefore we infer from (III.20) that $\mathscr{J}$ is in the domain of R, and that $(R\mathscr{J})(z; a)$ is an entire function of z. As a consequence, we obtain a simple representation for the pairing (III.13).

III.3. *The Pairing Expressed in Terms of $\mathscr{J}(z; a)$*

We now express the pairing $\langle \tau, p \rangle$ in terms of the generating functional $\mathscr{J}(z; a)$ of (III.15). This leads us to the Gaussian transform $\mathfrak{Z}(a; g)$ of $\mathscr{J}(z; a)$, evaluated at the origin.

PROPOSITION III.4. *Let τ denote an entire cochain for $\mathfrak{A}$ and let $a \in$ $\mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ satisfy $a^2 = I$. Then the pairing* (III.13) *can be expressed in terms of the generating functional $\mathscr{J}$ as*

$$\mathfrak{Z}(a; g) = \langle \tau, a \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \mathscr{J}(t; a)\, dt. \tag{III.24}$$

Proof. Since $\mathscr{J}(z; a)$ is even, and an element of $\mathscr{E}_{1/2}$, we have from (III.21–22) that

$$(R\mathscr{J})(0; a) = \sum_{n=0}^{\infty} (-1/4)^n \frac{(2n)!}{n!} \operatorname{tr} \tau_{2n}(a, a, ..., a). \tag{III.25}$$

Comparing with (III.13), we find

$$(R\mathscr{J})(0; a) = \langle \tau, a \rangle.$$

As remarked above, the operator R can be expressed as convolution by a Gaussian. In particular,

$$(Rf)(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(s-t)^2} f(t)\, dt, \tag{III.26}$$

which also can be seen from

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}t^{2n}\,dt=\frac{(2n)!}{n!\,4^n}. \tag{III.27}$$

Hence we have the representation (III.24).

III.4. *Pairing for Families*

In Section II.2 we introduced the notion of a family $\mathfrak{F}=\{\tau(\lambda)\}\subset\mathscr{C}$ of cochains depending on a parameter λ belonging to an index set Λ. An important consequence of the topology introduced for $\mathscr{C}$, is that the pairing $\langle\tau(\lambda), a\rangle$ of a family inherits the convergence properties from $\mathscr{C}$. Associated with the family $\{\tau(\lambda)\}\subset\mathscr{C}$, we have a family of generating functions $\{\mathscr{J}(\lambda)\}$ defined by

$$\mathscr{J}(\lambda)=\mathscr{J}(z;a;\lambda)=\sum_{n=0}^{\infty}(-z^2)^n\,\tau_{2n}(\lambda)(a,a,...,a;g)=\sum_{n=0}^{\infty}\mathscr{J}_n(\lambda)\,z^n. \tag{III.28}$$

For each $a\in\mathfrak{A}$, and $\lambda\in\Lambda$, the function $\mathscr{J}(\cdot;a;\lambda)$ is a function in $\mathscr{E}_{1/2}$, as defined in (III.18). We consider $\{\mathscr{J}(\lambda)\}\subset\mathscr{E}_{1/2}$ as a family of functions in $\mathscr{E}_{1/2}$ parametrized by $\lambda\in\Lambda$.

PROPOSITION III.6. *Let the family $\{\tau(\lambda)\}\subset\mathscr{C}$ be bounded, continuous at λ_0, or differentiable at λ_0 in the sense of Section* II.4 *as a map from Λ to $\mathscr{C}$. Then*

(i) *The family $\{\mathscr{J}(\lambda)\}\subset\mathscr{E}_{1/2}$ is respectively*: *bounded, continuous at λ_0, or differentiable at λ_0 in the sense of a family in $\mathscr{E}_{1/2}$.*

(ii) *For $a^2=I\in\mathrm{Mat}_m(\mathfrak{A})$, pair $\tau(\lambda)$ with a as defined by* (III.13). *This defines the λ-dependent pairing*

$$\langle\tau(\lambda), a\rangle, \tag{III.29}$$

which is respectively: *bounded uniformly for $\lambda\in\Lambda$, continuous at λ_0, or differentiable at λ_0.*

Proof. If the family defined by $\tau(\lambda)$ is bounded, then the bound (II.31–32) ensures that the coefficients $\mathscr{J}_n(\lambda)$ in (III.28) satisfy

$$\sup_{\lambda\in\Lambda}|\mathscr{J}_n(\lambda)|\leqslant\alpha_n \tag{III.30}$$

where $n^{1/2}\alpha_n^{1/n}\to 0$ as $n\to\infty$. Thus $\{\mathscr{J}(\lambda)\}$ is a bounded family in $\mathscr{E}_{1/2}$. Likewise, convergence of the uniformly bounded coefficients $\mathscr{J}_n(\lambda)$ as

$\lambda \to \lambda_0$, ensures convergence of $\mathscr{J}(\lambda)$ to a function $\mathscr{J} \in \mathscr{E}_{1/2}$. This is a consequence of the uniform bound on $\mathscr{J}(\lambda)$ as a function of z, and of the Vitali convergence theorem for holomorphic functions. Likewise, if the difference quotient $(\tau(\lambda_0) - \tau(\lambda))/(\lambda_0 - \lambda)$ converges in $\mathscr{C}$ as $\lambda \to \lambda_0$, this means that $(\mathscr{J}(\lambda_0) - \mathscr{J}(\lambda))/(\lambda - \lambda_0)$ converges in $\mathscr{E}_{1/2}$. This proves part (i) of the proposition. The proof of part (ii) is similar. It is a consequence of the uniform bound (II.31–32), along with the continuity or differentiability. The uniform bound (II.31–32) ensures that the sum

$$\langle \tau(\lambda), a \rangle = \sum_{n=0}^{\infty} (-1/4)^n \frac{(2n)!}{n!} \operatorname{tr} \tau_{2n}(\lambda)(a, a, ..., a) \tag{III.31}$$

converges uniformly for $\lambda \in \Lambda$ and is bounded by the constant M defined by

$$M = \sum_{n=0}^{\infty} (\sup_{i,j} |||a_{ij}|||)^{2n+1} \frac{(2n)!}{n!} \alpha_{2n}. \tag{III.32}$$

Thus $\langle \tau(\lambda), a \rangle$ is a uniformly bounded function. If $\tau(\lambda)$ converges as $\lambda \to \lambda_0$, then $|||\tau_n(\lambda) - \tau_n(\lambda')|||$ is Cauchy for each n as $\lambda, \lambda' \to \lambda_0$. Define $\langle \tau(\lambda), a \rangle_N$ as (III.31), but with the sum over n limited to $n \leqslant N$. We write

$$\begin{aligned} &|\langle \tau(\lambda), a \rangle - \langle \tau(\lambda'), a \rangle| \\ &\quad \leqslant |\langle \tau(\lambda), a \rangle - \langle \tau(\lambda), a \rangle_N| + |\langle \tau(\lambda), a \rangle_n - \langle \tau(\lambda'), a \rangle_N| \\ &\qquad + |\langle \tau(\lambda'), a_n \rangle - \langle \tau(\lambda'), a \rangle|. \end{aligned} \tag{III.33}$$

The convergence of (III.32) ensures that there exists $N_0 < \infty$ such that for $N > N_0$, $|\langle \tau(\lambda), a \rangle - \langle \tau(\lambda), a \rangle_N| < \varepsilon$ uniformly in $\lambda \in \Lambda$. On the other hand, the fact that τ_n is Cauchy insures that for $N > N_0$ and fixed, we have $|\langle \tau(\lambda), a \rangle_N - \langle \tau(\lambda'), a \rangle_N| < \varepsilon$ for λ, λ' arbitrarily close to λ_0. Thus $\langle \tau(\lambda), a \rangle \to \langle \tau(\lambda_0), a \rangle$ as $\lambda \to \lambda_0$. The argument that differentiability of $\tau(\lambda)$ in $\mathscr{C}$ ensures differentiability of $\langle \tau(\lambda), a \rangle$ is similar, so we omit the details. This completes the proof of the proposition.

IV. THE JLO COCHAIN (DIFFERENTIABLE CASE)

IV.1. Heat Kernel Regularization and the Radon Transform

The description of the JLO cochain requires some more structure, in addition to the algebra $\mathfrak{A}$ and the functionals $\mathscr{C}$ on $\mathfrak{A}$. We begin by introducing a Hilbert space $\mathscr{H}$ and the algebra $\mathscr{B}(\mathscr{H})$ of bounded linear transformations on $\mathscr{H}$. We represent $\mathfrak{A}$ as a subalgebra of $\mathscr{B}(\mathscr{H})$. We let $\|\cdot\|$ denote the norm on $\mathscr{H}$ and $|||\cdot|||$ the inherited norm on $\mathfrak{A}$. We think of $\mathfrak{A}$ as the non-commutative generalization of an algebra of functions. In this section we formulate the case that $\mathfrak{A}$ is an algebra of differentiable functions; thus we call it the differentiable case. We also need to define a derivative on $\mathfrak{A}$. This is natural for a differential geometric interpretation of non-commutative geometry. Some other operators play a special role. They are the following:

$\mathbb{Z}_2$-Grading γ

A $\mathbb{Z}_2$-grading $\gamma \in \mathscr{B}(\mathscr{H})$ is a self-adjoint, unitary $\gamma = \gamma^* = \gamma^{-1}$. The grading γ determines a decomposition of $\mathscr{H}$, $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, into the $\pm$ eigenspaces of γ. The orthogonal projections of $\mathscr{H}$ onto $\mathscr{H}_\pm$ are $P_\pm = \frac{1}{2}(I \pm \gamma)$, and $P_+ + P_- = I$. We assume that $\mathfrak{A}$ is pointwise invariant under γ,

$$\gamma a = a\gamma, \qquad a \in \mathfrak{A}. \tag{IV.1}$$

In general, we denote the action of γ as

$$b^\gamma = \gamma b \gamma^{-1} = \gamma b \gamma, \qquad b \in \mathscr{B}(\mathscr{H}). \tag{IV.2}$$

Dirac Operator Q

The operator[6] $Q = Q^*$ is an (unbounded) operator on $\mathscr{H}$ whose sure $H = Q^2$ has the interpretation of Laplacian. It is assumed that Q and γ anticommute,

$$\gamma Q + Q\gamma = 0. \tag{IV.3}$$

Let $Q_\pm = QP_\pm$. Then $Q = Q_+ + Q_-$, where $Q_+^* = Q_-$ and $Q_+^2 = Q_-^2 = 0$. It follows that $H = Q_+ Q_- + Q_- Q_+$. We also assume that for $0 < \beta$,

$$\mathrm{Tr}(e^{-\beta Q^2}) < \infty. \tag{IV.4}$$

The condition (IV.4) is called Θ-summability by Connes [4].

Derivation d

The operator Q defines a graded derivation d. For operators $b \in \mathscr{B}(\mathscr{H})$ in the domain of d,

$$db = Qb - b^\gamma Q. \tag{IV.5}$$

[6] In the physics literature, the square root Q of the energy operator is called the *supercharge*. The Laplacian $H = Q^2$ is called the energy, or Hamiltonian. The relation between supersymmetry in physics and index theory was observed by Witten [32] in the context of the index of the exterior differential. In quantum field theory, the supercharge operator also has the interpretation of a Dirac operator on loop space.

We assume that all $a \in \mathfrak{A}$ are in the domain of d, and that the bilinear form $da = Qa - aQ$ defined on $\mathscr{D}(Q) \times \mathscr{D}(Q)$ uniquely determines a bounded linear operator

$$da \in \mathscr{B}(\mathscr{H}).$$

We assume that for all $a, b \in \mathfrak{A}$,

$$d(ab) = (da)\, b + a(db). \tag{IV.6}$$

Note that for b in the domain of d^2, or as a sesquilinear form,

$$d^2 b = [Q^2, b] = Q^2 b - bQ^2. \tag{IV.7}$$

In Section V we elaborate the properties of the domain of d.

Symmetry Group $\mathfrak{G}$

Equivariance arises through the existence of a continuous, unitary representation $U(g)$ of a compact Lie group $\mathfrak{G}$ on $\mathscr{H}$. We assume that

$$U(g)\, \gamma = \gamma U(g), \qquad U(g)\, Q = QU(g). \tag{IV.8}$$

We let

$$b^g = U(g)\, bU(g)^*, \qquad b \in \mathscr{B}(\mathscr{H}), \tag{IV.9}$$

denote the automorphism of $\mathscr{B}(\mathscr{H})$ induced by $\mathfrak{G}$. We assume that $\mathfrak{G}$ acts on $\mathfrak{A}$, namely that for all $a \in \mathfrak{A}$, $a^g \in \mathfrak{A}$. The group $\mathfrak{G}$ may be trivial, in which case the cochains are no longer functions on $\mathfrak{G}$. This is the ordinary (rather than equivariant) theory. (The representation $U(g)$ acting on $\mathscr{H}$ has no relation with, and should not be confused with, U in (II.7) which acts on $\mathscr{D}$.)

Heat Kernel Regularization

Consider $(n+1)$ operators $b_j \in \mathscr{B}(\mathscr{H})$, $j = 0, 1, ..., n$. Define the operator valued function $X(s)$ on $\mathbb{R}^{n+1}$ by

$$X(s) = \begin{cases} b_0 e^{-s_0 Q^2} b_1 e^{-s_1 Q^2} \cdots b_n e^{-s_n Q^2}, & \text{every} \quad s_j > 0 \\ 0, & \text{any} \quad s_j \leqslant 0 \end{cases}. \tag{IV.10}$$

The operator $X(s)$ is the *heat-kernel regularized density* of the ordered set of operators $\{b_0, b_1, ..., b_n\}$. See also [13]. Note that $X(s)$ is an $(n+1)$-multilinear function on the set $\mathscr{B}(\mathscr{H})$. We call $\{b_0, ..., b_n\}$ the set of *vertices* of $X(s)$. We sometimes use X to denote the set of vertices,

$$X = \{b_0, b_1, ..., b_n\},$$

as well as the heat kernel regularization, at least in cases where no confusion can arise.

The Radon Transform

We also consider the radon transform $\hat{X}(\beta)$ of $X(s)$ corresponding to the hyperplane $s_0 + s_1 + \cdots + s_n = \beta > 0$. In other words

$$\hat{X}(\beta) = \int X(s)\, d^n s(\beta). \tag{IV.11}$$

Here $d^n s(\beta)$ denotes a restriction of Lebesgue measure on $\mathbb{R}^{n+1}$ to the hyperplane $\sum_{j=0}^{n} s_j = \beta$, defined by

$$d^n s(\beta) = \delta(s_0 + \cdots + s_n - \beta)\, ds_0\, ds_1 \cdots ds_n, \tag{IV.12}$$

where δ denotes the Dirac measure. The measure $d^n s(\beta)$, integrated over the positive "quadrant" $\mathbb{R}^{n+1}_{+}$, equals $\beta^n/n!$, see (V.20, 24).

The heat-kernel-regularized density $X(s)$ and its Radon transform $\hat{X}(\beta)$ form the basic objects in the geometric theory we develop here. We refer to both as *heat-kernel regularizations* of $\{b_0, ..., b_n\}$. It often turns out in the geometric theory that hyperplanes for different values of β are equivalent, and this is always the case if $\exp(-\beta Q^2)$ is trace class for all $\beta > 0$, see Section VII.6. Thus in order to simplify notation, we restrict our attention to the plane

$$s_0 + s_1 + \cdots + s_n = 1.$$

Denote this value of the Radon transform by $\hat{X} = \hat{X}(\beta = 1)$. When there is no chance of confusion, we simply write for the corresponding measure

$$d^n s = d^n s(1). \tag{IV.13}$$

As a consequence of the trace class property of $e^{-\beta Q^2}$ for each $\beta > 0$, the heat kernel regularization $X(s)$ and also $\hat{X}$ is trace class. The trace norm $\|\hat{X}\|_1 = \mathrm{Tr}((\hat{X}^* \hat{X})^{1/2})$ of $\hat{X}$ satisfies

$$\|\hat{X}\|_1 \leqslant \frac{1}{n!} \mathrm{Tr}(e^{-Q^2}) \left(\prod_{j=0}^{n} \|b_j\| \right). \tag{IV.14}$$

We postpone the proof of (IV.14) to Section V, in conjunction with the proof of other related bounds, see Proposition V.3.iii and Corollary V.4.v.

Symmetries of $\hat{X}$

The group $\mathbb{Z}_2$, implemented by γ, and the group $\mathfrak{G}$, implemented by $U(g)$ both commute with $\exp(-\beta Q^2)$. Hence these groups act on $\hat{X}$ by acting on the vertices of X. Let $\hat{X}^\gamma$ denote the heat kernel regularization $\gamma \hat{X} \gamma^{-1}$ of X^γ with vertices

$$X^\gamma = \{b_0^\gamma, b_1^\gamma, ..., b_n^\gamma\}.$$

Similarly let $\hat{X}^g$ denote the heat kernel regularization $U(g)\, \hat{X} U(g)^*$ of X^g with vertices

$$X^g = \{b_0^g, b_1^g, ..., b_n^g\}.$$

IV.2. *Expectations and the Radon Transforms*

Expectations $\langle \hat{X}; g \rangle$

The expectation $\langle \hat{X}; g \rangle$ of a heat kernel regularization $\hat{X}$ is defined by

$$\langle \hat{X}; g \rangle = \operatorname{Tr}(\gamma U(g)\, \hat{X}). \tag{IV.15}$$

Since the expectation is a linear function of each vertex of X, we also use the notation which indicates this multilinearity, namely we denote $\langle \hat{X}; g \rangle$ by

$$\langle \hat{X}; g \rangle = \operatorname{Tr}(\gamma U(g)\, \hat{X}) = \langle b_0, b_1, ..., b_n; g \rangle = \langle b_0, b_1, ..., b_n; g \rangle_n. \tag{IV.16}$$

Here we use the subscript n, when it may be helpful to clarify the number of vertices. The bound (IV.14) ensures that the expectation is continuous in each vertex, and

$$|\langle b_0, b_1, ..., b_n; g \rangle| \leqslant \frac{1}{n!} \operatorname{Tr}(e^{-Q^2}) \left(\prod_{j=0}^{n} \|b_j\| \right). \tag{IV.17}$$

Symmetries of Expectations

We elaborate on [17]; see also [26]. As a consequence of cyclicity of the trace, and the commutativity of γ and $U(g)$,

$$\langle \hat{X}; g \rangle = \langle \hat{X}^\gamma; g \rangle = \langle \hat{X}^g; g \rangle = \langle \hat{X}^{g^{-1}}; g \rangle. \tag{IV.18}$$

More generally, for $h \in \mathfrak{G}$, $\langle \hat{X}^h; g \rangle = \langle \hat{X}; h^{-1} g h \rangle$.

Another symmetry of the expectation arises from cyclic permutation of the vertices,

$$\langle b_0, b_1, ..., b_n; g \rangle = \langle b_n^{g^{-1}\gamma}, b_0, ..., b_{n-1}; g \rangle. \tag{IV.19}$$

We also remark that the expectation is invariant under the infinitesimal d. This means that for all heat-kernel regularizations $\hat{X}$,

$$\langle d\hat{X};\, g\rangle = 0. \tag{IV.20}$$

In particular, if $\hat{X}^{\gamma} = \hat{X}$, then $(d\hat{X})^{\gamma} = -d\hat{X}$ and (IV.20) vanishes by (IV.18). On the other hand, if $\hat{X}^{\gamma} = -\hat{X}$, then $(d\hat{X}) = Q\hat{X} + \hat{X}Q$, and (IV.20) vanishes on account of (IV.3) and cyclicity of the trace. Writing out (IV.20) in detail for X with vertices $b_0, ..., b_n$, we infer that

$$\sum_{j=0}^{n} \langle b_0^{\gamma}, b_1^{\gamma}, ..., b_{j-1}^{\gamma}, db_j, b_{j+1}, ..., b_n;\, g\rangle_n = 0. \tag{IV.21}$$

Another interesting identity for expectations is

$$\langle b_0, ..., b_n;\, g\rangle_n = \sum_{j=1}^{n+1} \langle b_0, ..., b_{j-1}, I, b_j, ..., b_n;\, g\rangle_{n+1}. \tag{IV.22}$$

This follows from a change of variables in the Radon transform (IV.10), for $1 \leqslant j \leqslant n+1$. Let $\beta'_i = \beta_i$, $0 \leqslant i \leqslant j-2$, $\beta'_{j-1} = \beta_{j-1} + \beta_j$, $\beta'_i = \beta_{i-1}$, $j \leqslant i \leqslant n$, and $\beta'_{n+1} = \beta_{j-1}$. Then

$$\begin{aligned}
&\langle b_0, ..., b_{j-1}, I, ..., b_n;\, g\rangle_{n+1} \\
&\quad = \int_{\tilde{\sigma}_n} \operatorname{Tr}(\gamma U(g)\, b_0 e^{-\beta'_0 Q^2} \cdots b_{j-1} e^{-\beta'_{j-1} Q^2} b_j e^{-\beta'_j Q^2} \cdots b_n e^{-\beta'_n Q^2}) \\
&\quad\quad \times d\beta'_0 \cdots d\beta'_n\, d\beta'_{n+1},
\end{aligned} \tag{IV.23}$$

where $\tilde{\sigma}_n$ denotes the set

$$0 < \beta'_i, \qquad 0 < \beta'_{n+1} < \beta'_{j-1}, \qquad \text{and} \qquad \beta'_0 + \beta'_1 + \cdots + \beta'_n = 1. \tag{IV.24}$$

Note the integrand is independent of β'_{n+1}. Therefore the β'_{n+1} integration yields β'_{j-1} times an integrand common to every such term, $1 \leqslant j \leqslant n+1$. This latter integrand by itself would integrate to $\langle b_0, ..., b_n;\, g\rangle_n$. Summing over j results in $\sum_{j=0}^{n} \beta'_j$ times the integrand defining $\langle b_0, ..., b_n;\, g\rangle_n$. But the constraint $\sum_{j=0}^{n} \beta'_j = 1$ in this integral reduces the sum to $\langle b_0, ..., b_n;\, g\rangle_n$, so we infer (IV.22).

IV.3. *The Cochain* τ^{JLO}

The JLO cochain [17] is the expectation whose nth component is defined by

$$\tau_n^{\mathrm{JLO}}(a_0, ..., a_n;\, g) = \langle a_0, da_1, ..., da_n;\, g\rangle. \tag{IV.25}$$

In this section, we assume that the norm $|||\cdot|||$ on $a \in \mathfrak{A}$ dominates the first Sobolev norm

$$|||a|||_1 = \|a\| + \|da\|. \tag{IV.26}$$

In other words, we assume that

$$|||a|||_1 \leqslant |||a|||. \tag{IV.27}$$

Hence every element of $\mathfrak{A}$ has a bounded derivative, and thus $\mathfrak{A}$ is the non-commutative generalization of the space of differentiable functions. As a consequence, we infer that the expectation $\tau^{\mathrm{JLO}} = \{\tau_n^{\mathrm{JLO}}\}$ is a cochain:

Lemma IV.1. *The expectation τ^{JLO} is an element of the space $\mathscr{C}(\mathfrak{A})$ of cochains, as defined at the start of Section II. Furthermore τ^{JLO} extends by continuity and linearity from $\mathfrak{A}$ to the subalgebra $\mathscr{B}_1$ of operators $b \in \mathscr{B}(\mathscr{H})$ such that $|||b|||_1 < \infty$, and $\mathscr{B}_1$ is a Banach algebra.*

Proof. Clearly τ_n^{JLO} is $(n+1)$-linear in $\mathfrak{A}$. We show that $\tau_n^{\mathrm{JLO}} \in \mathscr{C}_n$. This requires the symmetry (II.1) of cochains, continuity in the norm of $\mathfrak{A}$ and continuity in $\mathfrak{G}$. In addition τ_n^{JLO} must vanish for $a_k = I$, $k = 1, 2, ..., n$. But this latter fact follows from $dI = 0$.

The required symmetry (II.1) for τ_n^{JLO} is a consequence established in (IV.18). We combine this fact with the assumption (IV.8) which ensures $d(a^{g^{-1}}) = (da)^{g^{-1}}$. Thus

$$\begin{aligned}\tau_n^{\mathrm{JLO}}(a_0^{g^{-1}}, ..., a_n^{g^{-1}}; g) &= \langle a_0^{g^{-1}}, d(a_1^{g^{-1}}), ..., d(a_n^{g^{-1}}); g\rangle \\ &= \langle a_0^{g-1}, (da_1)^{g^{-1}}, ..., (da_n)^{g^{-1}}; g\rangle \\ &= \langle a_0, da_1, ..., da_n; g\rangle = \tau_n^{\mathrm{JLO}}(a_0, ..., a_n)\end{aligned}$$

as desired.

The continuity of τ_n^{JLO} in $\mathfrak{A}$ follows from (IV.17) and (IV.26).

$$|\tau_n^{\mathrm{JLO}}(a_0, ..., a_n; g)| \leqslant \frac{1}{n!} \operatorname{Tr}(e^{-Q^2}) \left(\prod_{j=0}^{n} |||a_j|||_1 \right). \tag{IV.28}$$

Since $|||a|||_1 \leqslant |||a|||$, we have

$$|||\tau_n^{\mathrm{JLO}}||| \leqslant \frac{1}{n!} \operatorname{Tr}(e^{-Q^2}). \tag{IV.29}$$

Furthermore $U(g)$ is a continuous, unitary representation, so $\tau_n^{\mathrm{JLO}}(a_0, ..., a_n; g)$ is continuous (pointwise) in $\mathfrak{G}$. Thus $\tau_n^{\mathrm{JLO}} \in \mathscr{C}_n$.

We verify that the sequence $\tau^{\mathrm{JLO}} \in \mathscr{C}$. The factor $1/n!$ in (IV.29) ensures that

$$n\, |||\tau_n|||^{1/n} \leqslant O(1),$$

so the entire condition (II.3) is satisfied.

Thus τ_n^{JLO} extends by continuity to $\mathscr{B}_1$. Finally we verify that $\mathscr{B}_1$ is a Banach algebra. In fact for $a, b \in \mathscr{B}_1$, we infer from (IV.5) that $d(ab) = (da)\, b + a^{\gamma}(db)$. Hence

$$|||ab|||_1 = \|ab\| + \|d(ab)\| \leqslant \|a\|\ \|b\| + \|da\|\ \|b\| + \|a\|\ \|db\| \leqslant |||a|||_1\ |||b|||_1,$$

as asserted.

Other Symmetries of τ^{JLO}

Remark that for $a_j \in \mathfrak{A}$, the odd components of τ^{JLO} vanish, namely

$$\tau_{2n+1}^{\mathrm{JLO}}(a_0, ..., a_n;\, g) = 0. \tag{IV.30}$$

This is a consequence of (IV.1, 3, 5) which ensures $(da)^{\gamma} = -da$. Thus using (IV.18),

$$\begin{aligned} \langle a_0, da_1, ..., da_n;\, g\rangle &= \langle a_0^{\gamma}, (da_1)^{\gamma}, ..., (da_n)^{\gamma};\, g\rangle \\ &= (-1)^n \langle a_0, da_1, ..., da_n;\, g\rangle, \end{aligned}$$

so τ_n^{JLO} vanishes for odd n.

Another elementary identity, a consequence of (IV.21) and the choice of $a_j \in \mathfrak{A}$ is that

$$\tau_n^{\mathrm{JLO}}(a_0, a_1, ..., a_n) + \sum_{j=1}^{n} (-1)^{j-1}\, \tau_n^{\mathrm{JLO}}(a_0, ..., a_{j-1}, da_j, a_{j+1}, ..., a_n;\, g) = 0. \tag{IV.31}$$

VI.4. *The JLO Pairing and the Generating Functional*

The Generating Functional $\mathscr{J}(z; a)$

We evaluate the generating functional $\mathscr{J}(z; a)$ of (III.15) for $\tau = \tau^{\mathrm{JLO}}$. Since $\tau_{2n+1}^{\mathrm{JLO}} = 0$, we can also write for $a \in \mathfrak{A}$,

$$\mathscr{J}^{\mathrm{JLO}}(z; a) = \sum_{n=0}^{\infty} (-z^2)^n \operatorname{tr} \tau_{2n}^{\mathrm{JLO}}(a, a, ..., a;\, g). \tag{IV.32}$$

Hence by the Hille–Phillips perturbation formula for semigroups,

$$e^{-(Q^2+X)} = e^{-Q^2} - \int_0^1 e^{-sQ^2} X e^{-(1-s)(Q^2+X)}\, ds,$$

it follows that

$$\mathscr{J}^{\mathrm{JLO}}(z; a) = \mathrm{Tr}(\gamma U(g)\, a e^{-Q^2 + iz\, da}). \tag{IV.33}$$

A Formula for the Pairing

Using (IV.33), we infer that the pairing can be expressed simply.

PROPOSITION IV.2. *The pairing* $\langle \tau^{\mathrm{JLO}}, a \rangle$ *of the cochain* τ^{JLO} *with* $a \in \mathrm{Mat}_m(\mathfrak{A})$ *satisfying* $a^2 = I$ *can be written*

$$\mathfrak{Z}^Q(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \mathrm{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da})\, dt. \tag{IV.34}$$

Here Tr *denotes both the trace on* $\mathscr{H}$ *and the matrix trace in* $\mathrm{Mat}_m(\mathfrak{A})$.

V. FRACTIONALLY DIFFERENTIABLE STRUCTURES

We use the name *quantum harmonic analysis* for the study of fractional differentiability of operator valued functions. An interpolation space, in the quantum context, is a Banach algebra of operator-valued functions with fractional derivatives. We distinguish quantum harmonic analysis from "non-commutative harmonic analysis," a term used to denote the study of harmonic analysis on non-commutative groups.

V.1. *The Classical Picture*

Let us digress on a simple case—we refer to it as the classical case—for purposes of motivation. Take $\mathscr{E} = \bigoplus_{k=0}^n \mathscr{E}_k$ to be the exterior algebra of smooth differential forms on the n-torus $\mathbb{T}^n$. Let $\mathscr{E}_k$ denote k-forms with the standard L^2 inner product. We let $\mathscr{H}$ denote the Hilbert space of L^2 forms obtained by completing $\mathscr{E}$ as an inner product space with the inner product on $\mathscr{E}$ given by the sum of the inner products on $\mathscr{E}_k$. Thus $\mathscr{H} = \bigoplus_{k=0}^n \mathscr{H}^{(k)}$. Also define γ as $(-1)^k$ on $\mathscr{H}^{(k)}$, and let $\mathfrak{G}$ denote the group of translations on T^n. Thus $\mathfrak{G}$ acts unitarily on $\mathscr{H}$, and for $f \in \mathscr{H}$, $(U(y)\, f)(x) = f(x-y)$.

Define d to be the exterior derivative with domain $\mathscr{E} \subset \mathscr{H}$. Then define $Q = d + d^*$. Clearly $Q\gamma + \gamma Q = 0$ on $\mathscr{E}$, and $U(g)\, Q = QU(g)$. The operator Q with domain $\mathscr{E}$ is essentially self-adjoint. Then $Q^2 = dd^* + d^*d = -\Delta$ is the Laplacian. Also $\exp(-\beta Q^2)$, $\beta \geqslant 0$, commutes with γ and $U(a)$. Furthermore $\exp(-\beta Q^2) = e^{\beta \Delta}$ is trace class for every $\beta > 0$.

Alternatively, we can consider $\mathscr{E}$ as a subalgebra of $B(\mathscr{H})$, the bounded, linear operators on $\mathscr{H}$. An element in $\mathscr{E}_k$ maps $\mathscr{H}_{k'}$ to $\mathscr{H}_{k+k'}$ by exterior multiplication. We give this algebra the norm

$$|||b|||_1 = \|b\| + \|db\|, \qquad b \in \mathscr{E}, \tag{V.1}$$

where $\|\cdot\|$ denotes the operator norm on $\mathscr{H}$. This agrees with the L^∞ norm defined on the coefficients of the form b. Define $\mathfrak{A}_1$ as the completion of the smooth functions $\mathscr{E}_0$ (the smooth zero forms) in the norm (V.1). Hence $\mathfrak{A}_1$ is the algebra of Lipschitz continuous functions,

$$\sup_x |a(x+y) - a(x)| \leqslant M\,|y|. \tag{V.2}$$

Since (V.1) has the same form as the norm (IV.26), we can regard this example as a special case of Section IV where we take $|||\cdot||| = |||\cdot|||_1$ and $\mathfrak{A} = \mathfrak{A}_1$. From this point of view, the material in Section IV belongs to the study of the non-commutative Lipschitz class.

In order to distinguish the differentiable structure from the continuous structure in the non-commutative case, one wants to study the analogs of Hölder continuous classes, which in the classical case would satisfy

$$\sup_x |a(x+y) - a(x)| \leqslant M\,|y|^\alpha, \qquad 0 < \alpha \leqslant 1, \tag{V.3}$$

for α the exponent of continuity.

Related to such classes are functions with fractional derivatives of order α. The derivative da of a Hölder continuous function is unbounded. However fractional derivatives may be bounded. One way to define an L^p fractional derivative of order α of the function a is to suppose that

$$(-\Delta + I)^{\alpha/2}\, a(x) \in L^p(\mathbb{T}^n), \tag{V.4}$$

for which an extensive theory exists in the classical case, see [31]. If $a(x)$ is bounded, then a natural norm on such functions is, $\|a\|_{L^\infty} + \|(-\Delta + I)^{\alpha/+2}\, a\|_{L^p}$. If the norm with $p = \infty$ exists, then one is ensured that the function $a(x)$ is Hölder continuous for all continuity exponents $\alpha' < \alpha$. For $\alpha \leqslant 1$ this norm

$$\|a\|_{L^\infty} + \|(-\Delta + I)^{\alpha/2}\, a\|_{L^\infty} \tag{V.5}$$

is equivalent to

$$\|a\|_{L^\infty} + \|(-\Delta + I)^{-(1-\alpha)/2}\, da\|_{L^\infty}. \tag{V.6}$$

The norm (V.6) provides a natural measure of functions with derivatives of order α, or of functions which are Hölder continuous with exponent $\alpha' < \alpha$. Other norms of classical analysis could also be studied.

In this section we pose these questions in the non-commutative case. Thus function space norms need to be replaced by operator norms. Connes [3] proposed a translation from classical concepts to quantum ones. Adapted to our context we have:

classical	**non-commutative**
function space	algebra $\mathfrak{A}$ of linear transformations
exterior derivative d	graded commutator with Q
Laplacian	Q^2
$(-1)^{\text{degree}}$	γ
L^∞-norm	operator norm
L^p-norm	I_p-Schatten norm
generalized function	operator-valued generalized function
Sobolev norms of a function	norms of maps between Sobolev spaces
tempered distribution	bounded map between Sobolev spaces
space of fractionally differentiable functions	interpolation space $\mathfrak{J}_{\beta,\alpha}$
exterior derivative	graded commutator with Q
degree of regularity	local regularity exponent η_{local}
regularity as a function of dimension	global regularity exponent η_{global}
regularized current	heat kernel regularization
integral of (current)	JLO-cochain τ_n^{JLO}

It is natural to define fractional derivatives in terms of the scales determined by Q. Thus we say that a bounded operator a has a derivative of order μ if $(Q^2+I)^{\mu/2}\,a(Q^2+I)^{-\mu/2}$ is also bounded. We take the norm

$$\|a\| + \|(Q^2+I)^{\mu/2}\,a(Q^2+I)^{-\mu/2}\| \tag{V.7}$$

as the non-commutative version of (V.5). Since da is fundamental for the theory of invariants, we prefer to pose our assumptions in terms like (V.6), rather than (V.7). We show in Section V.3 that (V.7) with $\mu > 1$ leads us to assume a bound on

$$\|(Q^2+I)^{-\beta/2}\,da(Q^2+I)^{-\alpha/2}\| < \infty \tag{V.8}$$

for some $0 \leqslant \alpha, \beta$ and $0 \leqslant \alpha + \beta < 1$. The order of differentiability is $\mu = 1 - \beta$. Thus we obtain not one space, but a family of non-commutative spaces which generalize the space of bounded functions with fractional derivatives in the sense of (V.6). In fact, each of these spaces gives rise to a theory of geometric invariants. In Section V.5 we introduce a family of *interpolation spaces* $\mathfrak{I}_{\beta,\alpha}$ that provides a natural framework in which to generalize the construction of Section IV.

V.2. *Sobolev Spaces in* $\mathscr{H}$

We start with a Hilbert space $\mathscr{H}$ and a fundamental Dirac operator $Q=Q^*$ on $\mathscr{H}$ with domain $\mathscr{D}$, as in Section IV. We define Sobolev spaces $\mathscr{H}_p \subset \mathscr{H}$ for $0 \leqslant p < \infty$ which are the domain of $|Q|^p$ with a Hilbert space structure. In order to simplify the discussion of embeddings, let us consider the domain of the operator $(Q^2+I)^{p/2}$, which we denote $\mathscr{D}((Q^2+I)^{p/2})$ or $\mathscr{D}_p = \mathscr{D}_p(Q)$ for short. The Sobolev space $\mathscr{H}_p = \mathscr{H}_p(Q)$, $0 \leqslant p < \infty$, is the domain $\mathscr{D}_p$, considered as a Hilbert space with inner product

$$\langle f, g\rangle_{\mathscr{H}_p} = \langle (Q^2+I)^{p/2} f, (Q^2+I)^{p/2} g\rangle. \tag{V.9}$$

The corresponding negative Sobolev space $\mathscr{H}_{-p}$, for $p>0$, is the completion of $\mathscr{H}$ in the norm determined by the inner product

$$\langle f, g\rangle_{\mathscr{H}_{-p}} = \langle (Q^2+I)^{-p/2} f, (Q^2+I)^{-p/2} g\rangle. \tag{V.10}$$

For $\alpha > \beta$ there is a natural embedding $\mathscr{H}_\alpha \subset \mathscr{H}_\beta$. With respect to this embedding the spaces $\mathscr{H}_p$ and $\mathscr{H}_{-p}$ are dual, and for $p>0$ they define a Gelfand triple

$$\mathscr{H}_p \subset \mathscr{H} \subset \mathscr{H}_{-p}. \tag{V.11}$$

This is a standard device in the study of classical generalized functions or distributions, see [11]. We also introduced the square root of the resolvent of the "Laplacian" Q^2,

$$R = (Q^2+I)^{-1/2}, \tag{V.12}$$

so

$$\langle f, g\rangle_{\mathscr{H}_p} = \langle R^{-p} f, R^{-p} g\rangle_{\mathscr{H}}. \tag{V.13}$$

In the classical case, the integral operator given by R^{2p}, $p>0$, is called the Bessel transform operator of order p. We define

$$\mathscr{H}_\infty = \bigcap_p \mathscr{H}_p, \qquad \text{and} \qquad \mathscr{H}_{-\infty} = \bigcup_p \mathscr{H}_p. \tag{V.14}$$

Then for $p \geqslant 0$

$$\mathscr{H}_\infty \subset \mathscr{H}_p \subset \mathscr{H}_0 \subset \mathscr{H}_{-p} \subset \mathscr{H}_{-\infty}, \tag{V.15}$$

and for $s>0$,

$$e^{-sQ^2}\colon \mathcal{H}_{-\infty} \to \mathcal{H}_{\infty}. \tag{V.16}$$

V.3. *Some Facts*

Schatten Classes I_p

The analog in the non-commutative case of l_p spaces are the Schatten ideals $I_p = I_p(\mathcal{H})$. This is the subspace of compact operators on $\mathcal{H}$ for which the norm $\|\cdot\|_p$ is finite. Here

$$\|b\|_p = \|b\|_{I_p} = (\mathrm{Tr}((b^*b)^{p/2}))^{1/p}. \tag{V.17}$$

When there may be a chance of confusion, we write $\|\cdot\|_{I_p}$ for $\|\cdot\|_p$. For $p=1$ this is the trace norm, and if $\|b\|_p<\infty$ for some p, then $\|b\| = \lim_{p\to\infty} \|b\|_p$. It is clear that $\|b\|_p \leqslant \|b\|_{p'}$ if $p' \leqslant p$.

The Schatten norms satisfy a Hölder inequality for $1 \leqslant r$, namely

$$\|ab\|_r \leqslant \|a\|_p \, \|b\|_q, \qquad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}. \tag{V.18}$$

More generally,

$$\|a_0 \cdots a_n\|_r \leqslant \prod_{j=0}^{n} \|a_j\|_{p_j}, \qquad \sum_{j=0}^{n} \frac{1}{p_j}=\frac{1}{r}. \tag{V.19}$$

The Beta Function B_n

Let $\eta_j>0$, $j=0, 1, ..., n$. Then define the beta function B_n as

$$B_n(\eta_0, \eta_1, ..., \eta_n) = \frac{\prod_{j=0}^{n} \Gamma(\eta_j)}{\Gamma(\sum_{j=0}^{n} \eta_j)}. \tag{V.20}$$

Here $\Gamma(\cdot)$ denotes the gamma function.

We also define $\sigma_n \subset \mathbb{R}^{n+1}$ as the subset

$$\sigma_n = \left\{ s\colon s \in \mathbb{R}^{n+1}, 0<s_j, \sum_{j=0}^{n} s_j = 1 \right\}. \tag{V.21}$$

A natural measure on σ_n is $d^n s(1)$, as given in (IV.12), namely Lebesgue measure restricted to the n-hyperplane $s_0 + \cdots + s_n = 1$. Then we claim that

$$B_n(\eta_0, ..., \eta_n) = \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(1), \tag{V.22}$$

namely B_n is a Radon transform given by the hyperplane σ_n. For $0<\beta$, define the Radon transform

$$B_n(\eta_0, ..., \eta_n; \beta) = \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(\beta). \tag{V.23}$$

Changing variables, we have

$$B_n(\eta_0, ..., \eta_n; \beta) = \beta^{-1+\sum_{j=0}^{n} \eta_j} B_n(\eta_0, ..., \eta_j; 1), \tag{V.24}$$

or

$$\int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(\beta) = \beta^{-1+\sum_{j=0}^{n} \eta_j} \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(1).$$

Using the representation $\Gamma(\eta) = \int_0^\infty e^{-t} t^{-1+\eta}\, dt$, we have

$$\begin{aligned}
&\Gamma(\eta_0) \cdots \Gamma(\eta_n) \\
&\quad = \int_{\mathbb{R}_+^{n+1}} (e^{-\sum_{j=0}^{n} s_j}) \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) ds_0\, ds_1 \cdots ds_n \\
&\quad = \int_0^\infty d\beta\, e^{-\beta} \int_{\mathbb{R}_+^{n+1}} ds_0\, ds_1 \cdots ds_n\, \delta(s_0 + s_1 + \cdots + s_n - \beta) \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) \\
&\quad = \int_0^\infty d\beta\, e^{-\beta} \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(\beta) \\
&\quad = \int_0^\infty d\beta\, e^{-\beta} \beta^{(-1+\sum_{j=0}^{n} \eta_j)} \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(1) \\
&\quad = \Gamma\left(\sum_{j=0}^{n} \eta_j \right) \int_{\mathbb{R}_+^{n+1}} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) d^n s(1),
\end{aligned}$$

where in the second to last equality we use (V.24). Hence we have proved (V.22).

We remark that with $|\sigma_n|$ the measure of σ_n, and $\beta > 0$, we infer from (V.20, 24) that

$$\begin{gathered}
|\sigma_n| = B_n(1, ..., 1) = \frac{1}{n!}, \qquad |\beta\sigma_n| = \frac{\beta^n}{n!}, \\
B_n\left(\frac{1}{2}, 1, ..., 1\right) = \frac{4^n n!}{(2n)!}, \qquad \text{and} \qquad B_n\left(\frac{1}{2}, \frac{1}{2}, 1, ..., 1\right) = \frac{\pi}{(n-1)!}.
\end{gathered} \tag{V.25}$$

V.4. *Operator-Valued Generalized Functions*

Consider the natural definition of an (operator-valued) *generalized function* or *operator-valued distribution* in the non-commutative case. This is a linear transformation on $\mathscr{H}$ whose domain and whose range are a Sobolev space of the type above. Such an unbounded transformation on $\mathscr{H}$ is a bounded transformation when viewed as a map between the different Sobolev spaces.

Consider a bounded linear transformation x with domain $\mathscr{H}_{p_1}$ and range contained in $\mathscr{H}_{p_2}$. If $p_1=0$ and $p_2\geqslant 0$, then x is a bounded linear transformation on $\mathscr{H}$. If $p_1>0$ and $p_2\geqslant 0$, then x is an unbounded operator on $\mathscr{H}$ with domain $\mathscr{D}_{p_1}$. If $p_1\geqslant 0$ and $p_2<0$, then x is a sesquilinear form with domain $\mathscr{D}_{p_2}\times\mathscr{D}_{p_1}$. The operator $(Q^2+I)^{-p/2}$ defines a unitary isomorphism between $\mathscr{H}_{p_1}$ and $\mathscr{H}_{p_1+p}$. The norm of a bounded, linear transformation x from $\mathscr{H}_{p_1}$ to $\mathscr{H}_{p_2}$ is given on $\mathscr{H}$ by

$$\|x\|_{(p_2,\,p_1)}=\|x\|_{\mathscr{H}_{p_1}\to\mathscr{H}_{p_2}}=\|R^{-p_2}xR^{p_1}\|. \tag{V.26}$$

In case that the domain $\mathscr{H}_{p_1}$ has a positive Sobolev index p_1 and the target space $\mathscr{H}_{p_2}$ has a negative Sobolev index p_2, one says that x is a *generalized function.* Then (with $p_2>0$),

$$\|x\|_{(-p_2,\,p_1)}=\|x\|_{\mathscr{H}_{p_1}\to\mathscr{H}_{-p_2}}=\|R^{p_2}xR^{p_1}\|. \tag{V.27}$$

Let $\mathscr{T}(p_2,p_1)=\mathscr{T}(p_2,p_1;Q)$ denote the space of bounded, linear transformations from $\mathscr{H}_{p_1}(Q)$ to $\mathscr{H}_{p_2}(Q)$. If $x\in\mathscr{T}(p_3,p_2)$ and $y\in\mathscr{T}(p_2,p_1)$ then $xy\in\mathscr{T}(p_3,p_1)$. Clearly Q is an element of $\mathscr{T}(p-1,p)$, with norm $\|Q\|_{(p-1,\,p)}=1$.

For $\alpha,\beta>0$, any $x\in\mathscr{T}(-\beta,\alpha)$ defines a sesquilinear form on $\mathscr{D}_\beta\times\mathscr{D}_\alpha$ in $\mathscr{H}$. For all p, the space $\mathscr{D}_p$ contains the space $\mathscr{H}_\infty$, so x is defined on $\mathscr{H}_\infty\times\mathscr{H}_\infty$. Thus for $0<s,t$, we infer from (V.16) that

$$e^{-sQ^2}xe^{-tQ^2}$$

is bounded. In fact for $0<s\leqslant 1$, $0<\alpha\leqslant 1$,

$$\|R^{-\alpha}e^{-sQ^2}\|\leqslant 2s^{-\alpha/2},\qquad\text{and}\qquad \|e^{-sQ^2}xe^{-tQ^2}\|\leqslant 4s^{-\beta/2}t^{-\alpha/2}\,\|x\|_{(-\beta,\,\alpha)}. \tag{V.28}$$

On this account we could define an alternative norm

$$|||x|||_{(-p_2,\,p_1)}=\sup_{0<s,\,t\leqslant 1}\left(s^{\beta/2}t^{\alpha/2}\,\|e^{-sQ^2}xe^{-tQ^2}\|\right), \tag{V.29}$$

and use (V.29) to define a slightly larger space of generalized functions including $\mathscr{T}(-\beta, \alpha)$, namely the completion of $\mathscr{T}(-\beta, \alpha)$ in the norm $|||x|||_{(-\beta, \alpha)} \leqslant 4 \|x\|_{(-\beta, \alpha)}$.

DEFINITION V.1. a. A non-commutative generalized function x is an element of $\mathscr{T}(-\beta, \alpha)$ for some α, β. If $\alpha, \beta \geqslant 0$, we call x a *vertex* of type (β, α) with respect to Q, (or *vertex* for short).

b. A regular set of $(n+1)$ ordered vertices $X = \{x_0, ..., x_n\}$, with respect to Q, (for short, a regular set of vertices) is a set of vertices x_j of type (β_j, α_j), where α_j, β_j satisfy the following conditions:

$$0 < \eta_j = 1 - \tfrac{1}{2}(\alpha_j + \beta_{j+1}), \qquad j = 0, 1, ..., n. \tag{V.30}$$

Here β_{n+1} is defined by $\beta_{n+1} = \beta_0$.

c. The *local regularity exponent* η_{local} of the set X is defined by

$$0 < \eta_{\text{local}} = \min_{1 \leqslant j \leqslant n} \{\eta_j\}, \tag{V.31}$$

and the *global regularity exponent* η_{global} of the set X is defined by the mean exponent

$$\eta_{\text{global}} = \frac{1}{n+1} \sum_{j=0}^{n} \eta_j. \tag{V.32}$$

A given vertex x_j is generally an element of several different spaces $\mathscr{T}(-\beta, \alpha)$; for instance, Q is an element of $\mathscr{T}(\mu - 1, \mu)$ for every real μ. We say that a set of vertices $X = \{x_0, ..., x_n\}$ is regular if it satisfies Definition V.1.b for some given set of $\{(\beta_j, \alpha_j)\}$. Note that

$$0 < \eta_{\text{local}} \leqslant \eta_{\text{global}} \leqslant 1. \tag{V.33}$$

Furthermore, if $\{x_0, x_1, ..., x_n\}$ is a regular set of vertices, then so is any cyclic permutation

$$\{x_j, x_{j+1}, ..., x_n, x_0, x_1, ..., x_{j-1}\}. \tag{V.34}$$

The transformations $U(g)$ and γ commute with Q^2 on $\mathscr{H}$. Thus $U(g)$: $\mathscr{H}_p \to \mathscr{H}_p$ and γ: $\mathscr{H}_p \to \mathscr{H}_p$. We infer that $x \in \mathscr{T}(-\beta, \alpha)$ ensures

$$x^\gamma = \gamma x \gamma \in \mathscr{T}(-\beta, \alpha), \qquad \text{and} \qquad x^g = U(g)\, x U(g)^* \in \mathscr{T}(-\beta, \alpha). \tag{V.35}$$

It follows that if $\{x_0, ..., x_n\}$ is a regular set of vertices, then so is

$$\{x_0^{g_0}, x_1^{g_1}, ..., x_n^{g_n}\} \tag{V.36}$$

for $g_0, g_1, ..., g_n \in \mathfrak{G}$. Similarly any of the x_j's may be replaced by x_j^{γ}.

DEFINITION V.2. The *heat kernel regularization* of a regular set of vertices $X = \{x_0, x_1, ..., x_n\}$ with respect to Q is defined for $s \in \sigma_n$ by the following sesquilinear form on $\mathscr{H} \times \mathscr{H}$,

$$X(s) = R^{\beta_0} x_0 e^{-s_0 Q^2} x_1 e^{-s_1 Q^2} \cdots x_n e^{-s_n Q^2} R^{-\beta_0}. \tag{V.37}$$

We take $X(s) = 0$ for $s \notin \sigma_n$.

Note that $s \in \sigma_n$ ensures that each $s_j > 0$. Hence the form (V.37) is bounded on $\mathscr{H} \times \mathscr{H}$, and $X(s)$ uniquely determines a bounded, linear operator on $\mathscr{H}$, which we also denote by $X(s)$. Furthermore, $\exp(-\beta Q^2)$ is trace class for all $\beta > 0$, so $X(s)$ is a trace class operator on $\mathscr{H}$.

PROPOSITION V.3. *Assume that $X(s)$ is the heat kernel regularization* (V.37) *of a regular set of vertices with respect to Q. Then for any μ in the interval* $0 < \mu < 1$:

(i) *The trace norm of $X(s)$ is bounded for $s \in \sigma_n$, as defined in* (V.21), *by*

$$\|X(s)\|_1 \leqslant (2\mu^{-(1-\eta_{\text{global}})})^{n+1} \operatorname{Tr}(e^{-(1-\mu) Q^2}) \left(\prod_{j=0}^{n} s_j^{-(1-\eta_j)} \|x_j\|_{(-\beta_j, \alpha_j)} \right), \tag{V.38}$$

with η_{global} defined in (V.32) *and η_j in* (V.31).

(ii) *The map $s \mapsto X(s)$ is continuous from σ_n to I_1, the Schatten ideal of trace class operators. In fact the map is Hölder continuous with exponent η' less than η_{local}, up to the boundary of σ_n. For the Euclidean distance $|s - s'|$ sufficiently small, and with m_1, m_2 the constant defined in* (V.57),

$$\|X(s) - X(s')\|_1 \leqslant m_1 m_2^{n+1} |s - s'|^{\eta'} \left(\prod_{j=0}^{n} s_j^{-1+\eta_j} \right) \left(\sum_{j=0}^{n} s_j^{-\eta'} \right) \times \left(\prod_{j=0}^{n} \|x_j\|_{(-\beta_j, \alpha_j)} \right). \tag{V.39}$$

Since $\eta' < \eta_{\text{local}}$ the right-hand side of (V.39) *is integrable over $s \in \sigma_n$.*

(iii) *If each* $x_j \in \mathscr{B}(\mathscr{H})$, *then for* $s \in \sigma_n$,

$$\|X(s)\|_1 \leqslant \operatorname{Tr}(e^{-Q^2}) \left(\prod_{j=0}^{n} \|x_j\| \right). \tag{V.40}$$

Proof. Define the following operators T_j, S_j, $j=0, 1, ..., n$:

$$T_j = R^{\beta_j} x_j R^{\alpha_j} \qquad S_j = R^{-\alpha_j - \beta_{j+1}} e^{-s_j Q^2}, \tag{V.41}$$

where $\beta_{n+1} := \beta_0$. Then $X(s) = T_0 S_0 T_1 S_1 \cdots T_n S_n$. Each T_j is bounded, and

$$\|T_j\| = \|x_j\|_{(-\beta_j, \alpha_j)}. \tag{V.42}$$

Each S_j is in the Schatten class $I_{s_j^{-1}}$. In fact, for $0 < \mu < 1$, by the Hölder inequality for Schatten norms (V.18),

$$\begin{aligned} \|S_j\|_{s_j^{-1}} &\leqslant \|R^{-\alpha_j - \beta_{j+1}} e^{-\mu s_j Q^2}\|_\infty \, \|e^{-(1-\mu) s_j Q^2}\|_{s_j^{-1}} \\ &\leqslant 2(\mu s_j)^{-(\alpha_j + \beta_{j+1})/2} \, (\operatorname{Tr}(e^{-(1-\mu) Q^2}))^{s_j}, \end{aligned} \tag{V.43}$$

where we use the bound (V.28) for the $\|\cdot\|_{I_\infty}$ (operator) norm. Thus using Hölders inequality (V.19) on $X(s)$ with the exponent ∞ for T_j and the exponent s_j^{-1} for S_j, and using $\sum_{j=0}^n s_j = 1$, we have (with the exponents η_j and η_{global} defined in (V.31, 32))

$$\|X(s)\|_1 \leqslant 2^{n+1} \mu^{-(n+1)(1-\eta_{\text{global}})} \operatorname{Tr}(e^{-(1-\mu) Q^2}) \left(\prod_{j=0}^{n} s_j^{-(1-\eta_j)} \|x_j\|_{(-\beta_j, \alpha_j)} \right),$$

which is the bound (V.38). Note that if $x_j \in \mathscr{B}(\mathscr{H})$, $j=0, 1, ..., n$, then $\alpha_j = \beta_j = 0$, $j=0, 1, ..., n$ and we can take $\mu = 0$ in the bound on S_j. In fact, we have $\|T_j\| = \|x_j\|$ and $\|S_j\|_{I_{s_j^{-1}}} = (\operatorname{Tr}(e^{-Q^2}))^{s_j}$. Thus the factor 2^{n+1} in (V.38) can be replaced by 1. Also $\eta_j = 1 = \eta_{\text{global}}$, for all j, so in this case we have (V.40). This completes the proof of (i) and (iii).

(ii) In order to establish continuity of $s \to X(s)$ at s, we consider $X(s) - X(s')$ where $s, s' \in \sigma_n$ and where s' is sufficiently close to s. Let $\tilde{s} = \min_j s_j$; note $s \in \sigma_n$ ensures $\tilde{s} > 0$. We suppose that s' lies in the neighborhood of s defined by

$$\sup_j |s_j - s_j'| < \varepsilon \tilde{s}. \tag{V.44}$$

We take $0 < \varepsilon < 1$. Thus

$$|s_j - s_j'| \, s_j^{-1} \leqslant \varepsilon, \qquad \text{and} \qquad s_j' \geqslant (1-\varepsilon) \, s_j, \qquad j=0, 1, ..., n. \tag{V.45}$$

The first inequality in (V.45) is a consequence of

$$|s_j - s'_j| \leqslant \varepsilon \tilde{s} \leqslant \varepsilon s_j,$$

while the second inequality follows by

$$s'_j = s_j + (s'_j - s_j) \geqslant s_j - |s_j - s'_j| \geqslant (1-\varepsilon)\, s_j.$$

We now show that on this set, and for any $\eta' < \eta_{\text{local}}$,

$$\|X(s) - X(s')\|_1 \leqslant O(|s - s'|^{\eta'}). \tag{V.46}$$

In other words, $s \mapsto X(s)$ is Hölder continuous with any exponent $\eta' < \eta_{\text{local}}$.

We require a slightly different set of bounds from (V.43). Let us denote $S_j(s)$ by S_j and $S_j(s')$ by S'_j. The bound (V.45) ensures

$$\begin{aligned}
\|S'_j\|_{s_j^{-1}} &\leqslant \|R^{-\alpha_j - \beta_{j+1}} e^{-\mu s'_j Q^2}\|_\infty \, \|e^{-(1-\mu)\, s'_j Q^2}\|_{s_j^{-1}} \\
&\leqslant 2(\mu s'_j)^{-(\alpha_j + \beta_{j+1})/2}\, (\operatorname{Tr}(e^{-(1-\mu)(s'_j/s_j)\, Q^2}))^{s_j} \\
&\leqslant 2((1-\varepsilon)\, \mu s_j)^{-(\alpha_j + \beta_{j+1})/2}\, (\operatorname{Tr}(e^{-(1-\mu)(1-\varepsilon)\, Q^2}))^{s_j}.
\end{aligned} \tag{V.47}$$

Furthermore we establish for $\eta' < \eta_{\text{local}}$

$$\|S_j - S'_j\|_{s_j^{-1}} \leqslant M s_j^{-1+(\eta_j - \eta')}\, |s_j - s'_j|^{\eta'}\, (\operatorname{Tr}(e^{-(1-\varepsilon)(1-\mu)\, Q^2}))^{s_j}, \tag{V.48}$$

where

$$M = 2\mu^{-2+\eta_j\varepsilon - \eta'}. \tag{V.49}$$

Let $s_j(\alpha) = \alpha s_j + (1-\alpha)\, s'_j$ interpolate between s_j and s'_j. Then

$$\begin{aligned}
S_j - S'_j = S_j(s_j(\alpha))|_0^1 &= \int_0^1 \frac{dS_j(s_j(\alpha))}{d\alpha}\, d\alpha \\
&= (s'_j - s_j) \int_0^1 Q^2 S_j(s_j(\alpha))\, d\alpha \\
&= (s'_j - s_j) \int_0^1 Q^2 R^{-(\alpha_j + \beta_{j+1})} e^{-s_j(\alpha)\, Q^2} d\alpha.
\end{aligned} \tag{V.50}$$

Thus

$$\|S_j - S'_j\|_{s_j^{-1}} \leqslant |s_j - s'_j| \int_0^1 \|Q^2 R^{-(\alpha_j+\beta_{j+1})} e^{-\mu s_j(\alpha) Q^2}\| \, \|e^{-(1-\mu) s_j(\alpha) Q^2}\|_{s_j^{-1}} \, d\alpha$$

$$\leqslant |s_j - s'_j| \int_0^1 2(\mu s_j(\alpha))^{-(2+\alpha_j+\beta_{j+1})/2}$$

$$\times (\mathrm{Tr}(e^{-(1-\mu) s_j^{-1} s_j(\alpha) Q^2}))^{s_j} \, d\alpha. \qquad \text{(V.51)}$$

For $0 \leqslant \alpha \leqslant 1$,

$$s_j(\alpha) \geqslant (1-\varepsilon) \, s_j. \qquad \text{(V.52)}$$

Therefore

$$\|S_j - S'_j\|_{s_j^{-1}} \leqslant 2 \, |s_j - s'_j| \, (\mu s_j)^{-(2+\alpha_j+\beta_{j+1})/2} \, (\mathrm{Tr}(e^{-(1-\varepsilon)(1-\mu) Q^2}))^{s_j}. \qquad \text{(V.53)}$$

Write $s_j^{-(2+\alpha_j+\beta_{j+1})/2} = s_j^{-(1-\eta')} s_j^{-(\alpha_j+\beta_{j+1}+2\eta')/2}$, and use (V.47). Thus

$$\|S_j - S'_j\|_{s_j^{-1}} \leqslant 2 \, |s_j - s'_j|^{\eta'} \, s_j^{-(\alpha_j+\beta_{j+1}+2\eta')/2} \varepsilon^{(1-\eta')} \mu^{-(1+\alpha_j+\beta_{j+1})/2}$$

$$\times (\mathrm{Tr}(e^{-(1-e)(1-\mu) Q^2}))^{s_j}. \qquad \text{(V.54)}$$

With η_j given in (V.30) and M of (V.49) we have (V.48).

Now write

$$X(s) - X(s') = \sum_{j=0}^{n} T_0 S_0 T_1 S_1 \cdots T_j (S_j - S'_j) \, T_{j+1} S'_{j+1} \cdots T_n S'_n. \qquad \text{(V.55)}$$

Estimate $\|X(s) - X(s')\|_1$ using Hölder's inequality in the I_p norms, as in the derivation of the bound on $X(s)$. Use the operator norm on each T_j and the $\|\cdot\|_{s_j^{-1}}$-Schatten norm on S_j, on S'_j, or on $S_j - S'_j$.

We obtain from (V.42, 43, 47, 48) the following bound on (V.55):

$$\|X(s) - X(s')\|_1 \leqslant m_1 m_2^{n+1} \, |s - s'|^{\eta'} \left(\prod_{j=0}^{n} s_j^{(-1+\eta_j)} \right) \left(\sum_{j=0}^{n} s_j^{-\eta'} \right)$$

$$\times \left(\prod_{j=0}^{n} \|x_j\|_{(-\beta_j, \alpha_j)} \right). \qquad \text{(V.56)}$$

Here

$$m_1 = \mu^{-1} \varepsilon^{(1-\eta')} \, \mathrm{Tr}(e^{-(1-\varepsilon)(1-\mu) Q^2}), \qquad \text{and}$$
$$m_2 = 2((1-\varepsilon) \, \mu)^{-(1-\eta_{\mathrm{global}})}. \qquad \text{(V.57)}$$

This completes the proof of the proposition.

COROLLARY V.4. *Assume* $X(s)$ *is the heat kernel regularization* (V.37) *for a regular set of vertices with respect to* Q, *and that* $\exp(-\beta Q^2)$ *is trace class for all* $\beta > 0$. *Then with* $d^n s = d^n s(1)$ *defined in* (IV.12–13),

(i) *The Radon transform*

$$\hat{X} = \int_{\mathbb{R}_+^{n+1}} X(s)\, d^n s \tag{V.58}$$

exists and is a trace class operator on $\mathscr{H}$.

(ii) *The trace and integration of* $\gamma U(g)\, X(s)$ *commute, namely*

$$\operatorname{Tr}(\gamma U(g)\, \hat{X}) = \int_{\mathbb{R}_+^{n+1}} \operatorname{Tr}(\gamma U(g)\, X(s))\, d^n s. \tag{V.59}$$

(iii) *For* $s \in \sigma_n$, *defined in* (*V*.21), *and for* $0 < \mu < 1$, *the quantity*

$$\begin{aligned} &\operatorname{Tr}(\gamma U(g)\, X(s)) \\ &\quad = \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} x_0 e^{-s_0 Q^2} x_1 e^{-s_1 Q^2} \cdots e^{-s_{n-1} Q^2} x_n e^{-(1-\mu)\, s_n Q^2}), \end{aligned} \tag{V.60}$$

is independent of μ. *Thus we define*

$$\begin{aligned} &\operatorname{Tr}(\gamma U(g)\, x_0 e^{-s_0 Q^2} \cdots x_n e^{-s_n Q^2}) \\ &\quad = \lim_{\mu \to 0+} \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} x_0 e^{-s_0 Q^2} \cdots x_n e^{-(1-\mu)\, s_n Q^2}) \\ &\quad = \operatorname{Tr}(\gamma U(g)\, X(s)). \end{aligned} \tag{V.61}$$

In summary, we write $\operatorname{Tr}(\gamma U(g)\, \hat{X})$ *as*

$$\langle x_0, x_1, ..., x_n; g \rangle_n = \int_{\mathbb{R}_+^{n+1}} \operatorname{Tr}(\gamma U(g)\, x_0 e^{-s_0 Q^2} \cdots x_n e^{-s_n Q^2})\, d^n s. \tag{V.62}$$

(iv) *Given* $0 < \mu < 1$, *the expectation* (V.62) *satisfies*

$$|\langle x_0, x_1, ..., x_n; g \rangle_n| \leqslant m_1 m_2^{n+1} \Gamma((n+1)\, \eta_{\text{global}})^{-1} \left(\prod_{j=0}^{n} \|x_j\|_{(-\beta_j, \alpha_j)} \right), \tag{V.63}$$

for constants

$$m_1 = \operatorname{Tr}(e^{-(1-\mu)\, Q^2}), \qquad m_2 = 2\Gamma(\eta_{\text{local}})\, \mu^{-(1-\eta_{\text{global}})}. \tag{V.64}$$

(v) *If all* $x_i \in \mathscr{B}(\mathscr{H})$, *so* $\alpha_i = \beta_i = 0$, $i = 0, 1, 2, ..., n$, *then*

$$|\langle x_0, ..., x_n; g\rangle_n| \leqslant \frac{1}{n!} \operatorname{Tr}(e^{-Q^2}) \left(\prod_{j=0}^{n} \|x_j\| \right). \tag{V.65}$$

(vi) *The expectation* $\langle x_0, x_1, ..., x_n; g\rangle_n$ *satisfies the symmetries* (IV.15–19). *Thus,*

$$\langle x_0, ..., x_n; g\rangle_n = \langle x_0^g, ..., x_n^g; g\rangle_n = \langle x_0^\gamma, ..., x_n^\gamma; g\rangle_n, \tag{V.66}$$

$$\langle x_0, ..., x_n; g\rangle_n = \langle x_n^{g^{-1}\gamma}, x_0, x_1, ..., x_{n-1}\rangle_n, \tag{V.67}$$

$$\langle x_0, ..., x_n; g\rangle_n = \sum_{j=1}^{n+1} \langle x_0, ..., x_{j-1}, I, x_j, ..., x_n; g\rangle_{n+1}, \tag{V.68}$$

and if both $\{x_0, x_1, ..., x_{j-1}, Qx_j, x_{j+1}, ..., x_n\}$ *and* $\{x_0, x_1, ..., x_{j-1}, x_jQ, x_{j+1}, ..., x_n\}$ *are also a regular set of vertices for* $j = 0, 1, ..., n$, *then*

$$\langle d\hat{X}; g\rangle = \sum_{j=0}^{n} \langle x_0^\gamma, x_1^\gamma, ..., x_{j-1}^\gamma, dx_j, x_{j+1}, ..., x_n; g\rangle_n = 0. \tag{V.69}$$

Proof. (i–ii) We showed in Proposition V.3i–ii, that $s \mapsto X(s)$ is Hölder continuous from σ_n to the Schatten class I_1. Using this bound, we infer that the Radon transform $\int_{\mathbb{R}_+^{n+1}} X(s)\, d^n s$ exists on the unit hyperplane, and the integral and trace commute

$$\int_{\mathbb{R}_+^{n+1}} \operatorname{Tr}(X(s))\, d^n s = \operatorname{Tr}\left(\int_{\mathbb{R}_+^{n+1}} X(s)\, d^n s \right).$$

This is also the case with $X(s)$ replaced by $TX(s)$, for $T \in \mathscr{B}(\mathscr{H})$. In particular (V.59) holds.

(iii) We evaluate $\operatorname{Tr}(\gamma U(g)\, X(s))$ for $s \in \sigma_n$. With the notation (V.41), $\operatorname{Tr}(\gamma U(g)\, X(s)) = \operatorname{Tr}(\gamma U(g)\, T_0 S_0 \cdots T_n S_n)$. Each S_j is trace class, and for $0 < \mu < 1$, both $e^{-\mu Q^2}$ and $e^{-(1-\mu) Q^2}$ are trace class. Thus

$$\begin{aligned}
\operatorname{Tr}(\gamma U(g)\, X(s)) &= \operatorname{Tr}(\gamma U(g)\, T_0 S_0 \cdots T_n S_n e^{\mu s_n Q^2} e^{-\mu s_n Q^2}) \\
&= \operatorname{Tr}(e^{-\mu s_n Q^2} \gamma U(g)\, T_0 S_0 \cdots T_n S_n e^{\mu s_n Q^2}) \\
&= \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} R^{\beta_0} x_0 R^{\alpha_0} S_0 \cdots T_n R^{-\alpha_n} R^{-\beta_0} e^{-(1-\mu)\, s_n Q^2}).
\end{aligned}$$

Here we use the fact that Q^2 commutes with γ and with $U(g)$. Also, R^{β_0} commutes with Q^2, with γ, and with $U(g)$. Therefore we can also cyclically permute R^{β_0} in the trace to yield

$$\begin{aligned}
&\operatorname{Tr}(\gamma U(g)\, X(s))\\
&\quad = \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} R^{\beta_0} x_0 R^{\alpha_0} S_0 \cdots T_n R^{-\alpha_n} R^{-\beta_0} e^{-(1-\mu)\, s_n Q^2})\\
&\quad = \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} x_0 R^{\alpha_0} S_0 \cdots T_n R^{-\alpha_n} e^{-(1-\mu)\, s_n Q^2})\\
&\quad = \operatorname{Tr}(\gamma U(g)\, e^{-\mu s_n Q^2} x_0 e^{-s_0 Q^2} x_1 e^{-s_1 Q^2} \cdots e^{-s_{n-1} Q^2} x_n e^{-(1-\mu)\, s_n Q^2}).
\end{aligned}$$

Since this is true for any μ in the range, $\operatorname{Tr}(\gamma U(g)\, X(s))$ is independent of μ, and we have established (V.61). This completes the proof of (iii).

(iv–v) Note that

$$\begin{aligned}
|\langle x_0, x_1, ..., x_n; g\rangle| = |\operatorname{Tr}(\gamma U(g)\, \hat{X})| &\leqslant \|\gamma U(g)\|\ \|\hat{X}\|_1\\
&\leqslant \int_{\mathbb{R}_+^{n+1}} \|X(s)\|_1\, d^n s. \qquad \text{(V.70)}
\end{aligned}$$

Thus the bounds (V.63–65) are established by integrating (V.38) and using the definition of B_n, see (V.20). Note that $\eta_j < 1$, and $\Gamma(\eta_j)$ is monotonic decreasing on (0, 1). Thus $\Gamma(\eta_j) \leqslant \Gamma(\eta_{\text{local}})$.

(vi) The symmetries (V.66, 67, 69) can be established as the corresponding symmetries for $\operatorname{Tr}(\gamma U(g)\, X(s))$, expressed as (V.60). Then we integrate over σ_n. In the case of (V.68), we follow the argument in Section IV leading to (IV.22).

V.5. *Interpolation Spaces*

In this section we define certain Banach algebras $\mathfrak{J}_{\beta,\alpha}$ consisting of operators $b \in \mathscr{B}(\mathscr{H})$ with a bounded fractional derivative. We call these spaces *interpolation spaces*. These spaces are a natural framework for the study of the JLO cochain, and in Section VI we introduce algebras $\mathfrak{A} \subset \mathfrak{J}_{\beta,\alpha}$ to study τ^{JLO} on $\mathscr{C}(\mathfrak{A})$.

As in previous sections let $Q = Q^*$ with domain $\mathscr{D} = \mathscr{H}_1 \subset \mathscr{H}$, and let $R = (Q^2 + I)^{-1/2}$. We say that b has a bounded derivative of order $\alpha > 0$, if b is a bounded linear transformation on $\mathscr{H}_\alpha$. In other words, the form $R^{-\alpha} b R^{\alpha}$ defines a bounded operator which we denote

$$R^{-\alpha} b R^{\alpha} \in \mathscr{B}(\mathscr{H}). \qquad \text{(V.71)}$$

In the notation of Section V.2, $b \in \mathscr{T}(\alpha, \alpha)$. Let us define $\mathfrak{J}_\alpha$ as $\mathscr{B}(\mathscr{H}) \cap \mathscr{T}(\alpha, \alpha)$ with the norm

$$\|b\|_{\mathfrak{J}_\alpha} = (\|b\| + \|R^{-\alpha} b R^{\alpha}\|). \qquad \text{(V.72)}$$

If $b \in \mathfrak{J}_\alpha$, then a bilinear form on $\mathscr{D}_\alpha \times \mathscr{D}_\alpha$,

$$\langle R^{-\alpha} g, b R^\alpha f \rangle = \langle g, R^{-\alpha} b R^\alpha f \rangle, \tag{V.73}$$

so

$$|\langle R^{-\alpha} g, b R^\alpha f \rangle| \leqslant \|b\|_{\mathfrak{J}_\alpha} \|g\| \, \|f\|. \tag{V.74}$$

We now show that $\mathfrak{J}_\alpha$ is a Banach algebra. The range of b is contained in $\mathscr{H}_\alpha$, and hence is in the domain of a. Hence

$$\begin{aligned} \|ab\|_{\mathfrak{J}_\alpha} &= \|ab\| + \|R^{-\alpha} a R^\alpha R^{-\alpha} b R^\alpha\| \\ &\leqslant \|a\| \, \|b\| + \|R^{-\alpha} a R^\alpha\| \, \|R^{-\alpha} b R^\alpha\| \leqslant \|a\|_{\mathfrak{J}_\alpha} \|b\|_{\mathfrak{J}_\alpha}, \end{aligned} \tag{V.75}$$

showing that desired property.

It is useful to characterize fractional differentiability not by the property (V.71), but rather by some properties of $db = Qb - b^\gamma Q$. The reason is that the expression db arises in the JLO-cochain of Section IV, and this is central to our geometric interpretation in Section VII.

We now define a family of subalgebras $\mathfrak{J}_{\beta, \alpha}$ of $\mathfrak{J}_{1-\beta}$, and we extend the definition of our cochain from the differentiable elements treated in Section IV, to these algebras of fractionally differentiable functions. Let $0 \leqslant \alpha, \beta$ and $0 \leqslant \alpha + \beta < 1$. If $b \in \mathscr{B}(\mathscr{H})$, then $b \in \mathfrak{J}_{\beta, \alpha}$ if $db \in \mathscr{T}(-\beta, \alpha)$. In other words, $\mathfrak{J}_{\beta, \alpha}$ consists of elements b of $\mathscr{B}(\mathscr{H})$ such that (in the notation of Section V.3) db is a vertex of type (β, α). We give $\mathfrak{J}_{\beta, \alpha}$ the norm

$$\begin{aligned} \|b\|_{\mathfrak{J}_{\beta, \alpha}} &= \|b\| + c_{\alpha+\beta} \|db\|_{(-\beta, \alpha)} \\ &= \|b\| + c_{\alpha+\beta} \|R^\beta \, db \, R^\alpha\|. \end{aligned} \tag{V.76}$$

Here we define c_μ for $0 \leqslant \mu < 1$ by

$$c_\mu = \sup_{0 \leqslant \delta \leqslant 1} 2\delta \int_0^\infty t^{-\delta/2} (1+t)^{-1-(1-\mu)/2+\delta/2} \, dt. \tag{V.77}$$

Note that the function $c_\mu(\delta, t) = 2\delta t^{-\delta/2} (1+t)^{-1-(1-\mu)/2+\delta/2} \geqslant 0$ satisfies

$$\frac{\partial c_\mu(\delta, t)}{\partial \delta} = \left(\delta^{-1} + \frac{1}{2} \log(1+t^{-1}) \right) c_\mu(\delta, t) \geqslant 0, \qquad \text{and}$$

$$\frac{\partial c_\mu(\delta, t)}{\partial \mu} = \frac{1}{2} \log(1+t) \, c_\mu(\delta, t) \geqslant 0.$$

Hence $c_\mu(\delta, t)$ increases monotonically in δ for fixed μ, and in μ for fixed δ. Therefore

$$c_\mu = 2\int_0^\infty t^{-1/2}(1+t)^{-1+\mu/2}\, dt \geqslant c_0 = 2\int_0^\infty t^{-1/2}(1+t)^{-1}\, dt = 2\pi. \quad \text{(V.78)}$$

In addition, c_μ diverges logarithmically as $\mu \nearrow 1$.

PROPOSITION V.5. *Let* $0 \leqslant \alpha, \beta$ *and* $0 \leqslant \alpha + \beta < 1$. *Then*

(i) $\mathfrak{I}_{\beta, \alpha} \subset \mathfrak{I}_\delta$ *for all*

$$0 \leqslant \delta \leqslant 1 - \beta. \quad \text{(V.79)}$$

In this case

$$\|b\|_{\mathfrak{I}_\delta} \leqslant 2\, \|b\|_{\mathfrak{I}_{\beta, \alpha}}.$$

(ii) *Let* δ *satisfy* $-(1-\alpha) \leqslant \delta \leqslant 1 - \beta$. *Then*

$$\mathfrak{I}_{\beta, \alpha} \subset \mathscr{T}(\delta, \delta). \quad \text{(V.80)}$$

Furthermore,

$$\|b\|_{(\delta, \delta)} = \|R^{-\delta} b R^{\delta}\| \leqslant \|b\| + \tfrac{1}{2} c_{\alpha+\beta}\, \|db\|_{(-\beta, \alpha)} \leqslant \|b\|_{\mathfrak{I}_{\beta, \alpha}}. \quad \text{(V.81)}$$

(iii) *If* $a \in \mathfrak{I}_{\beta, \alpha}$, *then* $a \in \mathscr{T}(-\beta, -\beta) \cap \mathscr{T}(\alpha, \alpha)$. *Also if* $a, b \in \mathfrak{I}_{\beta, \alpha}$, *then both* $(da)\, b$ *and* $a(db)$ *are elements of* $\mathscr{T}(-\beta, \alpha)$. *Also*

$$\|(da)\, b\|_{(-\beta, \alpha)} \leqslant \|da\|_{(-\beta, \alpha)}\, \|b\|_{(\alpha, \alpha)} \quad \text{(V.82)}$$

and

$$\|a(db)\|_{(-\beta, \alpha)} \leqslant \|a\|_{(-\beta, -\beta)}\, \|db\|_{(-\beta, \alpha)}. \quad \text{(V.83)}$$

COROLLARY V.6. *Let* $a, b \in \mathfrak{I}_{\beta, \alpha}$, *with* $0 \leqslant \alpha, \beta$ *and* $\alpha + \beta < 1$. *Then*

(i) *Graded Leibniz Rule*: *The relation*

$$d(ab) = (da)\, b + a^{\gamma}(db), \quad \text{(V.84)}$$

is an identity of elements in $\mathscr{T}(-\beta, \alpha)$, *namely between vertices* $d(ab)$, $(da)\, b$, *and* $a^{\gamma}(db)$ *of type* (β, α).

(ii) *The space* $\mathfrak{I}_{\beta, \alpha}$ *is a Banach algebra, so*

$$\|ab\|_{\mathfrak{I}_{\beta, \alpha}} \leqslant \|a\|_{\mathfrak{I}_{\beta, \alpha}}\, \|b\|_{\mathfrak{I}_{\beta, \alpha}}. \quad \text{(V.85)}$$

For $t \geqslant 0$, introduce the operators $R(t) = (Q^2 + (1+t)\, I)^{-1/2}$ and $R = R(0)$. Then the spectral theorem ensures that for $\mu \geqslant 0$,

$$\|R(t)^\mu\| \leqslant (1+t)^{-\mu/2}, \qquad \|R^{-\mu} R(t)^\mu\| \leqslant 1, \qquad \text{and} \qquad \|QR(t)\| \leqslant 1. \tag{V.86}$$

We use a standard representation for R^μ, $0 < \mu < 2$, which is a consequence of the Cauchy integral theorem applied to the function $z^{-\mu/2}$, namely

$$R^\mu = \frac{\sin(\pi\mu/2)}{\pi} \int_0^\infty t^{-\mu/2} R(t)^2 \, dt. \tag{V.87}$$

Proof of Proposition V.5 and Corollary V.6. (i) We estimate the $\mathfrak{I}_\delta$ norm of $b \in \mathfrak{I}_{\beta,\alpha}$. Let $\mathscr{D}_2 = \mathscr{D}(Q^2) = \mathrm{Range}(R(t)^2)$. On the form domain $\mathscr{D}_2 \times \mathscr{D}_2$, and for $0 \leqslant \delta \leqslant 1 - \beta$ we write

$$\|b\|_{\mathfrak{I}_\delta} = \|b\| + \|R^{-\delta} b R^\delta\| \leqslant 2\, \|b\| + \|R^{-\delta}[b, R^\delta]\|. \tag{V.88}$$

We now study $R^{-\delta}[b, R^\delta]$. Using (V.87),

$$[b, R^\delta] = \frac{\sin(\pi\delta/2)}{\pi} \int_0^\infty t^{-\delta/2} [b, R(t)^2] \, dt. \tag{V.89}$$

On $\mathscr{D}_2 \times \mathscr{D}_2$,

$$\begin{aligned} [b, R(t)^2] &= R(t)^2\, [R(t)^{-2}, b]\, R(t)^2 \\ &= R(t)^2\, (Q\, db + db^\gamma Q)\, R(t)^2. \end{aligned} \tag{V.90}$$

Here $b^\gamma = \gamma b \gamma$ and $(db)^\gamma = -db^\gamma$. Hence

$$R^{-\delta}[b, R^\delta] = \frac{\sin(\pi\delta/2)}{\pi} \int_0^\infty t^{-\delta/2} R^{-\delta} R(t)^2\, (Q\, db + db^\gamma Q)\, R(t)^2 \, dt. \tag{V.91}$$

We can bound (V.91) using (V.86). We use the following to estimate the first term on the right of (V.91),

$$\begin{aligned} \|R^{-\delta} R(t)^2\, Q\, db\, R(t)^2\| &\leqslant \|R^{-\delta} R(t)^2\, Q R^{-\beta}\|\ \|db\|_{(-\beta,\alpha)}\ \|R^{-\alpha} R(t)^2\| \\ &\leqslant (1+t)^{-(1-\beta-\delta)/2}\, (1+t)^{-(2-\alpha)/2}\ \|db\|_{(-\beta,\alpha)}, \end{aligned} \tag{V.92}$$

provided $\delta \leqslant 1 - \beta$, and $\alpha \leqslant 2$, as assumed in (V.79). Furthermore $db^\gamma = -(db)^\gamma$, and the unitarity of γ, along with $\gamma R = R\gamma$ ensures that for all (p, q),

$$\|db^\gamma\|_{(p,q)} = \|db\|_{(p,q)}. \tag{V.93}$$

We therefore estimate in a similar fashion the second term on the right of (V.91) namely

$$\|R^{-\delta}R(t)^2\, db^\gamma Q R(t)^2\| \leqslant (1+t)^{-(2-\delta-\beta)/2}\,(1+t)^{-(1-\alpha)/2}\,\|db\|_{(-\beta,\alpha)}, \tag{V.94}$$

provided $\beta+\delta\leqslant 2$ and $\alpha\leqslant 1$. But $\beta+\delta\leqslant 1$ by (V.79); also $0\leqslant\alpha, \beta$ and $\alpha+\beta<1$ ensures $\alpha<1$. Hence (V.94) does hold.

Using (V.92, 94), we bound (V.91). There are two similar bounds for the two terms in (V.91). We also use $\sin x\leqslant x$ for $0\leqslant x\leqslant\pi/2$. Thus

$$\begin{aligned}\|R^{-\delta}[b, R^\delta]\| &\leqslant \delta\int_0^\infty t^{-\delta/2}(1+t)^{-1-(1-\alpha-\beta)/2+\delta/2}\,dt\,\|db\|_{(-\beta,\alpha)}\\ &\leqslant \tfrac{1}{2}c_{\alpha+\beta}\,\|db\|_{(-\beta,\alpha)}.\end{aligned} \tag{V.95}$$

Here $c_{\alpha+\beta}$ is defined in (V.77), and is relevant since both $0\leqslant\alpha+\beta<1$ and $0\leqslant\delta\leqslant 1-\beta\leqslant 1$. Hence we conclude that $b\in\mathfrak{I}_\delta$ for $0\leqslant\delta\leqslant 1-\beta$.

To estimate the norm $\|b\|_{\mathfrak{I}_\delta}$, using (V.88) we have

$$\begin{aligned}\|b\|_{\mathfrak{I}_\delta} &\leqslant 2\,\|b\| + \|R^{-\delta}[b, R^\delta]\|\\ &\leqslant 2\,\|b\| + \tfrac{1}{2}c_{\alpha+\beta}\,\|db\|_{(-\beta,\alpha)}\\ &\leqslant 2\,\|b\|_{\mathfrak{I}_{\beta,\alpha}}.\end{aligned} \tag{V.96}$$

This completes the proof of part (i) of the proposition.

(ii) We have also proved part (ii) in case $0\leqslant\delta\leqslant 1-\beta$. In fact using (V.95)

$$\begin{aligned}\|b\|_{(\delta,\delta)} = \|R^{-\delta}bR^\delta\| &\leqslant \|b\| + \|R^{-\delta}[b, R^\delta]\|\\ &\leqslant \|b\| + \tfrac{1}{2}c_{\alpha+\beta}\,\|db\|_{(-\beta,\alpha)} \leqslant \|b\|_{\mathfrak{I}_{\beta,\alpha}}.\end{aligned} \tag{V.97}$$

Thus to complete the proof of (ii), we need to verify the case $\alpha-1\leqslant\delta\leqslant 0$. In that case, we show equivalently that $b\in\mathscr{T}(-\delta, -\delta)$ for $0\leqslant\delta\leqslant 1-\alpha$. Thus we need to verify that $R^\delta bR^{-\delta}$ is bounded. Write

$$\begin{aligned}R^\delta bR^{-\delta} &= b + [R^\delta, b]\,R^{-\delta}\\ &= b - \frac{\sin(\pi\delta/2)}{\pi}\int_0^\infty t^{-\delta/2}R(t)^2\,(Q\,db + db^\gamma\,Q)\,R(t)^2\,R^{-\delta}dt.\end{aligned} \tag{V.98}$$

Now we use the estimates (V.86), which yield

$$\| R(t)^2\, Q\, db\, R(t)^2\, R^{-\delta} \| \leqslant (1+t)^{-1-(1-\alpha-\beta)/2+\delta/2}\, \| db \|_{(-\beta,\,\alpha)}, \quad \text{(V.99)}$$

as long as both $\beta \leqslant 1$ and $\alpha + \delta \leqslant 2$. Since we assume $0 \leqslant \alpha, \beta$ and $\alpha + \beta < 1$, it follows that $\beta < 1$. Also $\alpha < 1$, and since $0 \leqslant \delta \leqslant 1 - \alpha$, we infer that $\alpha + \delta \leqslant 1$. Thus both conditions are met and (V.99) holds. Likewise

$$\| R(t)^2\, db^{\gamma}\, QR(t)^2\, R^{-\delta} \| \leqslant (1+t)^{-1-(1-\alpha-\beta)/2+\delta/2}\, \| db \|_{(-\beta,\,\alpha)}, \quad \text{(V.100)}$$

if both $\beta \leqslant 2$ and $\alpha + \delta \leqslant 1$. Both these conditions also hold. Thus from (V.98–100) we infer that

$$\begin{aligned} \| R^{\delta} b R^{-\delta} \| &= \| b \|_{(-\delta,\,-\delta)} \\ &\leqslant \| b \| + \delta \int_0^{\infty} t^{-\delta/2} (1+t)^{-1-(1-(\alpha+\beta))/2+\delta/2}\, dt\, \| db \|_{(-\beta,\,\alpha)} \\ &\leqslant \| b \| + \tfrac{1}{2} c_{\alpha+\beta}\, \| db \|_{(-\beta,\,\alpha)}. \end{aligned} \quad \text{(V.101)}$$

Hence $b \in \mathscr{T}(-\delta, -\delta)$ and (V.80–81) hold as claimed.

(iii) Let us assume $a \in \mathfrak{J}_{\beta,\,\alpha}$. Then from (ii), and the restrictions $0 \leqslant \alpha, \beta$ and $\alpha + \beta < 1$, we infer

$$a \in \mathscr{T}(-\beta, -\beta) \cap \mathscr{T}(\alpha, \alpha). \quad \text{(V.102)}$$

Thus we can estimate $(da)\, b$ as a map from $\mathscr{H}_{\alpha}$ to $\mathscr{H}_{-\beta}$ as

$$\| (da)\, b \|_{-\beta,\,\alpha} \leqslant \| da \|_{(-\beta,\,\alpha)}\, \| b \|_{(\alpha,\,\alpha)},$$

showing (V.82). Likewise, since γ commutes with Q,

$$\| a^{\gamma} \|_{(-\beta,\,-\beta)} = \| a \|_{(-\beta,\,-\beta)}, \quad \text{(V.103)}$$

and

$$\| a^{\gamma}(db) \|_{(-\beta,\,\alpha)} \leqslant \| a^{\gamma} \|_{(-\beta,\,-\beta)}\, \| db \|_{(-\beta,\,\alpha)} = \| a \|_{(-\beta,\,-\beta)}\, \| db \|_{(-\beta,\,\alpha)}, \quad \text{(V.104)}$$

which is (V.83). We therefore conclude that $(da)\, b$ and $a^{\gamma}(db)$ are both elements of $\mathscr{T}(-\beta, \alpha)$. This completes the proof of the proposition.

To establish the corollary, note $b \in \mathscr{T}(\alpha, \alpha)$, $Q \in \mathscr{T}(\alpha - 1, \alpha)$ and $a^{\gamma} \in \mathscr{T}(\alpha - 1, \alpha - 1)$, according to (V.80). Therefore $a^{\gamma} Q b \in \mathscr{T}(\alpha - 1, \alpha)$. The important conclusion here is that $a^{\gamma} Q b$ is defined as a sesquilinear form on $\mathscr{H} \times \mathscr{H}$ with some domain; in fact the domain is $\mathscr{D}_{1-\alpha} \times \mathscr{D}_{\alpha}$. But $\beta < 1 - \alpha$, so $\mathscr{D}_{1-\alpha} \subset \mathscr{D}_{\beta}$, and $\mathscr{D}_{1-\alpha} \times \mathscr{D}_{\alpha} \subset \mathscr{D}_{\beta} \times \mathscr{D}_{\alpha}$, which is contained in the domain of $(da)\, b$ and $a^\gamma(db)$. Furthermore Qab and $(ab)^\gamma\, Q$ are both forms on the domains $\mathcal{D}_1 \times \mathcal{D}_1 \subset \mathcal{D}_{1-\alpha} \times \mathcal{D}_\alpha$. Thus on $\mathcal{D}_1 \times \mathcal{D}_1$, we have the identity

$$\begin{aligned} d(ab) &= Qab - (ab)^\gamma\, Q \\ &= Qab - a^\gamma Q b + a^\gamma Q b - a^\gamma b^\gamma Q \\ &= (da)\, b + a^\gamma(db). \end{aligned} \tag{V.105}$$

However, by Proposition V.5.iii, each term on the right-side of (V.105) extends by continuity to $\mathcal{D}_\beta \times \mathcal{D}_\alpha$. Thus $d(ab)$ also extends by continuity to this domain, and the identity (V.84) holds in $\mathcal{T}(-\beta, \alpha)$. We have therefore demonstrated the Leibniz rule (V.84) as an identity on $\mathcal{T}(-\beta, \alpha)$.

Finally we estimate $\|ab\|_{\mathfrak{J}_{\beta,\alpha}}$ for $a, b \in \mathfrak{J}_{\beta,\alpha}$. Using (V.81–84), and the definition (V.76) of the norm on $\mathfrak{J}_{\beta,\alpha}$, we conclude

$$\begin{aligned} \|ab\|_{\mathfrak{J}_{\beta,\alpha}} &= \|ab\| + c_{\alpha+\beta}\, \|d(ab)\|_{(-\beta,\alpha)} \\ &\leqslant \|a\|\, \|b\| + c_{\alpha+\beta}(\,\|(da)\, b\|_{(-\beta,\alpha)} + \|a(db)\|_{(-\beta,\alpha)}) \\ &\leqslant \|a\|\, \|b\| + c_{\alpha+\beta}\, \|da\|_{(-\beta,\alpha)}\, (\|b\| + \tfrac{1}{2} c_{\alpha+\beta}\, \|db\|_{(-\beta,\alpha)}) \\ &\quad + c_{\alpha+\beta}\, \|db\|_{(-\beta,\alpha)}\, (\|a\| + \tfrac{1}{2} c_{\alpha+\beta}\, \|da\|_{(-\beta,\alpha)}) \\ &= (\,\|a\| + c_{\alpha+\beta}\, \|da\|_{(-\beta,\alpha)})(\,\|b\| + c_{\alpha+\beta}\, \|db\|_{(-\beta,\alpha)}) \\ &= \|a\|_{\mathfrak{J}_{\beta,\alpha}}\, \|b\|_{\mathfrak{J}_{\beta,\alpha}}. \end{aligned} \tag{V.106}$$

Thus $\mathfrak{J}_{\beta,\alpha}$ is a Banach algebra, and the proof of the corollary is complete.

V.6. *Generalized Schatten Classes*

In Section V.4 we introduced the spaces $\mathcal{T}(p_2, p_1)$ of generalized functions as bounded, linear transformations from $\mathcal{H}_{p_1}$ to $\mathcal{H}_{p_2}$. It is convenient to introduce subspaces of $\mathcal{T}(p_2, p_1)$ which are Schatten I_p classes, with $\mathcal{T}(p_2, p_1)$ being the I_∞ case. We measure I_p size in terms of the Schatten norm (V.17). We say $R^{-p_2} x R^{p_1} \in I_p$, if the bilinear form $R^{-p_2} x R^{p_1}$ uniquely determines an operator in $\mathcal{B}(\mathcal{H})$ which belongs to the Schatten ideal I_p. Thus for $1 \leqslant p$, define the generalized Schatten class

$$\mathcal{T}(p_2, p_1; p) = \{x \colon x \in \mathcal{T}(p_2, p_1),\, R^{-p_2} x R^{p_1} \in I_p\}. \tag{V.107}$$

Let $\mathcal{T}(p_2, p_1; p)$ be a normed space with norm

$$\|x\|_{\mathcal{T}(p_2, p_1; p)} = \|R^{-p_2} x R^{p_1}\|_{I_p}. \tag{V.108}$$

The norms $\mathcal{T}(p_2, p_1; p)$ satisfy a Hölder inequality, as a consequence of the inequality (V.18) for Schatten class I_p norms.

Hölder Inequality

Let $x_j \in \mathscr{T}(\alpha_j, \alpha_{j+1}; p_j)$, $j=0, 1, ..., n$, where $1 \leqslant p_j$, $\sum_{j=0}^{n} p_j^{-1} = p^{-1} \leqslant 1$. Then $x_0 x_1 \cdots x_n \in \mathscr{T}(\alpha_0, \alpha_{n+1}; p)$ and

$$\|x_0 x_1 \cdots x_n\|_{\mathscr{T}(\alpha_0, \alpha_{n+1}; p)} \leqslant \prod_{j=0}^{n} \|x_j\|_{\mathscr{T}(\alpha_j, \alpha_{j+1}; p_j)}. \tag{V.109}$$

Using the results of Section V.5, we arrive at certain relations between $\mathfrak{I}_{\beta, \alpha}$ and $\mathscr{T}(-\beta, \alpha; p)$.

PROPOSITION V.7. *Let $Q = Q^*$ and $e^{-sQ^2} \in I_1$, for all $s > 0$. Let $b \in \mathfrak{I}_{\beta, \alpha}$ for $0 \leqslant \alpha, \beta$, and $0 \leqslant \alpha + \beta < 1$. Then*

(i)

$$Q\, db \in \mathscr{T}(-\beta - 1, \alpha), \qquad db^{\gamma} Q \in \mathscr{T}(-\beta, \alpha + 1), \tag{V.110}$$

and

$$d^2 b = Q\, db + (db^{\gamma})\, Q = [Q^2, b] \in \mathscr{T}(-\beta - 1, \alpha + 1). \tag{V.111}$$

(ii) *Let $0 \leqslant s$. Then as a form on $\mathscr{H}_\infty \times \mathscr{H}_\infty$,*

$$[b, e^{-sQ^2}] = \int_0^s e^{-tQ^2}\, d^2 b e^{-(s-t)\, Q^2}\, dt. \tag{V.112}$$

Both sides of (V.112) *also define operators in $\mathscr{B}(\mathscr{H})$.*

(iii) *For $0 < \varepsilon < s$, define*

$$H_\varepsilon := \int_\varepsilon^{s-\varepsilon} e^{-tQ^2}\, d^2 b e^{-(s-t)\, Q^2}\, dt \in \mathscr{T}(\alpha, -\beta; s^{-1}). \tag{V.113}$$

Furthermore, as $\varepsilon, \varepsilon' \to 0$,

$$\|H_\varepsilon - H_{\varepsilon'}\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \to 0. \tag{V.114}$$

The corresponding limit $H_0 = \lim_{\varepsilon \to 0} H_\varepsilon$ is (V.112). *Thus*

$$[b, e^{-sQ^2}] \in \mathscr{T}(\alpha, -\beta; s^{-1}). \tag{V.115}$$

(iv) *Let $0 < \mu < 1$, and let*

$$\begin{aligned} M &= M(\alpha, \beta, \mu, s) \\ &= 8\mu^{-(1/2) - (\alpha+\beta)} B_1((1-\alpha-\beta)/2, (2-\alpha-\beta)/2)(\mathrm{Tr}(e^{-(1-\mu)\, Q^2}))^s. \end{aligned}$$

Then

$$\|[b, e^{-sQ^2}]\|_{\mathcal{T}(\alpha,\, -\beta;\, s^{-1})} \leqslant M s^{(1/2)-(\alpha+\beta)} \, \|b\|_{\mathfrak{I}_{\beta,\alpha}}. \tag{V.116}$$

Proof. (i) Let $\mathscr{D}_2 = D(Q^2)$. The identity (V.111) can be established on the domain $\mathscr{D}_2 \times \mathscr{D}_2$, where

$$Q^2 b - bQ^2 = Q(Qb - b^\gamma Q) + (Qb^\gamma - bQ)\, Q = d(db) = d^2 b.$$

This can be written $d^2 b = Q\, db + (db^\gamma)\, Q$, which is the algebraic relation (V.111). Since $b \in \mathfrak{I}_{\beta,\alpha}$, in particular $db \in \mathcal{T}(-\beta, \alpha)$. Hence $Q\, db \in \mathcal{T}(-\beta-1, \alpha) \subset \mathcal{T}(-\beta-1, \alpha+1)$. Also $db^\gamma = -(db)^\gamma \in \mathcal{T}(-\beta, \alpha)$. Thus $db^\gamma Q \in \mathcal{T}(-\beta, \alpha+1) \subset \mathcal{T}(-\beta-1, \alpha+1)$. Hence the domain inclusions (V.110–111) hold.

(ii) As a form on $\mathscr{H}_\infty \times \mathscr{H}_\infty$, and using (V.110–111), we infer that

$$[b, e^{-sQ^2}] = -e^{-tQ^2} b e^{-(s-t)\,Q^2}\big|_{t=0}^{t=s} = \int_0^s e^{-tQ^2} d^2 b e^{-(s-t)\,Q^2}\, dt. \tag{V.117}$$

Thus (V.112) is an identity for sesquilinear forms. The left side is an element of $\mathscr{B}(\mathscr{H})$, and therefore so is the right side.

(iii–iv) As $e^{-tQ^2/2}$ is trace class and $d^2 b \in \mathcal{T}(-\beta-1, \alpha+1)$, clearly $e^{-tQ^2}\, d^2 b e^{-(s-t)\,Q^2}$ is trace class for $0 < t < s$. Furthermore, using Hölder's inequality on

$$(e^{-(1-\mu)\,tQ^2})(e^{-\mu tQ^2} R^{-\alpha}(d^2 b)\, R^{-\beta} e^{-\mu(s-t)\,Q^2})(e^{-(1-\mu)(s-t)\,Q^2}),$$

with exponents t^{-1}, ∞, and $(s-t)^{-1}$, respectively, we see that

$$\begin{aligned} &\|e^{-tQ^2}(d^2 b)\, e^{-(s-t)\,Q^2}\|_{\mathcal{T}(\alpha,\, -\beta;\, s^{-1})} \\ &\quad = \|e^{-tQ^2} R^{-\alpha}(d^2 b)\, R^{-\beta} e^{-(s-t)\,Q^2}\|_{s^{-1}} \\ &\quad \leqslant 4\mu^{-(1/2)-(\alpha+\beta)} (\mathrm{Tr}(e^{-(1-\mu)\,Q^2}))^s \, (t^{-(\alpha+\beta+1)/2}(s-t)^{-(\alpha+\beta)/2} \\ &\qquad \times \|R^{-\alpha} Q\, db\, R^{-\beta}\|_{(-\alpha-\beta-1,\, \alpha+\beta)} \\ &\qquad + t^{-(\alpha+\beta)/2}(s-t)^{-(\alpha+\beta+1)/2} \, \|R^{-\alpha}(db^\gamma)\, Q R^{-\beta}\|_{(-\alpha-\beta,\, 1+\alpha+\beta)}). \end{aligned} \tag{V.118}$$

Here we have used (V.110–111) as well as (V.28). Note

$$\|R^{-\alpha} Q\, db\, R^{-\beta}\|_{(-\alpha-\beta-1,\, \alpha+\beta)} \leqslant \|db\|_{(-\beta,\, \alpha)}, \tag{V.119}$$

and

$$\|R^{-\alpha}(db^\gamma)\, Q R^{-\beta}\|_{(-\alpha-\beta,\, 1+\alpha+\beta)} \leqslant \|db\|_{(-\beta,\, \alpha)}. \tag{V.120}$$

Thus integrating (V.118) and using (V.20, 24) we obtain

$$\|H_\varepsilon\|_{\mathscr{T}(\alpha,\,-\beta;\,s^{-1})} \leqslant 8\mu^{-(1/2)-(\alpha+\beta)} s^{(1/2)-(\alpha+\beta)} B_1((1-\alpha-\beta)/2,\,(2-\alpha-\beta)/2) \times (\operatorname{Tr}(e^{-(1-\mu)\,Q^2}))^s\, \|db\|_{(-\beta,\,\alpha)}. \tag{V.121}$$

This shows that $H_\varepsilon \in \mathscr{T}(\alpha, -\beta; s^{-1})$ and the bound on $\|H_\varepsilon\|_{\mathscr{T}(\alpha,\,-\beta;\,s^{-1})}$ is of the form (V.116), uniformly in ε. We now establish convergence of H_ε in this norm. In fact for $\varepsilon' > \varepsilon$, the expression $H_\varepsilon - H_{\varepsilon'}$ is just the integral (V.112) restricted to the intervals $t \in [\varepsilon, \varepsilon']$ and $t \in [s-\varepsilon', s-\varepsilon]$. We therefore obtain the bound

$$\|H_\varepsilon - H_{\varepsilon'}\|_{\mathscr{T}(\alpha,\,-\beta;\,s^{-1})} \leqslant o(1)\, \mu^{-(1/2)-(\alpha+\beta)} s^{(1/2)-(\alpha+\beta)}, \tag{V.122}$$

as $\varepsilon, \varepsilon' \to 0$. Thus we have established the convergence as $\varepsilon \to 0$ of H_ε in $\mathscr{T}(\alpha, -\beta; s^{-1})$. Since H_0 is equal to $[b, e^{-sQ^2}]$, as we saw in (V.112), we have the bound (V.116) also for the limit. This completes the proof of the proposition.

We end this section with a useful corollary.

COROLLARY V.8. (i) *Consider the set* $X = \{x_0, x_1, ..., x_n\}$. *Let* y_j *and* z_j *be elements of an interpolation space* $\mathfrak{I}_{\beta,\,\alpha}$, *where* $0 \leqslant \alpha, \beta$ *and* $0 \leqslant \alpha + \beta < 1$. *Suppose each* x_j *is one of the forms*

$$y_j, \quad dy_j, \quad d^2 y_j, \quad y_j(dz_j), \quad \text{or} \quad (dy_j)\, z_j, \tag{V.123}$$

but where no two adjacent x_j*'s are of the form* $d^2 y_j$. (*Here we consider* x_j *and* x_{j+1} *adjacent, as well as* x_0 *and* x_n *adjacent.*) *Then* X *is a regular set of vertices with respect to* Q.

(ii) *Let* $X_n^{\mathrm{JLO}} = \{a_0, da_1, ..., da_n\}$, *where* $a_j \in \mathfrak{I}_{\beta,\,\alpha}$. *Then there exist constants* $m_1, m_2 < \infty$ *such that the trace norm of the Radon transform* $\hat{X}_n^{\mathrm{JLO}}$, *defined in* (V.58), *of* $X_n^{\mathrm{JLO}}(s)$ *satisfies the bound*

$$\|\hat{X}_n^{\mathrm{JLO}}\|_1 \leqslant m_1 m_2^{n+1} \mu^{-(\alpha+\beta)\,n/2} \left(\frac{1}{n!}\right)^{(1/2)+((1-\alpha-\beta)/2)} \times \operatorname{Tr}(e^{-(1-\mu)\,Q^2}) \left(\prod_{j=0}^{n} \|a_j\|_{\mathfrak{I}_{\beta,\,\alpha}}\right). \tag{V.124}$$

(iii) *For* μ *fixed,*

$$n^{1/2}\, \|\hat{X}_n^{\mathrm{JLO}}\|_1^{1/n} \leqslant O(n^{-(1-\alpha-\beta)/2}) \left(\prod_{j=0}^{n} \|a_j\|_{\mathfrak{I}_{\beta,\,\alpha}}\right)^{1/n}, \tag{V.125}$$

where $(1-\alpha-\beta) > 0$.

(iv) *Let each* $a_i \in \mathfrak{I}_{\beta,\alpha}$, *and for* $2 \leqslant j \leqslant n$ *define two regular sets of* n *vertices by*

$$X_1 = \{a_0, da_1, ..., (da_{j-1})\, a_j, da_{j+1}, ..., da_n\}, \tag{V.126}$$

$$X_2 = \{a_0, da_1, ..., da_{j-1}, a_j da_{j+1}, da_{j+2}, ..., da_n\}, \tag{V.127}$$

and one set of $(n+1)$ *vertices by*

$$X_3 = \{a_0, da_1, ..., da_{j-1}, d^2 a_j, da_{j+1}, ..., da_n\}. \tag{V.128}$$

Then

$$X_1(s) - X_2(s) = \int_0^{s_{j-1}} X_3(s_0, ..., s_{j-2}, t, s_{j-1} - t, s_j, ..., s_{n-1})\, dt. \tag{V.129}$$

(v) *After integration over* $s \in \sigma_n$,

$$\hat{X}_1 - \hat{X}_2 = \hat{X}_3, \tag{V.130}$$

or in terms of the expectations (V.62)

$$\begin{aligned} &\langle a_0, da_1, ..., (da_{j-1})\, a_j, da_{j+1}, ..., da_n; g\rangle_{n-1} \\ &\qquad - \langle a_0, da_1, ..., da_{j-1}, a_j\, da_{j+1}, da_{j+1}, ..., da_n; g\rangle_{n-1} \\ &= \langle a_0, da_1, ..., da_{j-1}, d^2 a_j, da_{j+1}, ..., da_n; g\rangle_n. \end{aligned} \tag{V.131}$$

Similarly,

$$\begin{aligned} &\langle a_0 a_1, da_2, ..., da_n; g\rangle_{n-1} - \langle a_0, a_1\, da_2, ..., da_n; g\rangle_{n-1} \\ &\quad = \langle a_0, d^2 a_1, da_2, ..., da_n; g\rangle_n, \end{aligned} \tag{V.132}$$

and also

$$\begin{aligned} &\langle a_0, ..., (da_{n-1})\, a_n; g\rangle_{n-1} - \langle a_n^{g^{-1}} a_0, da_1, ..., da_{n-1}; g\rangle_{n-1} \\ &= \langle a_0, da_1, ..., d^2 a_n; g\rangle_n. \end{aligned} \tag{V.133}$$

(vi) *There are constants* $m_1, m_2 < \infty$ *such that for* $0 < \mu < 1$,

$$\begin{aligned} &|\langle a_0, da_1, ..., da_j, d^2 a_j, da_{j+1}, ..., da_n; g\rangle_n| \\ &\quad \leqslant m_1 m_2^{n+1} \mu^{-(\alpha+\beta) n/2 - 1}\, \mathrm{Tr}(e^{-(1-\mu) Q^2}) \\ &\qquad \times \left(\frac{1}{n!}\right)^{1-(\alpha+\beta)/2} \left(\prod_{j=0}^{n} \|a_j\|_{\mathfrak{I}_{\beta,\alpha}}\right). \end{aligned} \tag{V.134}$$

Proof. (i) For $y, z \in \mathfrak{J}_{\beta,\alpha}$, $y \in \mathscr{T}(0,0)$ and $dy \in \mathscr{T}(-\beta, \alpha)$. Furthermore

$$d^2 y \in \mathscr{T}(-\beta-1, \alpha+1), \quad y\, dz \in \mathscr{T}(-\beta, \alpha) \quad \text{and} \quad (dy)\, z \in \mathscr{T}(-\beta, \alpha).$$

This is a consequence of the definition of $\mathfrak{J}_{\beta,\alpha}$ and Proposition V.6. The most singular case occurs with $[(n+1)/2]$ vertices $x_j = d^2 y_j$, interspersed between vertices in $\mathscr{T}(-\beta, \alpha)$. (Here $[\cdot]$ denotes the integer part.) Thus with an even number of vertices (odd n)

$$0 < \eta_{\text{local}} = \eta_{\text{global}} = (1-\alpha-\beta)/2. \tag{V.135}$$

In the case of an odd number of vertices,

$$\eta_{\text{local}} = (1-\alpha-\beta)/2 < \eta_{\text{global}}. \tag{V.136}$$

In either case X is a regular set.

(ii–iii) In the case that $X = X_n^{\text{JLO}}$, we may take

$$\eta_0 = 1-\beta/2, \qquad \eta_j = \frac{1}{2} + \left(\frac{1-\alpha-\beta}{2}\right), \qquad j = 1, 2, ..., n-1,$$
$$\eta_n = 1-\alpha/2. \tag{V.137}$$

So

$$\eta_{\text{local}} = 1 - \tfrac{1}{2}(\alpha+\beta) > \tfrac{1}{2}, \qquad \text{and} \qquad (n+1)\,\eta_{\text{global}} = n\eta_{\text{local}} + 1. \tag{V.138}$$

The bounds (V.124–125) then follow from the bound (V.63) and the asymptotics of the Γ function.

(iv–vi) For $s \in \sigma_n$, $X_1(s)$ and $X_2(s)$ are trace class. Also

$$X_1(s) - X_2(s) = a_0 e^{-s_0 Q^2}\, da_1 \cdots e^{-s_{j-2} Q^2}\, da_{j-1} [a_j, e^{-s_{j-1} Q^2}]$$
$$\times\, da_{j+1}\, e^{-s_j Q^2} \cdots da_n e^{-s_{n-1} Q^2}. \tag{V.139}$$

Using Proposition V.7 iii–iv, the commutator in (V.137) is an element of $\mathscr{T}(\alpha, -\beta, s_{j-1}^{-1})$, with norm bounded by $M s_{j-1}^{(1/2)-(\alpha+\beta)} \|a_j\|_{\mathfrak{J}_{\beta,\alpha}}$. Therefore

$$da_{j-1}[a_j, e^{-s_{j-1} Q^2}]\, da_{j+1} \in \mathscr{T}(-\beta, \alpha; s_{j-1}^{-1}),$$

and

$$\begin{aligned}
&\|da_{j-1}[a_j, e^{-s_{j-1} Q^2}]\, da_{j+1}\|_{\mathscr{T}(-\beta,\alpha; s_{j-1}^{-1})} \\
&\quad \leqslant \|da_{j-1}\|_{\mathscr{T}(-\beta,\alpha;\infty)} \|[a_j, e^{-s_{j-1} Q^2}]\|_{\mathscr{T}(\alpha,-\beta; s_{j-1}^{-1})} \|da_{j+1}\|_{\mathscr{T}(-\beta,\alpha;\infty)} \\
&\quad \leqslant M s_{j-1}^{(1/2)-(\alpha+\beta)} \prod_{k=j-1}^{j+1} \|a_j\|_{\mathfrak{J}_{\beta,\alpha}}.
\end{aligned} \tag{V.140}$$

On the other hand, Proposition V.7 ii shows that

$$X_1(s) - X_2(s) = \int_0^{s_{j-1}} a_0 e^{-s_0 Q^2}\, da_1 e^{-s_1 Q^2} \cdots da_{j-1} e^{-tQ^2}\, d^2 a_j e^{-s_{j-1} - t)\, Q^2}$$
$$\times da_{j+1} e^{-s_j Q^2} \cdots da_n e^{-s_{n-1} Q^2}\, dt$$
$$= \int_0^{s_{j-1}} X_3(s_0, s_1, ..., s_{j-2}, t, s_{j-1} - t, s_j, ..., s_{n-1})\, dt, \quad \text{(V.141)}$$

which is (V.129). Integrating over σ_{n-1} yield (X.130–131). The similar identity (I.32) that follows has a similar proof. The identity (I.33) involving $d^2 a_n$ is a consequence of the cyclic symmetry (V.67) and the identity (V.131). In fact

$$\langle a_0, da_1, ..., (da_{n-1})\, a_n \rangle_{n-1} - \langle a_{2n}^{g^{-1}} a_0, da_1, ..., da_{n-1} \rangle_{n-1}$$
$$= \langle da_1, ..., (da_{n-1})\, a_n, a_0^g \rangle_{n-1} - \langle da_1, ..., da_{n-1}, a_{2n} a_0^g \rangle_{n-1}$$
$$= \langle da_1, ..., da_{n-1}, d^2 a_n, a_0^g \rangle_n$$
$$= \langle a_0, da_1, ..., da_{n-1}, d^2 a_n \rangle_n. \quad \text{(V.142)}$$

The estimate (V.134) then follows by an analysis of $\|X_3(s)\|_1$ similar to the proof of (V.124).

VI. COCYCLES

Throughout this section take $\mathfrak{A}$ to be a subalgebra of an interpolation space $\mathfrak{J}_{\beta, \alpha}$ introduced in Section V.5. We begin this section by showing that τ^{JLO} extends in this case to be an element of $\mathscr{C}(\mathfrak{A})$, including the formula for the pairing with a root of I. We define a fractionally-differentiable structure. Finally we show that τ^{JLO} is a cocycle. These facts are preliminary to the next section where we show that the pairing is actually a homotopy invariant.

VI.1. *The JLO-Cochain Extends to Interpolation Spaces*

In this sub-section we extend the JLO-cochain from the framework in Section IV where the space of cochains $\mathscr{C}(\mathfrak{A})$ lives over an algebra $\mathfrak{A}$ of differentiable functions (i.e., $da \in \mathscr{B}(\mathscr{H})$), to the case that $\mathfrak{A}$ is contained in one of the interpolation spaces $\mathfrak{J}_{\beta, \alpha}$. Thus $a \in \mathfrak{A}$ will have a fractional derivative of order $1-\beta$ and $da \in \mathscr{T}(-\beta, \alpha)$ will be a generalized function. We require

$$\mathfrak{A} \subset \mathfrak{J}_{\beta, \alpha} \text{ for some } 0 \leqslant \alpha, \beta, \qquad \text{and} \qquad 0 \leqslant \alpha + \beta < 1. \quad \text{(VI.1)}$$

We also require that the norm $|||\cdot|||$ on $\mathfrak{A}$ satisfy

$$\|a\|_{\mathfrak{J}_{\beta,\alpha}} \leqslant |||a|||. \tag{VI.2}$$

Otherwise, we retain the basic hypotheses of Section IV. The Hilbert space $\mathscr{H}$ is $\mathbb{Z}_2$ graded by γ and carries a continuous unitary representation $U(g)$. We assume that $Q = Q^*$ commutes with $U(g)$ and $Q\gamma + \gamma Q = 0$. We assume that $\exp(-\beta Q^2)$ is trace class for all $\beta > 0$. We assume that $\mathfrak{A}$ is pointwise invariant under the action of γ, and invariant under the action of $U(g)$, $U(g)\,\mathfrak{A}\,U(g)^* \subset \mathfrak{A}$.

Definition VI.1. We call the sextuple

$$\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$$

satisfying the above hypotheses a Θ-summable, fractionally-differentiable structure.

Proposition VI.2. *Let* $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ *be a* Θ*-summable, fractionally-differentiable structure. Then* $\tau^{\mathrm{JLO}} \in \mathscr{C}(\mathfrak{A})$. *There exists* $m < \infty$ *such that*

$$|||\tau_n^{\mathrm{JLO}}||| \leqslant m^{n+1} \left(\frac{1}{n!}\right)^{(1/2)+((1-\alpha-\beta)/2)} \operatorname{Tr}(e^{-Q^2/2}). \tag{VI.3}$$

Proof. For $a_j \in \mathfrak{J}_{\beta,\alpha}$, we have already established in Corollary V.8.ii, with $\mu = \frac{1}{2}$, that X_n^{JLO} is trace class with trace norm bounded by

$$m^{n+1} \operatorname{Tr}(e^{-Q^2/2}) \left(\frac{1}{n!}\right)^{(1/2)+((1-\alpha-\beta)/2)} \left(\prod_{j=0}^{n} \|a_j\|_{\mathfrak{J}_{\beta,\alpha}}\right).$$

Since $\|a_j\|_{\mathfrak{J}_{\beta,\alpha}} \leqslant |||a_j|||$, we conclude that τ_n^{JLO} is defined on $\mathfrak{A}^{n+1}$ and that $|||\tau_n^{\mathrm{JLO}}|||$ satisfies (VI.3). Since $\alpha + \beta < 1$, this entails the "entire" condition $n^{1/2}\,|||\tau_n^{\mathrm{JLO}}|||^{1/n} \to 0$. Thus $\tau^{\mathrm{JLO}} \in \mathscr{C}(\mathfrak{A})$.

Having extended the notion of τ^{JLO} to a fractionally-differentiable structure, we now observe that the pairing $\langle \tau^{\mathrm{JLO}}, a\rangle$ has the same representation as in the differentiable case.

Corollary VI.3. *Let* $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ *be a* Θ*-summable, fractionally-differentiable structure. Let* $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ *satisfy* $a^2 = I$. *Then*

$$\mathfrak{Z}^Q(a; g) = \langle \tau^{\mathrm{JLO}}, a\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, da})\, dt. \tag{VI.4}$$

Here Tr *denotes both the trace on* $\mathscr{H}$ *and the matrix trace in* $\mathrm{Mat}_m(\mathfrak{A})$, *in case* $m > 1$.

VI.2. *The JLO-Cochain is a Cocycle*

PROPOSITION VI.4. *Let* $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ *be a* Θ*-summable, fractionally-differentiable structure. Then the cochain* $\tau^{\mathrm{JLO}} \in \mathscr{C}(\mathfrak{A})$ *is a cocycle for* ∂, *namely*

$$\partial \tau^{\mathrm{JLO}} = 0. \tag{VI.5}$$

Remarks. 1. The cochain τ^{JLO} was originally defined in [17], for the differentiable $da \in \mathscr{B}(\mathscr{H})$, where the cocycle condition was also established. This cochain has been investigated again in several different contexts, see [12, 10, 22, 25, 30] for example. Our presentation is self-contained.

2. The only known cocycles for $\mathscr{C}(\mathfrak{A})$ are elements $[\tau^{\mathrm{JLO}}]$, where τ^{JLO} is defined by some Q. (Here Q gives rise either to the Θ-summable case considered above, or to the class of cochains satisfying the KMS-condition. See [18, 20, 22] for this extension, applicable in the differentiable case.) Taking the larger space of cochains $\mathscr{D}(\mathfrak{A}) \supset \mathscr{C}(\mathfrak{A})$, and the corresponding coboundary operator ∂, Connes earlier gave another cocycle τ^C, see [4]. This cocycle is convenient because it satisfies a "normalization" condition, central to Connes' analysis of pairings of cocycles in $\mathscr{D}$, and also used by him in other studies. Furthermore, Connes showed that any cocycle $\tau \in \mathscr{D}(\mathfrak{A})$ is cohomologous to a normalized cocycle in $\mathscr{D}$. The cocycle τ^C is determined by an operator F satisfying $F^2 = I$ and $F\gamma + \gamma F = 0$. With τ^{JLO} the cocycle determined by Q, and with τ^C the cocycle determined by an appropriate $F = F(Q)$, Connes has shown [5] (for the differentiable case) that τ^C and τ^{JLO} are cohomologous. In other words, there is a cochain $G \in \mathscr{D}$ such that $\tau^{\mathrm{JLO}} = \tau^C + \partial G$.

On the other hand, cocycles in $\mathscr{C}(\mathfrak{A})$ are not normalized in Connes' sense. As discussed in Section III, by working with the cochains $\mathscr{C}(\mathfrak{A})$ we avoid the need to consider this normalization. Furthermore, a pairing can be defined for all cochains in $\mathscr{C}(\mathfrak{A})$, rather than just for cocycles. The importance of pairing a cocycle then rests on the fact that the pairing yields a homotopy invariant, as discussed in Section VII.

Proof. It was shown in (IV.30) that if $\alpha = \beta = 0$, then evaluated on $\mathfrak{A}$, $\tau^{\mathrm{JLO}}_{2n+1} = 0$. By the symmetry (V.66), this extends to $\mathfrak{A} \subset \mathfrak{J}_{\beta,\alpha}$. Hence to establish the cocycle condition in $\mathscr{C}(\mathfrak{A})$, it is sufficient to show that for all odd n,

$$B\tau^{\mathrm{JLO}}_{n+1} = \langle da_0, ..., da_n; g \rangle_n = -b\tau^{\mathrm{JLO}}_{n-1}. \tag{VI.6}$$

By Corollary V.8.i, the expectation $\langle da_0, ..., da_n; g\rangle_n = \tau_n^{\mathrm{JLO}}(da_0, a_1, ..., a_n)$ that occurs in (VI.6) is well-defined. We prove below that for n odd,

$$(B\tau_{n+1}^{\mathrm{JLO}})(a_0, ..., a_n; g) = \tau_n^{\mathrm{JLO}}(da_0, a_1, ..., a_n; g), \tag{VI.7}$$

and

$$(b\tau_{n-1}^{\mathrm{JLO}})(a_0, ..., a_n; g) = \sum_{r=1}^{n} (-1)^{r-1}\, \tau_n^{\mathrm{JLO}}(a_0, a_1, ..., a_{r-1}, da_r, a_{r+1}, ..., a_n; g). \tag{VI.8}$$

Let us begin by evaluating $B\tau_n^{\mathrm{JLO}}$. Starting from the definition (II.19) of B, and the symmetry (V.67), we find that $B\tau_{n+1}^{\mathrm{JLO}}$ equals

$$\begin{aligned}(B\tau_{n+1}^{\mathrm{JLO}})(a_0, ..., a_n; g) &= \sum_{j=0}^{n} (-1)^j \langle I, da_{n-j+1}^{g^{-1}}, ..., da_n^{g^{-1}}, da_0, ..., da_{n-j}; g\rangle_{n+1} \\ &= \sum_{j=0}^{n} \langle da_0, ..., da_{n-j}, I, da_{n-j+1}, ..., da_n; g\rangle_{n+1} \\ &= \sum_{j=1}^{n} \langle da_0, ..., da_{j-1}, I, da_j, ..., da_n; g\rangle_{n+1} \\ &= \langle da_0, ..., da_n; g\rangle_n = \tau_n^{\mathrm{JLO}}(da_0, a_1, ..., a_n).\end{aligned} \tag{VI.9}$$

Here we have used (V.68). Note that this establishes the first equality in (VI.6), or (VI.7).

In order to compute $b\tau_{n-1}^{\mathrm{JLO}}$, we return to the definition (II.14) of b. Recalling (II.9), we evaluate each $V(r)$ acting on $\tau_{n-1}^{\mathrm{JLO}}$, for $r = 0, 1, ..., n$. Then

$$(V(0)\, \tau_{n-1}^{\mathrm{JLO}})(a_0, ..., a_n; g) = \langle a_0 a_1, da_2, ..., da_n; g\rangle_{n-1}.$$

Likewise for $1 \leqslant r \leqslant n-1$,

$$\begin{aligned}(V(r)\, \tau_{n-1}^{\mathrm{JLO}})(a_0, ..., a_n; g) &= (-1)^r \langle a_0, da_1, ..., da_{r-1}, d(a_r a_{r+1}), da_{r+2}, ..., da_n; g\rangle_{n-1} \\ &= (-1)^r \langle a_0, da_1, ..., a_r\, da_{r+1}, da_{r+2}, ..., da_n; g\rangle_{n-1} \\ &\quad + (-1)^r \langle a_0, da_1, ..., (da_r)\, a_{r+1}, da_{r+2}, ..., da_n; g\rangle_{n-1}.\end{aligned} \tag{VI.10}$$

Note that here we have expanded the vertex $d(a_r a_{r+1}) \in \mathscr{T}(-\beta, \alpha)$ into the sum of two vertices $(da_r)\, a_{r+1}$ and $a_r(da_{r+1})$, each of them in $\mathscr{T}(-\beta, \alpha)$.

This is justified in Corollary V.6.i. For $r=n$, we use the fact that a_n is even under γ. Thus we can apply (II.9) and then (V.67) to yield

$$(V(n)\,\tau_{n-1}^{\mathrm{JLO}})(a_0, ..., a_n; g) = (-1)^n \langle a_n^{g^{-1}} a_0, da_1, ..., da_{n-1}; g\rangle_{n-1}. \tag{VI.11}$$

This last term (VI.11) combines with the second term in (VI.10) for $r=n-1$. Using these remarks we obtain a representation for $b\tau_{n-1}^{\mathrm{JLO}}$ as a sum of exactly $2n$-terms. To do this, for $2 \leqslant r \leqslant n-1$, combine the second term from (VI.10) for $V(r-1)$ with the first term from (VI.10) for $V(r)$ into a new term indexed by r. Hence

$$\begin{aligned}(b\tau_{n-1}^{\mathrm{JLO}})&(a_0, ..., a_n; g)\\ &= \langle a_0 a_1, da_2, ..., da_n; g\rangle_{n-1} - \langle a_0, a_1\, da_2, ..., da_n; g\rangle_{n-1}\\ &\quad + \sum_{r=2}^{n-1} (-1)^{r-1} (\langle a_0, ..., (da_{r-1})\, a_r, da_{r+1}, ..., da_n; g\rangle_{n-1}\\ &\quad - \langle a_0, ..., da_{r-1}, a_r\, da_{r+1}, ..., da_n; g\rangle_{n-1})\\ &\quad + (-1)^{n-1} (\langle a_0, ..., (da_{n-1})\, a_n; g\rangle_{n-1}\\ &\quad - \langle a_n^{g^{-1}} a_0, da_1, ..., da_{n-1}; g\rangle_{n-1}).\end{aligned} \tag{VI.12}$$

Now use Corollary V.8.v. With the definition, $a(t) = e^{-tQ^2} a e^{tQ^2}$, this corollary is basically the analytic justification within our present context of the statement $ae^{-sQ^2} = e^{-sQ^2} a + \int_0^s d^2 a(t)\, e^{-sQ^2}\, dt$. We thus combine pairs of terms in (VI.12) to arrive at

$$\begin{aligned}(b\tau_{n-1}^{\mathrm{JLO}})&(a_0, ..., a_n; g)\\ &= \sum_{r=1}^{n} (-1)^{r-1} \langle a_0, da_1, ..., da_{r-1}, d^2 a_r, da_{r+1}, ..., da_n; g\rangle_n\\ &= \sum_{r=1}^{n} (-1)^{r-1}\, \tau_n^{\mathrm{JLO}}(a_0, a_1, ..., a_{r-1}, da_r, a_{r+1}, ..., a_n; g),\end{aligned} \tag{VI.13}$$

which is (VI.8).

The identity (VI.6) is now close at hand. Rewrite our current identity as

$$\begin{aligned}(b\tau_{n-1}^{\mathrm{JLO}})&(a_0, ..., a_n; g)\\ &= -\langle da_0, da_1, ..., da_n; g\rangle_n + \Bigg(\langle da_0, a_1, ..., da_n; g\rangle_n + \sum_{r=1}^{n} (-1)^{r-1}\\ &\quad \times \langle a_0, da_1, ..., da_{r-1}, d^2 a_r, da_{r+1}, ..., da_n; g\rangle_n\Bigg).\end{aligned} \tag{VI.14}$$

Define the set $X=\{x_0, ..., x_n\}$ by $x_0=a_0$, $x_1=da_1, ...,$ and $x_n=da_n$. We use the relation

$$\langle d\hat{X};\, g\rangle = 0,$$

which is a form of Corollary V.4.vi, and in particular of the identity (V.69). Since $x_0^\gamma = x_0$ and $x_j^\gamma = -x_j$ for $1 \leqslant j \leqslant n$, we infer that

$$\langle d\hat{X};\, g\rangle = \langle dx_0, x_1, ..., x_n;\, g\rangle_n + \sum_{r=1}^{n} (-1)^{r-1} \langle x_0, x_1, ..., dx_r, ..., x_n;\, g\rangle_n. \tag{VI.15}$$

This just equals the terms in the large parentheses on the right side of (VI.14), and hence these terms sum to zero. Thus we have established the second equality in (VI.6). This also proves (VI.5), and hence completes the proof of Proposition VI.4.

VII. HOMOTOPY INVARIANTS

VII.1. *The Main Result: JLO Pairing is Invariant*

In this section we consider the pairing of a family $\tau^{\mathrm{JLO}}(\lambda)$ of cocycles with some $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ satisfying $a^2 = I$. These cocycles are defined by a family of non-commutative, fractionally-differentiable structures on $\mathfrak{A}$. The pairing is defined in various forms in III.11, 28, 29, and 31, namely

$$\mathfrak{Z}^{Q(\lambda)}(a;\, g) = \langle \tau^{\mathrm{JLO}}(\lambda), a\rangle. \tag{VII.1}$$

The advantage to pairing $\tau^{\mathrm{JLO}}(\lambda)$ with $a \in \mathrm{Mat}_m(\mathfrak{A}^{\mathfrak{G}})$ rather than to pairing an arbitrary family of cochains $\tau(\lambda)$, is the fact that the pairing function is a constant function of λ. The continuous variation in λ is a *homotopy*. In other words, each $\mathfrak{Z}^{Q(\lambda)}(a;\, g)$ is a *homotopy invariant*. This invariant is in general not integer-valued, but may be in certain special cases.

Our basic result is to find conditions that are sufficient to prove that $\tau^{\mathrm{JLO}}(\lambda)$ is continuously differentiable in λ. Under these hypotheses, the pairing function $\langle \tau^{\mathrm{JLO}}(\lambda), a\rangle$ is actually constant. The considerations in this section are both algebraic and analytic. The algebraic considerations are universal; they show that if the functions have derivatives equal to their formal values, then the pairing is an invariant. The analytic conditions allow us to establish differentiability of the pairing, and furthermore to prove that the derivative of the pairing with respect to λ does equal its formal value. Much of the relevant analytic groundwork has already been prepared in Section V. Here we extend this analysis to families, and we formulate hypotheses on the variation of $Q(\lambda)$ that are sufficient to establish that we are within this framework.

We assume that the λ-dependence of $\tau^{\mathrm{JLO}}(\lambda)$, and hence that the pairing, arises from the λ-dependence of $Q(\lambda)$. Here $Q(\lambda)$ generates $\tau^{\mathrm{JLO}}(\lambda)$ as described in Section IV, and the parameter λ lies in an open interval $\Lambda = (\lambda_1, \lambda_2) \subset \mathbb{R}$. Our operator $Q(\lambda)$ is a self-adjoint operator on $\mathscr{H}$, and we suppose that $Q(\lambda)$ has the general form

$$Q(\lambda) = Q + q(\lambda). \tag{VII.2}$$

We regard $Q = Q^*$ as defining a basic τ^{JLO}, and $q(\lambda)$ as providing a deformation of Q and a perturbation $\tau^{\mathrm{JLO}}(\lambda)$ of τ^{JLO}.

Let us state the main result of this section.

THEOREM VII.1. (i) *Let $\{\mathscr{H}, Q(\lambda), \gamma, U(g), \mathfrak{A}\}$ be a regular family of Θ-summable, fractionally differentiable structures as defined in Section* VII.3. *Then the corresponding family of JLO-cocycles $\{\tau^{\mathrm{JLO}}(\lambda)\}$ is a continuously differentiable function $\Lambda \to \mathscr{C}(\mathfrak{A})$, and there is a continuous family $\{h(\lambda)\} \subset \mathscr{C}(\mathfrak{A})$ such that for all $\lambda \in \Lambda$,*

$$\frac{d}{d\lambda}\tau^{\mathrm{JLO}}(\lambda) = \partial h(\lambda). \tag{VII.3}$$

(ii) *The function $g \to \tau^{\mathrm{JLO}}(\lambda)$ is a continuous function of $g \in \mathfrak{G}$ from $\mathfrak{G}$ to $\mathscr{C}(\mathfrak{A})$, uniformly for λ in compact subsets of Λ.*

An immediate consequence of the fact that $d\tau^{\mathrm{JLO}}(\lambda)/d\lambda = \partial h$ is the invariance of $\mathfrak{Z}^Q(a; g)$. In particular, continuous differentiability of $\tau^{\mathrm{JLO}}(\lambda)$ ensures, cf. Proposition III.6, that the pairing $\langle \tau^{\mathrm{JLO}}(\lambda), a\rangle$ is also a continuously differentiable function. Hence

$$\frac{d}{d\lambda}\langle \tau^{\mathrm{JLO}}(\lambda), a\rangle = \langle \partial h(\lambda), a\rangle = 0. \tag{VII.4}$$

We established in Proposition III.1 the vanishing of the pairing function defined in (VI.4) on coboundaries. Thus we have

COROLLARY VII.2. (i) *For a regular family of non-commutative structures, the JLO-pairing $\langle \tau^{\mathrm{JLO}}(\lambda), a\rangle$ is constant. For $\lambda_1, \lambda_2 \in \Lambda$,*

$$\mathfrak{Z}^Q(a; g) = \langle \tau^{\mathrm{JLO}}(\lambda_1), a\rangle = \langle \tau^{\mathrm{JLO}}(\lambda_2), a\rangle. \tag{VII.5}$$

Furthermore

$$\tau^{\mathrm{JLO}}(\lambda_2) = \tau^{\mathrm{JLO}}(\lambda_1) + \partial H, \tag{VII.6}$$

where $H = H(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} h(\lambda)\, d\lambda$.

(ii) *In addition* (VII.5) *is a continuous function of* g.

We devote Section VII.2–5 to the precise formulation of a "regular deformation" and to the proof of the above theorem. Special cases of these results appear in the literature, for example [10]. But previously no general and easily verifiable conditions on $q(\lambda)$ had been identified, like the one we give here, which result in a homotopy. In fact, the analytic details we resolve here have not previously been made explicit, even in the case of bounded perturbations $q(\lambda)$ of Q.

VII.2. *Regular Linear Deformations*

In this subsection we outline a class of *regular linear deformations* $\{\mathscr{H}, Q(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ of the JLO cochain τ^{JLO} defined for a particular $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g)\mathfrak{A}\}$. We denote the family of Q's by $\{Q(\lambda)\}$ and the family of cochains by $\tau^{\mathrm{JLO}}(\lambda)$. These families depend on a fixed $\mathscr{H}$, γ, $U(g)$, and $\mathfrak{A}$.

We first compile a list of assumptions.

a. *The Starting Point*

The undeformed problem is given by the structure introduced earlier in Section V.7. It is defined by a self-adjoint $Q = Q^*$ with domain $\mathscr{D}$, acting on a Hilbert space $\mathscr{H}$. The heat kernel $\exp(-\beta Q^2)$ is assumed to be trace class for every $\beta > 0$. There is a $\mathbb{Z}_2$ grading γ on $\mathscr{H}$ for which $Q\gamma + \gamma Q = 0$. There is a continuous, unitary representation $U(g)$ on $\mathscr{H}$ of a compact Lie group $\mathfrak{G}$, and $U(g)\, Q = QU(g)$. There is a Banach algebra of observables $\mathfrak{A}$, with

$$\mathfrak{A} \subset \mathfrak{J}_{\beta, \alpha}, \tag{VII.7}$$

for some α and β satisfying $0 \leqslant \alpha, \beta$ and $0 \leqslant \alpha + \beta < 1$, and where $\mathfrak{J}_{\beta, \alpha}$ is an interpolation space introduced in Section V.5. We also require that the norm $|||\cdot|||$ of $\mathfrak{A}$ satisfy

$$\|a\|_{\mathfrak{J}_{\beta, \alpha}} \leqslant |||a|||, \tag{VII.8}$$

for all $a \in \mathfrak{A}$. Thus elements of $\mathfrak{A}$ have $0 < 1 - \beta$ fractional derivatives with respect to Q. We assume that the algebra $\mathfrak{A}$ is pointwise invariant under the action of γ, namely $a^\gamma = \gamma a \gamma = a$ for $a \in \mathfrak{A}$. Furthermore $\mathfrak{A}$ is invariant under the action of $U(g)$, namely $a^g = U(g)\, aU(g)^* \in \mathfrak{A}$ for $a \in \mathfrak{A}$. This structure defines a JLO cocycle τ^{JLO} and a non-commutative, fractionally-differentiable structure $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$.

b. *A Family of Regular Linear Perturbations*

A family of regular deformations of $\{\mathscr{H}, Q, \gamma, U(g), \mathfrak{A}\}$ is defined by a family $q(\lambda)$ of regular perturbations of Q on the space $\mathscr{H}$. Let q denote a symmetric operator on $\mathscr{H}$ with domain $\mathscr{D}$. We suppose there are constants $0 \leqslant a,\ M < \infty$ such that on the domain $\mathscr{D} \times \mathscr{D}$,

$$q^2 \leqslant a^2 Q^2 + M^2. \tag{VII.9}$$

In other words, q is a bounded map from the Sobolev space $\mathscr{H}_1$, defined in Section V.2, to $\mathscr{H}_{-1}$. We define the family $\{q(\lambda)\}$ of regular perturbations[7] parameterized by real λ, and the family $\{Q(\lambda)\}$ of perturbed operators, by

$$q(\lambda) = \lambda q, \qquad \text{and} \qquad Q(\lambda) = Q + q(\lambda). \tag{VII.10}$$

The linearity of $q(\lambda)$ in λ is the linearity in the sub-section title. Here λ belongs to a bounded open interval Λ,

$$\lambda \in \Lambda = (-\mu, \mu), \qquad \text{and} \qquad 0 < \mu < a^{-1}.$$

In addition to the bound (VII.9), we will need another bound: for some $0 < \varepsilon < 1$,

$$\|R^{1-\varepsilon} q(\lambda)\, R^{\varepsilon}\| + \|R^{\varepsilon} q(\lambda)\, R^{1-\varepsilon}\| \leqslant O(1), \tag{VII.11}$$

where $R = (Q^2 + I)^{-1/2}$. If $q(\lambda)$ is essentially self-adjoint on $\mathscr{D}$, then (VII.11) follows automatically from (VII.9), in fact for all $0 \leqslant \varepsilon \leqslant 1$. However, if $q(\lambda)$ is not essentially self-adjoint on $\mathscr{D}$, we also assume (VII.11) for some $0 < \varepsilon$.

c. *Symmetries*

We assume that γ and $U(g)$ of Assumption (a), also are symmetries of $\{Q(\lambda)\}$ in the sense that

$$Q(\lambda)\, \gamma + \gamma Q(\lambda) = 0, \qquad U(g)\, Q(\lambda) = Q(\lambda)\, U(g) \tag{VII.12}$$

[7] The condition (VII.9) ensures $\|qf\| \leqslant a\, \|Qf\| + M\, \|f\|$, a condition introduced by T. Kato to study $Q + q$, see [23]. The relevant case $a < 1$ corresponds in our present case to the bound $\mu a < 1$ of (VII.9). If $0 < a$ may be chosen arbitrarily small (which may require $M(a)$ large), then q is said to be infinitesimally small compared with Q. In that case μ may be chosen arbitrarily large.

for all $\lambda \in \Lambda$ and for all $g \in \mathfrak{G}$. Of course this is ensured by $\gamma q + q\gamma = 0$ and $U(g)\, q = qU(g)$.

d. *The Algebra* $\mathfrak{A}$

We assume that the algebra $\mathfrak{A} \subset \mathfrak{J}_{\beta, \alpha}$ is independent of λ. It is necessary that for $a \in \mathfrak{A}$, the derivation $d_\lambda a$, as λ varies, remains in $\mathfrak{J}_{\beta, \alpha}$, so we also require that the norm $|||\cdot|||$ on $\mathfrak{A}$ satisfies

$$|||a||| \geqslant \|a\| + \sup_{\lambda \in \Lambda} \|d_\lambda a\|_{(-\beta, \alpha)}$$

with α, β as above. Since $\Lambda = (-\mu, \mu)$ is an interval, this is ensured if both da and the commutator of q with a are bounded in the sense that

$$da \in \mathscr{T}(-\beta, \alpha) \qquad \text{and} \qquad [q, a] \in \mathscr{T}(-\beta, \alpha).$$

Definition VII.3. A family $\{\mathscr{H}, Q(\lambda)\}$ of operators satisfying Assumptions (a–b) is a regular (linear) Q-family. A family $\{\mathscr{H}, Q(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ which satisfies Assumptions (a–d) is a regular linear family of Θ-summable, fractionally-differentiable, non-commutative structures.

Remark. In Section VII.3 we replace linearity by an additional estimate.

Proposition VII.4. *Let $\{Q(\lambda)\}$ denote a regular (linear) Q-family. Then*

(i) *For each $\lambda \in \Lambda$, $Q(\lambda)$ is self-adjoint on the domain $\mathscr{D}$.*

(ii) *There are constants $\tilde{M}_1, \tilde{M}_2 < \infty$ such that*

$$Q^2 \leqslant \tilde{M}_1^2 (Q(\lambda)^2 + I) \tag{VII.13}$$

for all $|\lambda| \leqslant \mu$. Here $\tilde{M}_1 = \max\{2, 2\mu M, (1-\mu a)^{-1}, (1-\mu a)^{-1} \mu M\}$. Also for all $\lambda, \lambda' \in \Lambda$,

$$q(\lambda)^2 \leqslant \tilde{M}_2^2 (Q(\lambda')^2 + I), \tag{VII.14}$$

where $\tilde{M}_2^2 = \tilde{M}_1^2 + (\mu M)^2$.

(iii) *For all $\beta > 0$, and all $|\lambda| \leqslant \mu$, $\exp(-\beta Q(\lambda)^2)$ is trace class and*

$$\operatorname{Tr}(e^{-\beta Q(\lambda)^2}) \leqslant e^{\beta} \operatorname{Tr}(e^{-\beta Q^2 \tilde{M}_1^{-1}}). \tag{VII.15}$$

For given β, this bound is uniform for λ in a compact subset of Λ.

Corollary VII.5. *For λ in any compact subset of Λ, the regular linear family $\{\mathscr{H}, Q(\lambda), \gamma, U(g), \mathfrak{A}\}$ determines a bounded family $\{\tau^{\mathrm{JLO}}(\lambda)\}$ of JLO-cocycles in $\mathfrak{A}$.*

Proof. (i) A symmetric operator $Q(\lambda)$ on the domain $\mathscr{D}$ is self-adjoint if and only if for some $\alpha > 0$, $(Q(\lambda) \pm i\alpha)\, \mathscr{D} = \mathscr{H}$. This is the statement that the resolvents $(Q(\lambda) \pm i\alpha)^{-1}$ exist and are bounded. By the spectral theorem for $Q = Q^*$, we infer $\|(Q \pm i\alpha)^{-1}\| \leqslant \alpha^{-1}$ and $\|Q(Q \pm i\alpha))^{-1}\| \leqslant 1$. Thus (VII.9) ensures that for all $\lambda \in \Lambda$,

$$\|q(\lambda)(Q \pm i\alpha)^{-1}\|^2 \leqslant (\mu a)^2 + (\mu M/\alpha)^2.$$

Assumption (b) ensures $|\lambda|\, a < \mu a < 1$. Thus $\|q(\lambda)(Q \pm i\alpha)^{-1}\| < 1$, for α sufficiently large, uniformly in $\lambda \in \Lambda$. It follows that the series

$$(Q \pm i\alpha)^{-1} \sum_{n=0}^{\infty} (-q(Q \pm i\alpha)^{-1})^n \tag{VII.16}$$

converges in norm. This is $(Q + q \pm i\alpha)^{-1}$, as can be verified from the series expansion. Since the domain of (VII.16) is $\mathscr{H}$, the range of $(Q + q \pm i\alpha)$ is $\mathscr{H}$. The range of $(Q \pm i\alpha)^{-1}$ is $\mathscr{D}$, so the domain of $Q + q \pm i\alpha$ is contained in $\mathscr{D}$. However $Q + q \pm i\alpha$ is originally defined on all of $\mathscr{D}$, so that is its domain.

(ii) Remark that (VII.14) follows from (VII.13), using (VII.9). In fact

$$\begin{aligned} q(\lambda)^2 &\leqslant (\lambda a)^2\, Q^2 + (\lambda M)^2 \leqslant (\lambda a)^2\, \tilde{M}_1^2 (Q(\lambda')^2 + I) + (\lambda M)^2 \\ &\leqslant ((\mu a)^2\, \tilde{M}_1^2 + (\mu M)^2)(Q(\lambda')^2 + I) \\ &\leqslant (\tilde{M}_1^2 + (\mu M)^2)(Q(\lambda')^2 + I). \end{aligned}$$

So now we establish (VII.13). On the domain $\mathscr{D} \times \mathscr{D}$ for sesquilinear forms, it follows from the Schwarz inequality that for any $\varepsilon > 0$,

$$\pm (q(\lambda)\, Q + Q q(\lambda)) \leqslant \varepsilon Q^2 + \frac{1}{\varepsilon}\, q(\lambda)^2. \tag{VII.17}$$

Thus on $\mathscr{D} \times \mathscr{D}$, we infer from (VII.17) and (VII.9) that

$$\begin{aligned} Q^2 &= (Q(\lambda) - q(\lambda))^2 = Q(\lambda)^2 - q(\lambda)^2 - (q(\lambda)\, Q + Q q(\lambda)) \\ &\leqslant Q(\lambda)^2 + \varepsilon Q^2 + \left(\frac{1}{\varepsilon} - 1\right) q(\lambda)^2 \\ &\leqslant Q(\lambda)^2 + \left(\varepsilon + (\lambda a)^2 \left(\frac{1}{\varepsilon} - 1\right)\right) Q^2 + (\lambda M)^2 \left(\frac{1}{\varepsilon} - 1\right). \end{aligned} \tag{VII.18}$$

Case 1. $|\lambda| a \leqslant \frac{1}{2}$: In this case choose $\varepsilon = \frac{1}{2}$ in (VII.18). Then $\varepsilon + (\lambda a)^2 ((1/\varepsilon) - 1) \leqslant 3/4$, so collecting the Q^2 terms in (VII.18) gives

$$\tfrac{1}{4} Q^2 \leqslant Q(\lambda)^2 + (\lambda M)^2 \leqslant Q(\lambda)^2 + (\mu M)^2. \tag{VII.19}$$

Hence (VII.13) holds with $\tilde{M}_1 = 2 \max\{1, \mu M\}$.

Case 2. $\frac{1}{2} \leqslant |\lambda|\, a\mu a \leqslant 1$: In this case choose $\varepsilon = |\lambda| a$ in (VII.18). Then the coefficient of Q^2 in (VII.18) is $1 - \varepsilon - (\lambda a)^2 ((1/\varepsilon) - 1) = (1 - |\lambda|\, a)^2 \geqslant (1 - \mu a)^2$, and $((1/\varepsilon) - 1) \leqslant 1$. Thus (VII.18) ensures that

$$(1 - \mu a)^2\, Q^2 \leqslant Q(\lambda)^2 + \lambda^2 M^2 \leqslant Q(\lambda)^2 + \mu^2 M^2. \tag{VII.20}$$

Thus (VII.13) holds with $\tilde{M}_1 = (1 - \mu a)^{-1} \max\{1, \mu M\}$. Thus in both cases, (VII.13) holds with

$$\tilde{M}_1 = \max\{2, 2\mu M, (1 - \mu a)^{-1}, (1 - \mu a)^{-1}\, \mu M\}. \tag{VII.21}$$

This completes the proof of (b).

(iii) Using (VII.13),

$$\tilde{M}_1^{-2} Q^2 - I \leqslant Q(\lambda)^2.$$

We infer that if $E_i(Q^2)$ is the ith eigenvalue of Q^2, counting in increasing order, then by the minimax principle, $\beta \tilde{M}_1^{-2} E_i(Q^2) - \beta \leqslant \beta E_i(Q(\lambda)^2)$. Assumption (a) includes the assertion that $\exp(-\beta Q^2)$ is trace class for all β. Thus (VII.15) follows.

This completes the proof of the proposition. The corollary follows. In fact Assumptions (a–d) plus the fact that $Q(\lambda) = Q(\lambda)^*$ and $\mathrm{Tr}(e^{-\beta Q(\lambda)^2}) < \infty$, with a uniform bound for λ in a compact subset of Λ, ensure the existence of the family $\{\tau^{\mathrm{JLO}}(\lambda)\}$ of JLO-cocycles. The fact that this family is bounded then follows as a consequence of (VI.3), along with (VII.15).

Proposition VII.6. *If* $\{Q(\lambda)\}$ *denotes a regular, linear Q-family, then the Sobolev spaces* $\mathscr{H}_p(Q(\lambda))$, *with* $p \in [-1, 1]$ *and* λ *in a compact subset of* Λ, *are independent of* λ.

Proof. We require that if $f \in \mathscr{H}_p(Q(\lambda))$ then $f \in \mathscr{H}_p(Q(\lambda'))$, and if $f_n \to f \in \mathscr{H}_p(Q(\lambda))$, then $f_n \to f$ in $\mathscr{H}_p(Q(\lambda'))$. It is sufficient to establish this for $p \geqslant 0$, from which the result for $p \leqslant 0$ follows from the duality of $\mathscr{H}_p$ with $\mathscr{H}_{-p}$. Furthermore for $p = 1$, we have verified (Proposition VII.4.i) that $\mathscr{D}(Q(\lambda)) = \mathscr{D}(Q)$ for all $\lambda \in \Lambda$, and hence that $\mathscr{H}_p(Q(\lambda)) = \mathscr{D}((Q(\lambda)^2 + I)^{1/2}) = \mathscr{D}(Q(\lambda)) = \mathscr{H}_p(Q)$ is independent of λ.

The statement about convergence for $p=1$ is equivalent to the existence of constants $\tilde{M}_1$, $\tilde{M}_2$ such that for all $\lambda \in \Lambda$,

$$(Q^2+I) \leqslant \tilde{M}_1^2(Q(\lambda)^2+I), \qquad \text{and} \qquad Q(\lambda)^2+I \leqslant \tilde{M}_2^2(Q^2+I). \quad \text{(VII.22)}$$

The first inequality was proved in Proposition VII.4.ii, while the second follows from

$$Q(\lambda)^2 = (Q+q(\lambda))^2 = Q^2+q(\lambda)^2+q(\lambda)\,Q+Qq(\lambda) \leqslant 2(Q^2+q(\lambda)^2),$$

by Assumption (VII.9).

For $0 \leqslant p \leqslant 1$, the desired results for $\mathscr{H}_p$ follow from the inequalities

$$(Q^2+I)^p \leqslant \tilde{M}_1^p(Q(\lambda)^2+I)^p, \qquad (Q(\lambda)^2+I)^p \leqslant \tilde{M}_2^p(Q^2+I)^p.$$

But suppose $0 \leqslant A^2 \leqslant B^2$ is a monotonic relation for invertible operators on a domain $\mathscr{D} \times \mathscr{D}$, where A and B are essentially self-adjoint on $\mathscr{D}$. Then automatically

$$A^{2p} \leqslant B^{2p} \quad \text{(VII.23)}$$

for all $0 \leqslant p \leqslant 1$. This completes the proof.

VII.3. *Regular Deformations*

In Section VII.2 we studied regular linear deformations $Q(\lambda)=Q+\lambda q$ of Q, and the resulting family $\{\mathscr{H}, Q(\lambda), \gamma, U(g), \mathfrak{A}\}$ of Θ-summable, fractionally differentiable structures. In this section we replace λq by a family $q(\lambda)$. We require that $q(\lambda)$ satisfies the assumptions of Section VII.2, and in addition we make an assumption on the derivative of $q(\lambda)$ with respect to λ. Replacing Assumption (b) in Section VII.2, we formulate the following:

b′. *A Family of Regular Perturbations*

We assume that for each $\lambda \in \Lambda$, the operator $q(\lambda)$ is a symmetric operator on the domain $\mathscr{D}=\mathscr{D}(Q)$. We assume that there are constants $0 \leqslant a < 1$ and $0 \leqslant M < \infty$ such that for all λ in a compact subset of Λ, the inequality

$$q(\lambda)^2 \leqslant a^2Q^2+M^2 \quad \text{(VII.24)}$$

holds on $\mathscr{D} \times \mathscr{D}$. We define

$$Q(\lambda)=Q+q(\lambda). \quad \text{(VII.25)}$$

Note that if q is an operator with domain $\mathscr{D}$ which satisfies (VII.24) on $\mathscr{D} \times \mathscr{D}$, then q is an element of $\mathscr{T}(0, 1)$. Furthermore, if q is symmetric, then

q determines uniquely an element of $\mathscr{T}(-1,0)$ given by the adjoint sesquilinear form. Conversely, if q is a symmetric sesquilinear form on $\mathscr{D}_\infty \times \mathscr{D}_\infty$, and if furthermore $q \in \mathscr{T}(0,1) \cap \mathscr{T}(-1,0)$, then q uniquely determines a symmetric operator on the domain $\mathscr{D}$. Thus we may consider $q(\lambda)$ as an operator with domain $\mathscr{D}$ or as an element of $\mathscr{T}(0,1) \cap \mathscr{T}(-1,0)$.

According to (VII.24),

$$q(\lambda) \in \mathscr{T}(0,1) \cap \mathscr{T}(-1,0)$$

and $q(\lambda)$ varies over a bounded set for λ in a compact subset of Λ. As in Assumption (b), we also require that $q(\lambda) \in \mathscr{T}(-\varepsilon, 1-\varepsilon) \cap \mathscr{T}(-1+\varepsilon, \varepsilon)$ for some interval $0<\varepsilon<\varepsilon_0$. We combine these requirements by assuming that

$$\|q(\lambda)\|_{(-\varepsilon,\,1-\varepsilon)} \leqslant O(1) \tag{VII.26}$$

for all $\varepsilon \in [0, \varepsilon_0] \cup [1-\varepsilon_0, 1]$ with some $\varepsilon_0 > 0$. Furthermore the bound (VII.26) is uniform for λ in a compact subset of Λ. The assumption (VII.26) for all $0 \leqslant \varepsilon \leqslant 1$ follows automatically from (VII.24) in case $q(\lambda)$ is essentially self-adjoint on $\mathscr{D}$.

It is in the latter sense that we make an assumption about the differentiability of $q(\lambda)$. We assume that for $\lambda, \lambda' \in \Lambda$, the difference quotient

$$\delta(\lambda, \lambda') = \frac{q(\lambda) - q(\lambda')}{\lambda - \lambda'},$$

which is an element of $\mathscr{T}(0,1) \cap \mathscr{T}(-1,0)$, converges in both of these spaces as $\lambda' \to \lambda$.

Thus we assume that there exists a symmetric operator $\dot{q}(\lambda)$ with domain $\mathscr{D}$, which is the derivative of $q(\lambda)$ in the sense that for λ, λ' in a compact set of Λ,

$$\lim_{\lambda' \to \lambda} \|\delta(\lambda, \lambda') - \dot{q}(\lambda)\|_{(0,1)} + \lim_{\lambda' \to \lambda} \|\delta(\lambda, \lambda') - \dot{q}(\lambda)\|_{(-1,0)} = 0. \tag{VII.27}$$

We also want $\dot{q}(\lambda)$ to be continuous in λ, in the space $\mathscr{T}(0,1) \cap \mathscr{T}(-1,0)$. Thus we suppose that

$$\lim_{\lambda' \to \lambda} \|\dot{q}(\lambda) - \dot{q}(\lambda)\|_{(0,1)} + \lim_{\lambda' \to \lambda} \|\dot{q}(\lambda) - \dot{q}(\lambda')\|_{(-1,0)} = 0. \tag{VII.28}$$

Define *Assumption* b' to be (VII.27–28), replacing *Assumption* b of the previous subsection. We retain assumptions (a, c, d). In the linear case of Section VII.2, $q(\lambda) = \lambda q$, so $\delta(\lambda, \lambda') = q = \dot{q}(\lambda)$. As $q \in \mathscr{T}(0,1)$, the limits (VII.27–28) hold trivially, and b' is an automatic consequence of b.

DEFINITION VII.7. A family $Q(\lambda)$ satisfying the Assumptions (a) of Section VII.2 and (b′) above (including (VII.24–28)) is a regular Q-family. The family $\{\mathscr{H}, Q(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ which satisfies Assumptions (a, b′, c, d) is a regular family of Θ-summable, fractionally-differentiable, non-commutative structures.

PROPOSITION VII.8. *Let $\{Q(\lambda)\}$ denote a regular Q-family. Then for λ in any compact subset of Λ, the conclusions of Proposition* VII.4 *and Corollary* VII.5 *hold with $a<1$ replacing $\mu a<1$, and with M replacing μM in all estimates.*

The proof of this proposition parallels that of Proposition VII.4 and Corollary VII.5. We just replace the bound $\lambda^2 q^2 \leqslant \mu^2 a^2 Q^2 + \mu^2 M^2$ by (VII.24). Self-adjointness of $Q(\lambda)$ follows as before. Furthermore the three inequalities all follow from the inequality $q(\lambda)^2 \leqslant a^2 Q^2 + M^2$. Thus we end up with the modified form of (VII.13–15), where a replaces μa and where M replaces μM. Similarly we can derive the inequalities (VII.22). Thus we also have proved

PROPOSITION VII.9. *Let $Q(\lambda)$ denote a regular Q-family. Then the Sobolev spaces $\mathscr{H}_p(Q(\lambda))$, with $p \in [-1, 1]$, and λ in any compact subset of Λ, are independent of λ.*

We now proceed to study the differentiability of $\exp(-sQ(\lambda)^2)$. Let us define the difference quotient of heat kernels by

$$\Delta(\lambda, \lambda') = (\lambda - \lambda')^{-1} \left(e^{-sQ(\lambda)^2} - e^{-sQ(\lambda')^2}\right). \tag{VII.29}$$

PROPOSITION VII.10. *Let $Q(\lambda)$ be a regular Q-family and let $0 < s \leqslant 1$. Let λ belong to a compact subset of Λ. Let*

$$Y(\lambda) = \int_0^s e^{-tQ(\lambda)^2}\, d_\lambda \dot{q}(\lambda)\, e^{-(s-t)\,Q(\lambda)^2}\, dt, \tag{VII.30}$$

where $d_\lambda \dot{q}(\lambda) = Q(\lambda)\, \dot{q}(\lambda) + \dot{q}(\lambda)\, Q(\lambda)$. Also let $X = \{I, d_\lambda \dot{q}(\lambda)\}$ be a two-vertex set. Then

(i) *X is a regular set with respect to Q.*

(ii) *The operator $Y(\lambda)$ of* (*VII*.30) *is related to the heat kernel regularization of X by*

$$Y(\lambda) = \int_0^s X(t, s-t)\, dt. \tag{VII.31}$$

(iii) *Let* $0 \leqslant \alpha, \beta$ *and* $0 \leqslant \alpha + \beta < 1$ *and* $0 < s < 1$. *Consider* $Y(\lambda)$ *and* $\Delta(\lambda, \lambda')$ *for* λ, λ' *in a compact subset of* Λ. *They are bounded uniformly in* $\mathscr{T}(\alpha, -\beta; s^{-1})$, *as defined in Section V.6. There exists* $M < \infty$ *such that*

$$\| Y\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} + \|\Delta\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant M s^{-(\alpha+\beta)/2}. \tag{VII.32}$$

(iv) *The* λ*-derivative of* $e^{-sQ(\lambda)^2}$ *in* $\mathscr{T}(\alpha, -\beta; s^{-1})$ *exists and equals* $-Y$. *In fact for any* $\varepsilon > 0$,

$$\|\Delta + Y\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant o(1)\, s^{-(\alpha+\beta+\varepsilon)/2}, \tag{VII.33}$$

where $o(1) \to 0$ *as* $|\lambda - \lambda'| \to 0$.

We write

$$\frac{d}{d\lambda} e^{-sQ(\lambda)^2} = -\int_0^s e^{-tQ(\lambda)^2}\, d_\lambda \dot{q}(\lambda)\, e^{-(s-t)\, Q(\lambda)^2}\, dt. \tag{VII.34}$$

Remark. The fact that the derivative (VII.34) exists not just as a limit of difference quotients in $\mathscr{B}(\mathscr{H})$, but also as a limit in the space $\mathscr{T}(\alpha, -\beta; s^{-1})$ is crucial. It is this fact which will allow us to differentiate the expression for $\tau_n^{\mathrm{JLO}}(\lambda)$ in terms of the expectations which define $\tau_n^{\mathrm{JLO}}(\lambda)$. In other words, it establishes the commutativity of differentiation with respect to λ and the trace and integration over s.

Proof. (i) We assume in (b′) that $\dot{q}(\lambda) \in \mathscr{T}(0, 1) \cap \mathscr{T}(-1, 0)$. Furthermore as explained in (b), $q(\lambda) \in \mathscr{T}(0, 1) \cap \mathscr{T}(-1, 0)$, so also $Q(\lambda) \in \mathscr{T}(0, 1) \cap \mathscr{T}(-1, 0)$. As a consequence, both $\dot{q}(\lambda)\, Q(\lambda)$ and $Q(\lambda)\, \dot{q}(\lambda)$ and therefore $d_\lambda \dot{q}(\lambda)$ are elements of $\mathscr{T}(-1, 1)$. Hence the two-vertex set has $\alpha_0 = \beta_0 = 0$ and $\alpha_1 = \beta_1 = 1$, giving $\eta_0 = \eta_1 = \frac{1}{2}$. Thus according to Definition V.1, X is a regular set with respect to Q.

(ii) Note $X(t, s-t) \in I_1$ for $t, s-t > 0$, and $\|X(t, s-t)\|_1 \leqslant O(t^{-1/2}(s-t)^{-1/2})$ as a consequence of (V.38), which is integrable over s. This defines $Y(\lambda)$ in (VII.31) and also proves (ii).

(iii) We next estimate $\| Y(\lambda)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})}$. In fact

$$\| Y(\lambda)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant \int_0^s \|X(t, s-t)\|_{\mathscr{T}(\alpha, -\beta, s^{-1})}\, dt,$$

so it is sufficient to estimate the integrand for $t, s-t > 0$. We have,

$$\begin{aligned}\|X(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} &= \|R^{-\alpha} X(t, s-t)\, R^{-\beta}\|_{I_{s^{-1}}} \\ &= \|R^{-\alpha} e^{-tQ(\lambda)^2}\, d_\lambda \dot{q}(\lambda)\, e^{-(s-t)\, Q(\lambda)^2} R^{-\beta}\|_{I_{s^{-1}}}.\end{aligned}$$

Here $r = (Q^2 + I)^{-1/2}$. By Proposition VII.9, there is a constant $\tilde{M}_3 < \infty$, independent of λ, in a compact subset of Λ, such that for $R(\lambda) = (Q(\lambda)^2 + I)^{-1/2}$,

$$\|R^{-1}R(\lambda)\| \leqslant \tilde{M}_3,$$

and hence

$$\|R^{-\alpha}R(\lambda)^{\alpha}\| \leqslant \tilde{M}_3^{\alpha}, \qquad 0 \leqslant \alpha \leqslant 1. \tag{VII.35}$$

Thus by Hölder's inequality, (V.28), and the fact that $\alpha + \beta < 1$,

$$\begin{aligned}
&\|X(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \\
&\quad \leqslant \tilde{M}_3^{\alpha+\beta} \, \|R(\lambda)^{-\alpha-1} \, e^{-tQ(\lambda)^2/2}\| \; \|R(\lambda)^{-\beta-1} e^{-(s-t)\, Q(\lambda)^2/2}\| \\
&\qquad \times \|R(\lambda)\, d_\lambda \dot{q}(\lambda)\, R(\lambda)\| \; \|e^{-tQ(\lambda)^2/2}\|_{I_{t^{-1}}} \, \|e^{-(s-t)\, Q(\lambda)^2/2}\|_{I_{(s-t)^{-1}}} \\
&\quad \leqslant 16 \tilde{M}_3^{\alpha+\beta} (\mathrm{Tr}(e^{-Q(\lambda)^2/2}))^s \, (s-t)^{-(\beta+1)/2} \, (t)^{-(\alpha+1)/2} \\
&\qquad \times \|R(\lambda)\, d_\lambda \dot{q}(\lambda)\, R(\lambda)\|.
\end{aligned} \tag{VII.36}$$

Since $d_\lambda \dot{q}(\lambda) \in \mathscr{T}(-1, 1)$, it follows that

$$\|R(\lambda)\, d_\lambda \dot{q}(\lambda)\, R(\lambda)\| \leqslant \tilde{M}_3^2 \, \|d_\lambda \dot{q}(\lambda)\|_{(-1, 1)}.$$

As we may take $\tilde{M}_3 > 1$,

$$\begin{aligned}
\|X(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} &\leqslant 16 \tilde{M}_3^3 t^{-(\alpha+1)/2} \, (s-t)^{-(\beta+1)/2} \\
&\qquad \times (\mathrm{Tr}(e^{-Q(\lambda)^2/2}))^s \, \|d_\lambda \dot{q}(\lambda)\|_{(-1, 1)}.
\end{aligned} \tag{VII.37}$$

Integrating over t we therefore obtain

$$\|Y(\lambda)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant \tilde{M}_4 s^{-(\alpha+\beta)/2} (\mathrm{Tr}(e^{-Q(\lambda)^2/2}))^s \, \|d_\lambda \dot{q}(\lambda)\|_{(-1, 1)}, \tag{VII.38}$$

with $\tilde{M}_4 = 16 \tilde{M}_3^3 B_1((1-\alpha)/2, (1-\beta)/2)$.

Next we derive a similar bound on $\|\Delta(\lambda, \lambda')\|_{\mathscr{T}(\alpha, -\beta; s^{-1})}$. In this case we recall from Proposition VII.9 with $p = 1$, that the domain $\mathscr{D}(Q(\lambda))$ of $Q(\lambda)$ is λ independent for λ in a compact subset of Λ. Thus $Q(\lambda)^2$ is a sesquilinear form on $\mathscr{D}(Q(\lambda')) \times \mathscr{D}(Q(\lambda'))$. Hence

$$\begin{aligned}
\Delta(\lambda, \lambda') &= (\lambda - \lambda')^{-1} \, e^{-tQ(\lambda)^2} e^{-(s-t)\, Q(\lambda')^2} \Big|_{t=0}^{t=s} \\
&= (\lambda - \lambda')^{-1} \int_0^s e^{-tQ(\lambda)^2} (Q(\lambda')^2 - Q(\lambda)^2) \, e^{-(s-t)\, Q(\lambda')^2} \, dt.
\end{aligned} \tag{VII.39}$$

Note $\mathrm{Range}(e^{-tQ(\lambda)^2}) \subset \mathscr{D}(Q(\lambda)) = \mathscr{D}(Q) = \mathscr{D}$. We have on $\mathscr{D} \times \mathscr{D}$ the form identity

$$\Delta(\lambda, \lambda') = -\int_0^s e^{-tQ(\lambda)^2}(Q(\lambda)\,\delta(\lambda, \lambda') + \delta(\lambda, \lambda')\,Q(\lambda'))\,e^{-(s-t)\,Q(\lambda')^2}\,dt. \tag{VII.40}$$

Since $\delta(\lambda, \lambda') \in \mathscr{T}(0, 1) \cap \mathscr{T}(-1, 0)$, we can repeat the proof of the bound on $Y(\lambda)$ to obtain

$$\|\Delta(\lambda, \lambda')\|_{\mathscr{T}(\alpha,\, -\beta;\, s^{-1})} \leqslant \tilde{M}_5 s^{-(\alpha+\beta)/2} (\mathrm{Tr}(e^{-\tilde{M}_1^{-2}Q^2/2}))^s \times (\|\delta(\lambda, \lambda')\|_{\mathscr{T}(0,\,1)} + \|\delta(\lambda, \lambda')\|_{\mathscr{T}(-1,\,0)}), \tag{VII.41}$$

where $\tilde{M}_5$ is a constant proportional to $B_1((1-\alpha)/2, (1-\beta)/2)$. Since s is bounded, and $\exp(-\beta Q^2)$ is trace class for all $\beta > 0$, this proves (VII.32).

(iv) Up to now we have only used the uniform bound on $\delta(\lambda, \lambda')$. However, in order to prove differentiability we need to establish convergence to zero of the sum $\Delta + Y$. We express $\Delta + Y$ as a sum of five terms, and we then show that each term converges to zero in $\mathscr{T}(\alpha, -\beta; s^{-1})$. With the notation

$$S(t) = e^{-tQ(\lambda)^2}, \qquad S'(t) = e^{-tQ(\lambda')^2}, \tag{VII.42}$$

use (VII.30, 40) to write

$$\Delta + Y = -\int_0^s S(t)\,Q(\lambda)(\delta S'(s-t) - \dot{q}S(s-t))\,dt - \int_0^s S(t)(\delta Q(\lambda')\,S'(s-t) - \dot{q}Q(\lambda)\,S(s-t))\,dt. \tag{VII.43}$$

We expand this as a sum of differences

$$\Delta + Y = \sum_{j=1}^{5} \int_0^s S_j(t, s-t)\,dt, \tag{VII.44}$$

with

$$Z_1(t, s-t) = S(t)\,Q(\lambda)(\delta - \dot{q})\,S'(s-t), \tag{VII.45}$$

$$Z_2(t, s-t) = S(t)\,Q(\lambda)\,\dot{q}(S'(s-t) - S(s-t)), \tag{VII.46}$$

$$Z_3(t, s-t) = S(t)(\delta - \dot{q})\,Q(\lambda')\,S'(s-t), \tag{VII.47}$$

$$Z_4(t, s-t) = S(t)\,\dot{q}\delta S'(s-t)(\lambda' - \lambda), \tag{VII.48}$$

$$Z_5(t, s-t) = S(t)\,\dot{q}Q(\lambda)(S'(s-t) - S(s-t)). \tag{VII.49}$$

We now show that for any $\varepsilon > 0$, each of these terms satisfies

$$\|Z_j(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant o(1)\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)/2}, \qquad \text{(VII.50)}$$

where $o(1) \to 0$ as $|\lambda - \lambda'| \to 0$, uniformly in s, t. We then integrate (VII.50) over $0 < t < s$. Since $0 \leqslant \alpha, \beta$ and $\alpha + \beta < 1$, it follows that $\beta < 1$. Therefore we can choose ε sufficiently small that $(1+\beta+\varepsilon)/2 < 1$. Then the integral converges, and we obtain

$$\|\Delta + Y\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \leqslant o(1)\, s^{-(\alpha+\beta+\varepsilon)/2}, \qquad \text{(VII.51)}$$

as claimed.

Let us begin by proving the bound on $Z_1(t, s-t)$. Observe that for $t > 0$, $s - t > 0$, by Hölder's inequality

$$\begin{aligned}
&\|Z_1(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \\
&\quad = \|R^{-\alpha} Z_1(t, s-t)\, R^{-\beta}\|_{I_{s^{-1}}} \\
&\quad \leqslant \|R^{-\alpha} S(t)\, Q(\lambda)\|_{I_{t^{-1}}}\, \|(\delta - \dot{q})\, R\|\, \|R^{-1} S'(s-t)\, R^{-\beta}\|_{I_{(s-t)^{-1}}} \\
&\quad \leqslant \|R^{-\alpha} S(t/3)\|\, \|S(t/3)\|_{I_{t^{-1}}}\, \|S(t/2)\, Q(\lambda)\| \\
&\qquad \times \|(\delta - \dot{q})\, R\|\, \|R^{-1} S'((s-t)/3)\|\, \|S'((s-t)/3)\|_{I_{(s-t)^{-1}}} \\
&\qquad \times \|S'((s-t)/3)\, R^{-\beta}\|. \qquad \text{(VII.52)}
\end{aligned}$$

We now bound the seven factors that occur after the final inequality. Using bounds (VII.35) and (V.28), we bound the first factor by

$$\|r^{-\alpha} S(t/3)\| \leqslant \tilde{M}_3^{\alpha}\, \|R(\lambda)^{-\alpha}\, S(t/3)\| \leqslant 2\tilde{M}_3^{\alpha}(t/3)^{-\alpha/2}. \qquad \text{(VII.53)}$$

We bound the third, fifth, and seventh terms similarly. Thus

$$\begin{aligned}
&\|Z_1(t, s-t)\|_{\mathscr{T}(\alpha, -\beta; s^{-1})} \\
&\quad \leqslant 3^2 2^3 (3\tilde{M}_3)^{(\alpha+\beta+1)/2}\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta)/2} \\
&\qquad \times (\mathrm{Tr}(e^{-Q(\lambda)^2/2}))^t\, (\mathrm{Tr}(e^{-Q(\lambda')^2/3}))^{s-t}\, \|(\delta - \dot{q})\|_{(0,1)}. \qquad \text{(VII.54)}
\end{aligned}$$

By Proposition VII.8, we have the bound (VII.15) at both points λ and λ' under consideration. Thus

$$(\mathrm{Tr}(e^{-Q(\lambda)^2/3}))^t\, (\mathrm{Tr}(e^{-Q(\lambda')^2/3}))^{s-t} \leqslant \tilde{M}_4^s,$$

where $\tilde{M}_4$ is bounded uniformly in λ, λ' in the compact subset of Λ. Hence we may include all constants together in one new constant $\tilde{M}_5$ to yield

$$\|Z_1(t, s-t)\|_{\mathscr{T}(\alpha - \beta; s^{-1})} \leqslant \tilde{M}_5 t^{-(1+\alpha)/2}(s-t)^{-(1+\beta)/2}\, \|\delta - \dot{q}\|_{(0,1)}. \qquad \text{(VII.55)}$$

The hypothesis (VII.27) ensures that $\|\delta - \dot{q}\|_{(0,1)}$ is $o(1)$ as $|\lambda - \lambda'| \to 0$. Thus (VII.55) is bounded by (VII.50) with $\varepsilon = 0$, and the bound on Z_1 has been proved. The bound on Z_3 is similar, except that we use (VII.27) to conclude $\|\delta - \dot{q}\|_{(-1,0)} = o(1)$.

Next we consider the bound on Z_4. Here we use the following bounds that are uniform in λ:

$$\|\dot{q}\|_{(-1,0)} \leqslant O(1), \qquad \text{and} \qquad \|\delta\|_{(0,1)} \leqslant O(1).$$

We proceed as above to establish

$$\|Z_4(t, s-t)\|_{\mathcal{T}(\alpha, -\beta; s^{-1})} \leqslant O(|\lambda - \lambda'|)\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta)/2}, \tag{VII.56}$$

so the explicit coefficient $\lambda' - \lambda$ provides Lipschitz continuity of this term. In particular, we have established (VII.50) for Z_1, Z_3, Z_4, and the bound holds for all three with $\varepsilon = 0$.

Let us now inspect Z_2 which requires a different method. Let P_n denote the orthogonal projection in $\mathscr{H}$ onto the subspace for which $Q^2 \leqslant n$. Decompose $\mathscr{H} = P_n \mathscr{H} \oplus (I - P_n)\, \mathscr{H}$ into two subspaces on which $Q^2 \leqslant n$ and $Q^2 > n$ respectively. Also write

$$Z_2(t, s-t) = Z_2(t, s-t)\, P_n + Z_2(t, s-t)(I - P_n). \tag{VII.57}$$

We claim that given $\varepsilon_1 > 0$, we can choose n_0 sufficiently large so that for $n > n_0$, we have

$$\|Z_2(t, s-t)(I - P_n)\|_{\mathcal{T}(\alpha, -\beta; s^{-1})} \leqslant \varepsilon_1 t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)/2} \tag{VII.58}$$

for all λ, λ' in the compact subset of Λ. In fact, choose $0 < \varepsilon$ sufficiently small so that $\beta + \varepsilon < 1$. We repeat the type of estimate in (VII.52) above to obtain the bound

$$\begin{aligned} &\|Z_2(t, s-t)\|_{\mathcal{T}(\alpha, -\beta-\varepsilon; s^{-1})} \\ &\qquad \leqslant O(t^{-(1+\alpha)/2})\, \|S'(s-t) - S(s-t)\|_{\mathcal{T}(1, -\beta-\varepsilon; (s-t)^{-1})} \\ &\qquad \leqslant O(1)\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)}. \end{aligned} \tag{VII.59}$$

We use here $\|r^{-1}S'(s-t)\, R^{-\beta-\varepsilon}\|_{I_{(s-t)^{-1}}} \leqslant O(1)$, and similarly we also use $\|R^{-1}S(s-t)\, R^{-\beta-\varepsilon}\|_{I_{(s-t)^{-1}}} \leqslant o(1)$.

On the other hand

$$\|R^{\varepsilon}(I - P_n)\| = \|(Q^2 + I)^{-\varepsilon/2}\, (I - P_n)\| \leqslant (n+1)^{-\varepsilon/2}. \tag{VII.60}$$

Thus we have

$$\begin{aligned}\|Z_2(t, s-t)(I-P_n)\|_{\mathscr{T}(\alpha,\ -\beta;\ s^{-1})} &\\ \leqslant \|Z_2(t, s-t)\ R^{-\varepsilon}R^{\varepsilon}(I-P_n)\|_{\mathscr{T}(\alpha,\ -\beta;\ s^{-1})} &\\ \leqslant \|Z_2(t, s-t)\|_{\mathscr{T}(\alpha,\ -\beta-\varepsilon;\ s^{-1})}\ \|R^{\varepsilon}(I-P_n)\| &\\ \leqslant O((n+1)^{-\varepsilon/2})\ t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)/2}. & \qquad \text{(VII.61)}\end{aligned}$$

Hence for n_0 sufficiently large and $n \geqslant n_0$, we have $O((n+1)^{-\varepsilon/2}) \leqslant \varepsilon_1$, and (VII.58) holds.

We also claim that if we choose a fixed $n > n_0$, then

$$\|Z_2(t, s-t)\ P_n\|_{\mathscr{T}(\alpha,\ -\beta;\ s^{-1})} \leqslant o(1)\ t^{-(1+\alpha)/2}. \qquad \text{(VII.62)}$$

In fact $\exp(-Q^2)$ is trace class so $P_n\mathscr{H}$ is a finite-dimensional subspace of $\mathscr{H}$. The dimension of $P_n\mathscr{H}$ is fixed once n is fixed. Furthermore, the operator T defined by

$$T = R^{-\alpha}Z_2(t, s-t)\ R^{-\beta}P_n,$$

yields

$$T^*T = P_nR^{-\beta}Z_2(t, s-t)^*\ R^{-2\alpha}Z_2(t, s-t)\ R^{-\beta}P_n, \qquad \text{(VII.63)}$$

which acts on the finite-dimensional subspace $P_n\mathscr{H}$. Thus the absolute value $|T| = (T^*T)^{1/2}$ of T also acts on $P_n\mathscr{H}$. We therefore can write

$$\|T\|_{I_{s^{-1}}} = (\mathrm{Tr}_{P_n\mathscr{H}}(|T|^{s^{-1}}))^{s}, \qquad \text{(VII.64)}$$

with the trace restricted to $P_n\mathscr{H}$. But

$$\|Z_2(t, s-t)\ P_n\|_{\mathscr{T}(\alpha-\beta;\ s^{-1})} = \|T\|_{I_{s^{-1}}}, \qquad \text{(VII.65)}$$

so we can evaluate the $\mathscr{T}(\alpha, \beta; s^{-1})$ norm of Z_2P_n on the subspace $P_n\mathscr{H}$.

Let f_j, for $j = 1, 2, ..., N$ be an orthonormal basis for $P_n\mathscr{H}$. We claim that for each j,

$$\langle f_j, |T|^{s^{-1}}\ f_j\rangle^{s} \leqslant o(1)\ t^{-(\alpha+1)/2}, \qquad \text{(VII.66)}$$

where $o(1) \to 0$ as $|\lambda - \lambda'| \to 0$. As a consequence of (VII.66) and of the fixed, finite dimensionality of $P_n\mathscr{H}$, we infer (VII.62).

To prove (VII.66), note that

$$\|R^{-\beta}P_n\| \leqslant \|(Q^2+I)^{\beta/2}\ P_n\| \leqslant (n+1)^{\beta/2}. \qquad \text{(VII.67)}$$

Also

$$R^{-\alpha} Z_2(t, s-t) = R^{-\alpha} S(t)\, Q(\lambda)\, \dot{q}(\lambda)(S'(s-t) - S(s-t)). \tag{VII.68}$$

As a consequence of Proposition VII.9, we have the bounds (VII.22) for all λ in a compact subset of Λ. In particular, we have

$$Q(\lambda)^2 \leqslant \tilde{M}_2^2 (Q^2 + I)$$

on the subspace $P_n \mathscr{H}$. We therefore conclude that $\|Q(\lambda)\, P_n\|$ is bounded by $\tilde{M}_2(n+1)$. In other words, if $P_m(\lambda)$ is the orthogonal projection in $\mathscr{H}$ onto the subspace on which $Q(\lambda)^2 \leqslant m$, we have

$$P_n \mathscr{H} \subset P_{\tilde{M}_2(n+1)}(\lambda)\, \mathscr{H},$$

for each λ in the compact subset of Λ. On the other hand the other inequality (VII.22) ensures

$$Q^2 \leqslant \tilde{M}_1^2 (Q(\lambda)^2 + I),$$

so

$$P_{\tilde{M}_2(n+1)}(\lambda)\, \mathscr{H} \subset P_{\tilde{M}_1(\tilde{M}_2(n+1)+1)} \mathscr{H}.$$

We conclude that $(S'(s-t) - S(s-t))\, P_n$ has a range in $P_{n_1} \mathscr{H}$, where $n_1 = \tilde{M}_1(\tilde{M}_2(n+1)+1)$. Thus

$$\begin{aligned} &\|\dot{q}(\lambda)(S'(s-t) - S(s-t))\, P_n\| \\ &\quad = \|\dot{q}(\lambda)\, R R^{-1} P_{n_1}(S'(s-t) - S(s-t))\, P_n\| \\ &\quad \leqslant \|\dot{q}\|_{(0,1)}\, (n_1+1)\, \|(S'(s-t) - S(s-t))\, P_n\|. \end{aligned} \tag{VII.69}$$

Combining (VII.68) with the bound

$$\|R^{-\alpha} S(t)\, Q(\lambda)\| \leqslant O(t^{-(1+\alpha)/2}), \tag{VII.70}$$

uniformly on a compact subset of Λ, we can bound $|T|$ in norm by

$$\begin{aligned} \|\,|T|\,\| &= \|(T^*T)^{1/2}\| \leqslant \|T^*T\|^{1/2} \\ &\leqslant O(t^{-(1+\alpha)/2})(n_1+1)(n+1)^{\beta/2}\, \|\dot{q}\|_{(0,1)}^{1/2}\, \|\dot{q}\|_{(-1,0)}^{1/2} \\ &\quad \times \|(S'(s-t) - S(s-t))\, P_n\|. \end{aligned} \tag{VII.71}$$

For $n \geqslant n_0$ fixed, the constants in (VII.71) are uniform in λ. However, $\|(S'(s-t)-S(s-t))\,P_n\| = o(1)$ as $|\lambda - \lambda'| \to 0$, for this norm is calculated on a given, finite dimensional subspace of $\mathscr{H}$. Thus

$$\|\,|T|\,\| = o(1)\, t^{-(1+\alpha)/2},$$

with $o(1) \to 0$ as $|\lambda - \lambda'| \to 0$. Likewise for $0 < s \leqslant 1$,

$$\|\,|T|^{s^{-1}}\| \leqslant (o(1)(t^{-(1+\alpha)/2}))^{s^{-1}}$$

and

$$\langle f_j, |T|^{s^{-1}} f_j\rangle^s \leqslant o(1)\, t^{-(1+\alpha)/2}.$$

Hence we have proved (VII.66) and (VII.62).

We now combine (VII.58) with (VII.62) to give

$$\begin{aligned}
&\|Z_2(t, s-t)\|_{\mathscr{T}(\alpha,\, -\beta;\, s^{-1})} \\
&\quad\leqslant \|Z_2(t, s-t)(I-P_n)\|_{\mathscr{T}(\alpha,\, -\beta;\, s^{-1})} + \|Z_2(t, s-t)\, P_n)\|_{\mathscr{T}(\alpha,\, -\beta;\, s^{-1})} \\
&\quad\leqslant o(1)\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)/2}. \qquad \text{(VII.72)}
\end{aligned}$$

Here we use $1 \leqslant (s-t)^{-(1+\beta+\varepsilon)/2}$ in the bound on $Z_2 P_n$. Thus we have established (VII.50) in the case of Z_2.

The proof of the bound (VII.50) for Z_5 is a minor modification on the proof for Z_2, and it also results in the bound

$$\|Z_5(t, s-t)\|_{\mathscr{T}(\alpha,\, -\beta;\, s^{-1})} \leqslant o(1)\, t^{-(1+\alpha)/2}(s-t)^{-(1+\beta+\varepsilon)/2}. \qquad \text{(VII.73)}$$

Hence we have completed the proof of (VII.33) and of the proposition.

In the course of establishing the proposition, we have used a method which gives a useful bound on Δ. We state this separately,

PROPOSITION VII.11. *Let $Q(\lambda)$ be a regular Q-family and let $0 < s \leqslant 1$. Let λ, λ' belong to a compact subset of Λ. Let $0 \leqslant \beta < 1$. Then for any $\varepsilon > 0$,*

$$\begin{aligned}
&\|e^{-sQ(\lambda)^2} - e^{-sQ(\lambda')^2}\|_{\mathscr{T}(\beta,\, -1;\, s^{-1})} + \|e^{-sQ(\lambda)^2} - e^{-sQ(\lambda')^2}\|_{\mathscr{T}(1,\, -\beta;\, s^{-1})} \\
&\quad\leqslant o(1)\, s^{-(1+\beta+\varepsilon)/2}, \qquad \text{(VII.74)}
\end{aligned}$$

where $o(1) \to 0$ as $|\lambda - \lambda'| \to 0$.

Proof. We inspect the second term in (VII.74). The bound on this difference in the norm $\mathscr{T}(1, -\beta; s^{-1})$ was proved in the course of our proof of the bound (VII.72) on $Z_2(, s-t)$. To be explicit,

$$\begin{aligned}
&\|e^{-sQ(\lambda)^2}-e^{-sQ(\lambda')^2}\|_{\mathscr{T}(1-\beta;\,s^{-1})}\\
&\quad=\|R^{-1}(e^{-sQ(\lambda)^2}-e^{-sQ(\lambda')^2})\,R^{-\beta}\|_{I_{s^{-1}}}\\
&\quad\leqslant\|R^{-1}(e^{-sQ(\lambda)^2}-e^{-sQ(\lambda')^2})\,R^{-\beta}(I-P_n)\|_{I_{s^{-1}}}\\
&\qquad+\|R^{-1}(e^{-sQ(\lambda)^2}-e^{-sQ(\lambda')^2})\,R^{-\beta}P_n\|_{I_{s^{-1}}},
\end{aligned}\tag{VII.75}$$

where P_n denotes the orthogonal projection onto the subspace of $\mathscr{H}$ on which $Q^2\leqslant n$. As in the proof of (VII.72), and with the notation used there,

$$\begin{aligned}
&\|R^{-1}(S(s)-S'(s))\,R^{-\beta}(I-P_n)\|_{I_{s^{-1}}}\\
&\quad\leqslant\|R^{-1}(S(s)-S'(s))\,R^{-\beta-\varepsilon}\|_{I_{s^{-1}}}\,\|R^{\varepsilon}(I-P_n)\|\\
&\quad\leqslant\|R^{-1}(S(s)-S'(s))\,R^{-\beta-\varepsilon}\|_{I_{s^{-1}}}\,(n+1)^{-\varepsilon}\\
&\quad\leqslant O(n+1)^{-\varepsilon}\,s^{-(1+\beta+\varepsilon)/2}\leqslant\varepsilon_1 s^{-(1+\beta+\varepsilon)/2}.
\end{aligned}\tag{VII.76}$$

Here $\varepsilon_1>0$ is given and n is chosen sufficiently large, depending on ε_1. Likewise for fixed n,

$$\|R^{-1}(S(s)-S'(s))\,R^{-\beta}P_n\|_{I_{s^{-1}}}\leqslant o(1),\qquad\text{as}\quad|\lambda-\lambda'|\to 0.\tag{VII.77}$$

This is a consequence of an analysis of T^*T, where

$$T=R^{-1}(S(s)-S'(s))\,R^{-\beta}P_n.\tag{VII.78}$$

We infer that

$$T^*T=P_nR^{-\beta}(S(s)-S'(s))\,R^{-2}(S(s)-S'(s))\,R^{-\beta}P_n\tag{VII.79}$$

is bounded using (VII.67) and the argument following, for n fixed, by

$$\|\,|T|\,\|\leqslant\|T^*T\|^{1/2}\leqslant O(1)\,\|(S(s)-S'(s))\,P_n\|\leqslant o(1).\tag{VII.80}$$

Hence

$$(\mathrm{Tr}_{P_n\mathscr{H}}(|T|^{s^{-1}}))^s\leqslant o(1)\tag{VII.81}$$

and (VII.77) holds. But (VII.76–77) show that

$$\|S(s)-S'(s)\|_{\mathscr{T}(1-\beta;\,s^{-1})}\leqslant o(1)\,s^{-(1+\beta+\varepsilon)/2},$$

which bounds the second term on the left of (VII.74). The bound on the first term of (VII.74) is just the corresponding dual bound on the adjoint of $S(s)-S'(s)$. Hence it follows and the proposition is proved.

VII.4. *The Basic Cochains $L(\lambda)$ and $h(\lambda)$*

We define two families of cochains $L(\lambda)$ and $h(\lambda)$ that characterize the λ-dependence of τ^{JLO} as follows. Let

$$\begin{aligned} L_n(\lambda)(a_0, ..., a_n; g) &= \sum_{j=1}^{n} \langle a_0, d_\lambda a_1, ..., d_\lambda a_{j-1}, [\dot{q}(\lambda), a_j], d_\lambda a_{j+1}, ..., d_\lambda a_n; g\rangle_n \\ &\quad - \sum_{j=0}^{n} \langle a_0, d_\lambda a_1, ..., d_\lambda a_j, d_\lambda \dot{q}(\lambda), d_\lambda a_{j+1}, ..., d_\lambda a_n; g\rangle_{n+1}, \\ &= \sum_{j=1}^{n} \langle a_0, d_\lambda a_1, ..., [\dot{q}(\lambda), a_j], ..., d_\lambda a_n; g\rangle_n \\ &\quad - \sum_{j=0}^{n} \tau_{n+1}^{\mathrm{JLO}}(a_0, a_1, ..., a_j, \dot{q}(\lambda), ..., a_n; g), \end{aligned} \tag{VII.82}$$

and let

$$\begin{aligned} h_n(\lambda)(a_0, ..., a_n; g) &= -\langle a_0, \dot{q}(\lambda), d_\lambda a_1, ..., d_\lambda a_n; g\rangle_{n+1} \\ &\quad + \sum_{k=1}^{n} (-1)^{k+1} \langle a_0, d_\lambda a_1, ..., d_\lambda a_k, \dot{q}(\lambda), d_\lambda a_{k+1}, ..., d_\lambda a_n; g\rangle_{n+1}. \end{aligned} \tag{VII.83}$$

The analytic properties of these cochains are a consequence of the groundwork we have laid.

PROPOSITION VII.12. *Let $\{\mathscr{H}, Q(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ be a regular family of Θ-summable, fractionally-differentiable structures. Then*

(i) *The families $\{L(\lambda)\}$ and $\{h(\lambda)\}$ are bounded families of cochains in $\mathscr{C}(\mathfrak{A})$.*

(ii) *As elements of $\mathscr{C}(\mathfrak{A})$, $L_{2n+1}(\lambda)=0$ and $h_{2n}(\lambda)=0$.*

(iii) *Let $a_j \in \mathfrak{A}$ and for $1 \leqslant k \leqslant n$ let $X^{(k)} = \{x_0^{(k)}, x_1^{(k)}, ..., x_{n+1}^{(k)}\}$ denote the regular sets of $(n+2)$-vertices*

$$x_0^{(k)} = a_0, \qquad x_1^{(k)} = da_1, ..., x_k^{(k)} = da_k, \qquad x_{k+1}^{(k)} = \dot{q}(\lambda), ..., x_{n+1}^{(k)} = da_n, \tag{VII.84}$$

while for $k=0$ let $X^{(k)} = X^{(k)}$ denote the regular set of vertices

$$x_0^{(k)} = a_0, \qquad x_1^{(k)} = \dot{q}(\lambda), \qquad x_2^{(k)} = da_1, ..., x_{n+1}^{(k)} = da_n. \tag{VII.85}$$

Then for $0 \leqslant j \leqslant n+1$ *and* $0 \leqslant k \leqslant n$, *each set of vertices*

$$X_j^{(k)} = \{x_0^{(k)}, x_1^{(k)}, ..., dx_j^{(k)}, ..., x_{n+1}^{(k)}\}, \tag{VII.86}$$

containing one additional derivative in the j*th-place, is also a regular set of vertices.*

(iv) *The function* $g \to h(\lambda)$ *is continuous in* g *as a map from* $\mathfrak{G}$ *to* $\mathscr{C}(\mathfrak{A})$, *uniformly for* λ *in compact subsets of* Λ.

Remark 1. In case there is need to specify the number of vertices explicitly, we denote the sets of vertices $X^{(k)}$ and $X_j^{(k)}$ above as $X^{(k)}(n)$ and $X_j^{(k)}(n)$.

Remark 2. The cochain $h(\lambda)$ of (VII.83) can be expressed in terms of the expectations of the sets of vertices $X^{(k)}$ defined in (VII.84–VII.85), namely

$$h_n(\lambda)(a_0, a_1, ..., a_n; g) = \sum_{k=0}^{n} (-1)^{k+1} \langle \hat{X}^{(k)}(n); g \rangle. \tag{VII.87}$$

Proof. (i–ii) In order to ensure that $L(\lambda)$ and $h(\lambda)$ are elements in $\mathscr{C}$, it is sufficient to show that these expectations exist, that they have the required symmetries, and that they satisfy a uniform bound of the form of (II.32). It is clear that for any $a_k = I$, $k \neq 0$, both $L_n(\lambda)$ and $h_n(\lambda)$ vanish, as they must for cochains in $\mathscr{C}$. Furthermore, the expressions (VII.22) have the invariance property (II.1). The symmetry property under γ ensures that $L_{2n+1} = 0$ and $h_{2n} = 0$, when evaluated on $\mathfrak{A}$.

To prove the uniform bound (II.32), we also show that the expectations defining $L_n(\lambda)$ and $h_n(\lambda)$ arise as expectations of regular sets of vertices. (For the cochain $h_n(\lambda)$ to exist in $\oplus \mathscr{C}_n$, it is sufficient to show that each $X^{(k)}$ defined in part (iii) of the proposition is a regular set of vertices.) Also introduce the sets of vertices for $1 \leqslant k \leqslant n$,

$$Y^{(k)} = \{a_0, d_\lambda a_1, ..., d_\lambda a_{k-1}, \dot{q}(\lambda)\, a_k, d_\lambda a_{k+1}, ..., d_\lambda a_n\},$$

and

$$Z^{(k)} = \{a_0, d_\lambda a_1, ..., d_\lambda a_{k-1}, a_k \dot{q}(\lambda), d_\lambda a_{k+1}, ..., d_\lambda a_n\}.$$

For a bounded, both $\dot{q}(\lambda)\, a$ and $a\dot{q}(\lambda)$ are vertices of type $(-1, 0)$ and $(0, 1)$ respectively, with

$$\|\dot{q}(\lambda)\, a\|_{(-1,0)} \leqslant \|\dot{q}(\lambda)\|_{(-1,0)}\, \|a\|,$$

and

$$\|a\dot{q}(\lambda)\|_{(0,1)} \leqslant \|a\| \, \|\dot{q}(\lambda)\|_{(0,1)}.$$

Here we use Proposition VII.6 which ensures that norms defined with $Q(\lambda)$ or with Q are equivalent. Thus for $a \in \mathfrak{A} \subset \mathfrak{J}_{\beta,\alpha}$, the sets $Y^{(k)}$ and $Z^{(k)}$ have regularity exponents (V.31, 32) given by

$$\eta^{Y}_{\text{local}} \geqslant \frac{1}{2} \min\{1-\alpha, 1-\beta\} > 0, \qquad \eta^{Y}_{\text{global}} > \frac{1}{2} + \frac{1}{2}(1-\alpha-\beta)\left(\frac{n}{n+1}\right).$$

Likewise,

$$\eta^{Z}_{\text{local}} \geqslant \frac{1}{2} \min\{1-\alpha, 1-\beta\} > 0, \qquad \eta^{Z}_{\text{global}} > \frac{1}{2} + \frac{1}{2}(1-\alpha-\beta)\left(\frac{n}{n+1}\right).$$

We can now apply Corollary V.4.iv to conclude that both $Y^{(k)}$ and $Z^{(k)}$ have expectations in $\mathscr{C}_n(\mathfrak{A})$, and define sequences in $\mathscr{C}(\mathfrak{A}$. Hence the first sum on the right of (VII.82) defines an element of $\mathscr{C}(\mathfrak{A})$.

The second sum on the right side of the expression (VII.82) for $L_n(\lambda)$ contains the vertex $d_\lambda \dot{q}(\lambda) = Q(\lambda)\,\dot{q}(\lambda) + \dot{q}(\lambda)\,Q(\lambda)$, that is of type $(-1, 1)$. (Even though this vertex is more singular than any vertex in the first sum, it does satisfy $\|d_\lambda \dot{q}(\lambda)\|_{-1,1} \leqslant O(1)$.) In this case η_{local} for the set of vertices

$$\tilde{Y} = \{a_0, d_\lambda a_1, ..., d_\lambda a_k, d_\lambda \dot{q}(\lambda), d_\lambda a_{k+1}, ..., d_\lambda a_n\}$$

is the same as η_{local} for the sets $Y^{(k)}$ and $Z^{(k)}$ above,

$$\eta^{\tilde{Y}}_{\text{local}} \geqslant \tfrac{1}{2} \min\{(1-\alpha), (1-\beta)\} > 0.$$

Furthermore, these vertices have a global exponent $\eta^{\tilde{Y}}_{\text{global}} > \frac{1}{2}$ for n sufficiently large. Thus the bound on this second sum proceeds just as the bound on the first sum. We use Proposition VII.6 and Corollary V.4.iv to infer that the n-fold sums of expectations obtained from these regular sets of vertices also define elements of $\mathscr{C}(\mathfrak{A})$. We add the various terms in question to conclude that $L(\lambda) \in \mathscr{C}(\mathfrak{A})$. The proof that $h(\lambda) \in \mathscr{C}(\mathfrak{A})$ proceeds similarly, by showing that the sets of vertices $X^{(k)}$ are regular sets and satisfy appropriate uniform bounds. The next bound on the $X^{(k)}_j$ is more singular, so we do not give details of the bounds on $h(\lambda)$.

(iii) Consider the expectation $\hat{X}^{(k)}_j$ of the set of vertices (VII.86). Here one vertex may be $d^2_\lambda a_j = Q(\lambda)\, d_\lambda a_j + d_\lambda a_j Q(\lambda)$ or one vertex may be $d\dot{q}(\lambda) = Q(\lambda)\,\dot{q}(\lambda) + \dot{q}(\lambda)\,Q(\lambda)$. Suppose that one vertex of the former type occurs. We use bounds for this vertex

$$\|Q(\lambda)\, d_\lambda a_j\|_{(-1-\beta,\alpha)} + \|d_\lambda a_j Q(\lambda)\|_{(-\beta,1+\alpha)} \leqslant O(1)\,\|a_j\|. \quad \text{(VII.88)}$$

The vertex adjacent to this vertex will be either a vertex $a_0, d_\lambda a_k$, or $\dot{q}(\lambda)$. The most singular configurations of three successive vertices are $\{..., d_\lambda a_j Q(\lambda), \dot{q}(\lambda), d_\lambda a_{j+1}, ...\}$ or $\{..., d_\lambda a_j Q(\lambda), d_\lambda a_{j+1}, \dot{q}(\lambda), ...\}$, or "adjoint" configurations $\{..., \dot{q}(\lambda), d_\lambda a_{j+1}, Q(\lambda)\, d_\lambda a_j, ...\}$ or $\{..., d_\lambda a_{j+1}, \dot{q}(\lambda, Q(\lambda)\, d_\lambda a_j, ...\}$.

Since the $\dot{q}(\lambda)$-vertex is in $\mathscr{T}(-1, 0) \cap \mathscr{T}(0, 1)$ we choose to estimate $\dot{q}(\lambda)$ with the index 0 on the side toward $Q(\lambda)$. Thus the local regularity index given by these configurations is at worst

$$\eta_{\text{local}} \geqslant \tfrac{1}{2} \min\{1 - \alpha, 1 - \beta, 1 - \alpha - \beta\} > 0. \tag{VII.89}$$

Thus in this case the set is a regular set of vertices. If on the other hand the differentiated vertex is $dx_j = d\dot{q}(\lambda)$, then the most singular three, successive vertex configurations are $\{..., d_\lambda a_k, \dot{q}(\lambda)\, Q(\lambda), d_\lambda a_l, ...\}$ or an "adjoint" configuration. In this case we have

$$\eta_{\text{local}} \geqslant \tfrac{1}{2} \min\{1 - \alpha, 1 - \beta\} > 0,$$

so the situation is slightly less singular. Both these bounds are uniform for $\lambda \in \Lambda$ in a compact set.

(iv) In order to establish the continuity of $h(\lambda)$ in g, we need to analyze the difference $h(\lambda; g) - h(\lambda; g')$, for g, g' nearby elements of $\mathfrak{G}$. We let P_r denote the orthogonal projection in $\mathscr{H}$ onto the subspace $Q^2 \leqslant r$. Let $\eta = (1 - \alpha - \beta)/2$, so

$$0 < \eta \leqslant \eta_{\text{local}},$$

and η is independent of n. Also let $R = (Q^2 + I)^{-1/2}$. Then

$$U(g) - U(g') = P_r(U(g) - U(g')) + R^{-\eta} R^{\eta} (I - P_r)(U(g) - U(g')).$$

We insert this relation in place of $U(g) - U(g')$ in each term in the difference

$$h_n(\lambda)(a_0, ..., a_n; g) - h_n(\lambda)(a_0, ..., a_n; g').$$

Expand into $2(n+1)$ terms using definition (VII.83). We will show that for g sufficiently close to g', the norm of each of these terms is small, with a coefficient $o(1)$ uniform for λ in a compact set, and with the large-n behavior given by $o(1)\, M^{n+1} (n!)^{-(1/2) - ((1 - \alpha - \beta)/2)}$. This being true, $h(\lambda)$ is continuous in $g \in \mathfrak{G}$, uniformly on compact sets of Λ.

In order to complete the proof, we show why each term is small. First choose r sufficiently large; the following argument shows that the contribution with a factor $(I-P_r)$ is small. The operator $R^{-\eta}$, which also commutes with γ can be moved cyclically through the trace around to the last factor $\exp(-s_{n+1}Q(\lambda)^2)$. We have established in Proposition VII.6, 9 that for λ in a compact subset of Λ, the operator norm $\|(Q(\lambda)^2+I)^{\eta/2}R^{-\eta}\|$ is bounded uniformly in λ. Thus the bound on $\|\exp(-s_{n+1}Q(\lambda)^2)R^{-\eta}\|$ can be estimated by $\|\exp(-s_{n+1}Q(\lambda)^2)(Q(\lambda)^2+I)^{-\eta/2}\|$ times a uniformly bounded constant. But $\eta \leqslant \eta_{\text{local}}$, so the extra factor $R(\lambda)^{-\eta} = (Q(\lambda)^2+I)^{\eta}/2$ can be absorbed into α_{n+1}, with the only effect being to change the combinatorial constant from one vertex in (V.63). We are thus left with the factor

$$\|R^{\eta}(I-P_r)\| \leqslant (r^2+1)^{-\eta/2},$$

which is $o(1)$ uniformly in all constants if r is sufficiently large.

Now fix r and consider the first term $P_r(U(g)-U(g'))$. Note that Q^2 and $U(g)$ commute, so $U(g)-U(g')$ acts on $P_r\mathscr{H}$. While $U(g)$ is only strongly continuous on $\mathscr{H}$, strong continuity and norm continuity agree on the finite dimensional subspace $P_r\mathscr{H}$. Thus

$$\|P_r(U(g)-U(g'))\| \leqslant o(1),$$

where $o(1)$ is bounded uniformly for fixed r, and where $o(1)\to 0$ as $g^{-1}g' \to e$, the identity in $\mathfrak{G}$. Thus both terms in the expansion of $U(g)-U(g')$ give a small coefficient times a uniform bound for λ contained in a compact subset of Λ, and the proof of Proposition VII.12 is complete.

PROPOSITION VII.13. *Assume that $Q(\lambda)$ is a regular family of perturbations and $\mathfrak{A} \in \mathfrak{I}_{\beta,\alpha}$ with $0 \leqslant \alpha, \beta$ and $0 \leqslant \alpha+\beta < 1$. Then*

(i) *The families of cochains $L(\lambda), h(\lambda) \in \mathscr{C}(\mathfrak{A})$ are continuous as maps from $\lambda \in \Lambda \to \mathscr{C}(\mathfrak{A})$.*

(ii) *The family $\tau^{\text{JLO}}(\lambda) \in \mathscr{C}(\mathfrak{A})$ is differentiable in λ, and $d\tau^{\text{JLO}}(\lambda)/d\lambda = L(\lambda)$. Hence τ^{JLO} is continuously differentiable.*

Proof. (i) The continuity of $L(\lambda)$ and of $h(\lambda)$ in λ follows from an analysis of the differences $L(\lambda)-L(\lambda')$ and $h(\lambda)-h(\lambda')$. Taking λ, λ' in a compact subset of Λ, we obtain uniform estimates

$$|||L_n(\lambda)-L_n(\lambda')||| + |||h_n(\lambda)-h_n(\lambda')||| \leqslant o(1)\, m^{n+1}\left(\frac{1}{n!}\right)^{(1/2)+\eta} \tag{VII.90}$$

for $\eta = \frac{1}{2}(1-\alpha-\beta) > 0$. Here $o(1)$ is independent of n and $o(1) \to 0$ as $|\lambda-\lambda'| \to 0$.

To obtain this bound, write out the differences of L_n or of h_n using the definitions (VII.83–VII.82). Each term in these sums has the form $\int \mathrm{Tr}(\gamma U(g)(X(s;\lambda)-X(s;\lambda')))\,ds$, where $X(s,\lambda)$ is a product of n or $n+1$ vertices (of the form a_0, $d_\lambda a_j$, $\dot{q}(\lambda)$, $d_\lambda \dot{q}(\lambda)$, or $[\dot{q}(\lambda), a_j]$) and an equal number of heat kernels. Thus each difference $X(s;\lambda)-X(s;\lambda')$ can be expanded further as a sum of $2n$ or $2(n+1)$ terms, with exactly one difference in each. Here this difference is either the difference of a vertex at two values of λ, or else a difference of heat kernels at two values of λ.

For each term, we repeat the uniform bounds as in the proof of Proposition VII.12. These bounds, however, can be improved through the presence of the difference, which will ultimately give a coefficient $o(1)$ as $|\lambda-\lambda'|\to 0$. If the difference is a difference of heat kernels, the bound (VII.74) proved in Proposition VII.11 can be used. For example, if the difference occurs in the jth heat kernel, the bound (VII.74) yields a factor $o(1)$, by introducing an arbitrarily small power $s_j^{-\varepsilon}$ from this one vertex. This small increase in the singularity from one vertex can be absorbed into the overall constant.

Next let us consider terms for which the difference is a difference of vertices. We need to consider each generic possibility. If the vertex is $d_\lambda a_j$ the difference has the form

$$d_\lambda a_j - d_{\lambda'} a_j = [q(\lambda)-q(\lambda'), q_j] = (\lambda-\lambda')[\delta, a_j],$$

where δ denotes the difference quotient for $q(\lambda)$. The only possible problem arises if δ is adjacent to a vertex containing $\dot{q}(\lambda)$. Each term has at most two vertices with q or $\dot{q}$'s. In this case we use (VII.26) for $0<\varepsilon<1$. For example, in the term

$$\cdots(\lambda-\lambda')\, e^{-s_{j-1}Q(\lambda)^2} a_j \delta e^{-s_j Q(\lambda')^2} \dot{q}(\lambda')\, a_{j+1} e^{-s_{j+1}Q(\lambda')^2}\cdots, \quad \text{(VII.91)}$$

we use the identity

$$R^\varepsilon a_j \delta R^{1-\varepsilon} = (R^\varepsilon a_j R^{-\varepsilon})(R^\varepsilon \delta R^{1-\varepsilon}) \quad \text{(VII.92)}$$

to transfer $R^{-\varepsilon}$ through a_j and away from the heat kernel $\exp(-s_j Q(\lambda')^2)$ which is sandwiched between the two factors q and $\dot{q}$. By Proposition V.5.ii, $\|R^\varepsilon a_j R^{-\varepsilon}\| = \|a_j\| \leqslant \|a_j\|_{\mathfrak{I}_{\beta,\alpha}}$ for $0<\varepsilon\leqslant 1-\alpha$. Also $\|R^\varepsilon \delta R^{1-\varepsilon}\| = \|\delta\|_{(-\varepsilon, 1-\varepsilon)} \leqslant O(1)$ by (VII.26). Thus (VII.91) can be bounded by $O(|\lambda-\lambda'|)$, times the usual bound. The special treatment of the two vertices with factors of q or $\dot{q}$ does not change η_{global} for n large. The $O(n^2)$ terms in L_n and h_n are bounded by m^n and do not affect the power of $1/n!$ in the overall estimate. Thus (VII.90) holds for any $\varepsilon \leqslant (1-\alpha-\beta)/2$, and part (i) is proved.

(ii) To establish differentiability of $\tau^{\mathrm{JLO}}(\lambda)$, we write out the difference quotient and subtract $L(\lambda)$. Then

$$\frac{\tau_n^{\mathrm{JLO}}(\lambda)-\tau_n^{\mathrm{JLO}}(\lambda')}{\lambda-\lambda'}-L_n(\lambda) \tag{VII.93}$$

can be expressed as a sum of differences, as in the proof of (i). Two types of terms occur in the difference quotient: the difference quotient for heat kernel factors and the difference quotient for vertices. These terms are in 1-1 correspondence with the terms in L_n; the difference quotients for heat kernels correspond to the $d_\lambda \dot{q}(\lambda)$ vertices in $L_n(\lambda)$ while the difference quotients for vertices correspond to the $[\dot{q}(\lambda), a_j]$ vertices in L_n. After combining these terms, the estimates of the terms with vertex differences proceed as in the proof of part (i). The terms with heat kernel difference quotients can be treated using the bound of Proposition VII.10.iv. The bounds are similar to bounds proved earlier, so we do not give the details.

VII.5. *Deformations of $\tau^{\mathrm{JLO}}(\lambda)$ Yield a Coboundary*

We establish that $d\tau^{\mathrm{JLO}}(\lambda)/d\lambda = L(\lambda) = \partial h(\lambda$, with $h(\lambda)$ defined in (VII.83) or (VII.87). In other words, we establish the constancy of $\langle \tau^{\mathrm{JLO}}(\lambda), a\rangle$ under a homotopy as explained in Section VII.1. This completes the proof of Theorem VII.1 and of Corollary VII.2.

PROPOSITION VII.14. *If $Q(\lambda)$ is a regular family of perturbations and $\mathfrak{A} \subset \mathfrak{I}_{\beta,\alpha}$, then $\tau^{\mathrm{JLO}}(\lambda) \in \mathscr{C}(\mathfrak{A})$ satisfies*

$$\frac{d}{d\lambda}\tau^{\mathrm{JLO}}(\lambda) = \partial h(\lambda).$$

Proof. In Proposition VII.13.ii we established that $d\tau^{\mathrm{JLO}}(\lambda)/d\lambda$ exists and equals $L(\lambda)$, and furthermore that $L(\lambda)$ is a continuous function from Λ to $\mathscr{C}(\mathfrak{A})$. We also have shown the existence and continuity of $h(\lambda)$. So now we need only show that $L = \partial h$.

In Proposition VII.12.ii, we have shown that both $h_{2n} = 0$ and $L_{2n+1} = 0$, when these cochains are evaluated on $\mathfrak{A}$. Thus we need only verify that

$$L_{2n}(\lambda) = (Bh_{2n+1})(\lambda) + (bh_{2n-1})(\lambda). \tag{VII.94}$$

The proof of this fact is a calculation that parallels the proof of Proposition VI.4. We begin by proving that

$$(Bh_{2n+1})(\lambda)(a_0, ..., a_{2n}; g) = -h_{2n}(da_0, a_1, ..., a_{2n}; g), \tag{VII.95}$$

where the right side is $\widetilde{h}_{2n}^{(0)}$ defined in Proposition VII.12.iii, and shown there to be an element of $\mathscr{C}_n(\mathfrak{A})$. In fact, we establish (VII.95) starting from the definition (II.20) of the operator B that yields on the odd elements of h the identity for B applies to odd components of a cochain,

$$(Bh_{2n+1})(a_0, ..., a_{2n}; g) = \sum_{j=0}^{2n} h_{2n+1}(I, a_{2n-j+1}^{g^{-1}}, ..., a_{2n}^{g^{-1}}, a_0, ..., a_{2n-j}; g).$$

Expand the right side using the definition (VII.83) of h. There are two sorts of terms depending on whether $\dot{q}(\lambda)$ appears to the left or to the right of da_0, namely

$$\begin{aligned}
&(Bh_{2n+1})(a_0, ..., a_{2n}; g)\\
&\quad = \sum_{j=0}^{2n} \Bigg(\sum_{k=0}^{j} (-1)^{k+1} \langle I, da_{2n-j+1}^{g^{-1}}, ..., da_{2n-j+k}^{g^{-1}}, \dot{q}(\lambda),\\
&\quad\quad da_{2n-j+k+1}^{g^{-1}}, ..., da_{2n}^{g^{-1}}, da_0, ..., da_{2n-j}; g\rangle_{2n+2}\\
&\quad\quad + \sum_{k=j+1}^{2n+2} (-1) \langle I, da_{2n-j+1}^{g^{-1}}, ..., da_{2n}^{g^{-1}}, da_0, ..., da_{k-j-1},\\
&\quad\quad \dot{q}(\lambda), ..., da_{2n-j}; g\rangle_{2n+2} \Bigg).
\end{aligned} \tag{VII.96}$$

Use the cyclic permutation symmetry of expectations (V.67) to permute da_0 into the zeroth position. In the first sum this introduces a factor $(-1)^{j+1}$, while in the second sum it introduces the factor $(-1)^j$. Thus

$$\begin{aligned}
&(Bh_{2n+1})(a_0, ..., a_{2n}; g)\\
&\quad = \sum_{j=0}^{2n} \Bigg(\sum_{k=0}^{j} (-1)^{k+j} \langle da_0, da_1, ..., da_{2n-j}, I, ..., da_{2n-j+k},\\
&\quad\quad \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+2}\\
&\quad\quad + \sum_{k=j+1}^{2n+1} (-1)^{k+j+1} \langle da_0, da_1, ..., da_{k-j-1}, \dot{q}(\lambda), ..., da_{2n-j},\\
&\quad\quad I, ..., da_{2n}; g\rangle_{2n+2} \Bigg).
\end{aligned} \tag{VII.97}$$

After redefining the summation variables, we obtain

$$\begin{aligned}
&(Bh_{2n+1})(a_0, ..., a_{2n}; g)\\
&\quad = \sum_{j=0}^{2n} \Bigg(\sum_{k=2n-j}^{2n} (-1)^k \langle da_0, da_1, ..., da_{2n-j},\\
&\qquad I, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+2}\\
&\qquad + \sum_{k=0}^{2n-j} (-1)^k \langle da_0, da_1, ..., da_k, \dot{q}(\lambda), da_k, ..., da_{2n-j},\\
&\qquad I, ..., da_{2n}; g\rangle_{2n+2} \Bigg). \qquad \text{(VII.98)}
\end{aligned}$$

Now combine the terms in (VII.98) that have a particular value of k. These terms all have the same sign $(-1)^k$. For fixed k there are $2n+2$ terms; the first sum contributes when $2n-k \leqslant j \leqslant 2n$ or $k+1$ terms, while the second sum contributes when $0 \leqslant j \leqslant 2n-k$ or $2n-k+1$ terms. In each of these terms the operators da_j occur in order of increasing j, and the operator $\dot{q}(\lambda)$ always follows da_k and precedes da_{k+1}. The difference between the terms is that the operator I occurs in positives 1, 2, ..., $2n+2$. (Note that the operator $\dot{q}(\lambda)$ immediately follows da_k, except in the second sum when $j=2n-k$, in which case the operator I intervenes between da_k and $\dot{q}(\lambda)$.) Consider these $2n+2$ terms as multilinear expectations of regular sets of $2n+3$-vertices chosen from the da_j's I $d\dot{q}(\lambda)$. Therefore one may use (V.68) to convert each sum of expectations $\langle \cdots \rangle_{2n+2}$ into one expectation (with the vertex I omitted) of the form $\langle \cdots \rangle_{2n+1}$. We obtain

$$\begin{aligned}
&(Bh_{2n+1})(a_0, ..., a_{2n}; g)\\
&\quad = \sum_{k=0}^{2n+1} (-1)^k \langle da_0, da_1, ..., da_k, \dot{q}(\lambda), da_{k+1}, ..., da_{2n}; g\rangle_{2n+1}\\
&\quad = -h_{2n}(da_0, a_1, ..., a_{2n}; g),
\end{aligned}$$

as claimed in (VII.95).

Next we evaluate bh_{2n-1}, along the lines of the proof (VI.8). Here the form of the expression we obtain is a bit more complicated than in Section VI, due to the presence of the vertex $\dot{q}(\lambda)$. Begin from the definition (II.15) of b applied to h. One obtains

$$\begin{aligned}
(bh_{2n-1})(a_0, ..., a_{2n}; g) &= \sum_{j=0}^{2n-1} (-1)^j h_{2n-1}(a_0, ..., a_j a_{j+1}, ..., a_{2n}; g)\\
&\quad + h_{2n-1}(a_{2n}^{g^{-1}} a_0, a_1, ..., a_{2n-1}; g). \qquad \text{(VII.99)}
\end{aligned}$$

Expand (VII.99) using the definition (VII.83) of h, giving a sum of $2n(2n+1)$ terms. Each of the terms coming from (VII.99) with $1 \leqslant j \leqslant 2n-1$ contains a vertex of the form $d(a_j a_{j+1})$. Expand this vertex further using the product rule for derivatives, $d(a_j a_{j+1}) = (da_j)\, a_{j+1} + a_j (da_{j+1})$, and express bh_{2n-1} as a sum of $2 * 2n(2n-1) + 4n = 8n^2$ expectations,

$$
\begin{aligned}
&(bh_{2n-1})(a_0, ..., a_{2n}; g) \\
&\quad = -\langle a_0 a_1, \dot{q}(\lambda), da_2, ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=2}^{2n} (-1)^k \langle a_0 a_1, da_2, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=2}^{2n} \sum_{j=1}^{k-1} (-1)^{k+1} \langle a_0, da_1, ..., a_j\, da_{j+1}, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=2}^{2n} \sum_{j=1}^{k-1} (-1)^{k+j} \langle a_0, da_1, ..., (da_j)\, a_{j+1}, ..., da_k, \\
&\qquad\quad \dot{q}(\lambda), ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=0}^{2n-2} \sum_{j=k+1}^{2n-1} (-1)^{k+j+1} \langle a_0, da_1, ..., da_k, \dot{q}(\lambda), ..., \\
&\qquad\quad a_j\, da_{j+1}, ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=0}^{2n-2} \sum_{j=k+1}^{2n-1} (-1)^{k+j+1} \langle a_0, da_1, ..., da_k, \dot{q}(\lambda), ..., \\
&\qquad\quad (da_j)\, a_{j+1}, ..., da_{2n}; g \rangle_{2n} \\
&\qquad + \sum_{k=1}^{2n-1} (-1)^{k+1} \langle a_{2n}^{g^{-1}} a_0, da_1, ..., da_k, \dot{q}(\lambda), ..., da_{2n-1}; g \rangle_{2n} \\
&\qquad - \langle a_{2n}^{g^{-1}} a_0, \dot{q}(\lambda), da_1, ..., da_{2n-1}; g \rangle_{2n}.
\end{aligned} \tag{VII.100}
$$

Now combine the terms in (VII.100) into a "telescoping sum" of pairs of expectations. Each pair of expectations differs only by the location of one operator a_j: at the end of position ℓ in one term, and at the start of position $(\ell+1) \bmod (2n+1)$ in the other. This requires the additional terms

$$
\sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, ..., a_j \dot{q}(\lambda) - \dot{q}(\lambda)\, a_j, ..., d_\lambda a_{2n}; g \rangle_{2n}. \tag{VII.101}
$$

So add and subtract the $2n$ commutators (VII.101) to (VII.100), yielding the desired $4n^2 + 2n$ pairs,

$$
\begin{aligned}
&(bh_{2n-1})(a_0, \ldots, a_{2n}; g) \\
&\quad = \sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, \ldots, [\dot{q}(\lambda), a_j], \ldots, d_\lambda a_{2n}; g \rangle_{2n} \\
&\qquad - (\langle a_0 a_1, \dot{q}(\lambda), da_2, \ldots, da_{2n}; g \rangle_{2n} - \langle a_0, a_1 \dot{q}(\lambda), \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad + \sum_{k=2}^{2n} (-1)^k (\langle a_0 a_1, da_2, \ldots, da_k, \dot{q}(\lambda), \ldots, da_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_0, a_1\, da_2, \ldots, da_k, \dot{q}(\lambda), \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad + \sum_{k=3}^{2n} \sum_{j=2}^{k-1} (-1)^{k+j+1} (\langle a_0, da_1, \ldots, (da_{j-1})\, a_j, da_{j+1}, \ldots, da_k, \\
&\qquad \dot{q}(\lambda), \ldots, da_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_0, da_1, \ldots, da_{j-1}, a_j\, da_{j+1}, \ldots, da_k, \dot{q}(\lambda), \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad - \sum_{k=2}^{2n} (\langle a_0, da_1, \ldots, (da_{k-1})\, a_k, \dot{q}(\lambda), da_{k+1}, \ldots, da_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_0, a_1, \ldots, da_{k-1}, a_k \dot{q}(\lambda), da_{k+1}, \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad - \sum_{k=0}^{2n-2} (\langle a_0, da_1, \ldots, da_k, \dot{q}(\lambda)\, a_{k+1}, da_{k+2}, \ldots, da_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_0, da_1, \ldots, da_k, \dot{q}(\lambda), a_{k+1}\, da_{k+2}, \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad + \sum_{k=0}^{2n-2} \sum_{j=k+2}^{2n-1} (-1)^{k+j} (\langle a_0, da_1, \ldots, da_k, \dot{q}(\lambda), \ldots, (a_{j-1})\, a_j, \\
&\qquad da_{j+2}, \ldots, da_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_0, da_1, \ldots, da_k, \dot{q}(\lambda), \ldots, da_{j-1}, a_j\, da_{j+1}, \ldots, da_{2n}; g \rangle_{2n}) \\
&\qquad + (\langle a_0, \dot{q}(\lambda), da_1, \ldots, (da_{2n-1})\, a_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_{2n}^{t^{-1}} a_0, da_1, \ldots, da_{2n-1}; g \rangle_{2n}) \\
&\qquad + \sum_{k=1}^{2n-1} (-1)^k (\langle a_0, da_1, \ldots, da_k, \dot{q}(\lambda), \ldots, (da_{2n-1})\, a_{2n}; g \rangle_{2n} \\
&\qquad - \langle a_{2n}^{g^{-1}} a_0, da_1, \ldots, da_k, \dot{q}(\lambda), \ldots, da_{2n-1}; g \rangle_{2n}).
\end{aligned}
\tag{VII.102}
$$

The commutator identities of Corollary V.8.v allow us to rewrite each of the pairs of expectations $\langle \cdots \rangle_{2n}$ as one expectation $\langle \cdots \rangle_{2n+1}$. In Proposition VII.2.iii we explicitly verify that each such term is well-defined. Thus (VII.102) takes the form

$$
\begin{aligned}
&(bh_{2n-1})(a_0, ..., a_{2n}; g) \\
&\quad = \sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, ..., [\dot{q}(\lambda), a_j], ..., d_\lambda a_{2n}; g\rangle_{2n} \\
&\qquad - \langle a_0, d^2 a_1, \dot{q}(\lambda), a_2, ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad + \sum_{k=2}^{2n} (-1)^k \langle a_0, d^2 a_1, da_2, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad + \sum_{k=3}^{2n} \sum_{j=2}^{k-1} (-1)^{k+j+1} \langle a_0, da_1, ..., da_{j-1}, d^2 a_j, da_{j+1}, ..., da_k, \\
&\qquad \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad - \sum_{k=2}^{2n} \langle a_0, da_1, ..., da_{k-1}, d^2 a_k, \dot{q}(\lambda), da_{k+1}, ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad - \sum_{k=0}^{2n-2} \langle a_0, da_1, ..., da_k, \dot{q}(\lambda), d^2 a_{k+1}, da_{k+2}, ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad + \sum_{k=0}^{2n-2} \sum_{j=k+2}^{2n-1} (-1)^{k+j} \langle a_0, da_1, ..., da_k, \dot{q}(\lambda), ..., \\
&\qquad d^2 a_j, ..., da_{2n}; g\rangle_{2n+1} \\
&\qquad + \langle a_0, \dot{q}(\lambda), da_1, ..., da_{2n-1}, d^2 a_{2n}; g\rangle_{2n+1} \\
&\qquad + \sum_{k=1}^{2n-1} (-1)^k \langle a_0, da_1, ..., da_k, \dot{q}(\lambda), ..., da_{2n-1}, d^2 a_{2n}; g\rangle_{2n+1}.
\end{aligned}
\tag{VII.103}
$$

In order to simplify the expression (VII.103) further, let us give a rule to determine the sign of each term with a second derivative $d^2 a_j$. In each such term, define $j^{\#} = j^{\#}(j, k)$ as the number of factors da_ℓ or $\dot{q}(\lambda)$ that occur to the left of $d^2 a_j$. Note that for the eight lines of (VII.103) that include a $d^2 a_j$, the values of $j^{\#}$ are respectively 0, 0, $j-1$, $k-1$, $k+1$, j, $2n$, $2n$. Furthermore the sign of each corresponding term is $(-1)^{k+j^{\#}}$. Thus (VII.103) can also be written

$$
\begin{aligned}
&(bh_{2n-1})(a_0, ..., a_{2n}; g) \\
&\quad = \sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, ..., [\dot{q}(\lambda), a_j], ..., d_\lambda a_{2n}; g\rangle_{2n} \\
&\qquad + \sum_{j=1}^{2n} \sum_{k=0}^{2n} (-1)^{k+j^{\#}} \langle a_0, da_1, ..., d^2 a_j, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1}.
\end{aligned}
\tag{VII.104}
$$

In the case $j=k$, it is $d^2 a_k$ that occurs. Likewise, write (VII.95) in the form

$$(Bh_{2n+1})(a_0, ..., a_{2n}; g) = \sum_{k=0}^{2n} (-1)^k \langle da_0, da_1, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1}. \quad \text{(VII.105)}$$

Add (VII.105) and (VII.104) to yield

$$\begin{aligned}(\partial h(\lambda))_{2n} (a_0, a_1, ..., a_{2n}; g) &= (bh_{2n-1})(a_0, ..., a_{2n}; g) + (Bh_{2n+1})(a_0, ..., a_{2n}; g) \\ &= \sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, ..., [\dot{q}(\lambda), q_j], ..., d_\lambda a_{2n}; g\rangle_{2n} \\ &\quad + \sum_{k=0}^{2n} (-1)^k \langle da_0, da_1, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1} \\ &\quad + \sum_{j=1}^{2n} \sum_{k=0}^{2n} (-1)^{k+j^\#} \langle a_0, da_1, ..., d^2 a_j, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1}.\end{aligned} \quad \text{(VII.106)}$$

Now we return to the sets of vertices $X^{(k)}$ and $X_j^{(k)}$ introduced in (VII.84–VII.86). Note that the Radon transform $\hat{X}_j^{(k)}(2n)$ of $X_j^{(k)}(2n)$ has the expectation $\langle \hat{X}_j^{(k)}(2n); g\rangle$ and that

$$\langle d\hat{X}^{(k)}(2n); g\rangle = 0. \quad \text{(VII.107)}$$

This is also the case for a linear combination, so

$$\begin{aligned}0 &= \sum_{k=0}^{2n} (-1)^k \langle d\hat{X}^{(k)}(2n); g\rangle \\ &= \sum_{k=0}^{2n} \langle a_0, d_\lambda a_1, ..., d_\lambda a_k, d_\lambda \dot{q}(\lambda), ..., d_\lambda a_{2n}; g\rangle_{2n+1} \\ &\quad + \sum_{k=0}^{2n} (-1)^k \langle da_0, da_1, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1} \\ &\quad + \sum_{j=1}^{2n} \sum_{k=0}^{2n} (-1)^{k+j^\#} \langle a_0, da_1, ..., d^2 a_j, ..., da_k, \dot{q}(\lambda), ..., da_{2n}; g\rangle_{2n+1}.\end{aligned} \quad \text{(VII.108)}$$

The last two sums in (VII.108) exactly equal the similar sums in (VII.106). Since

$$\langle a_0, d_\lambda a_1, ..., d_\lambda a_k, d_\lambda \dot{q}(\lambda), ..., d_\lambda a_{2n}; g\rangle_{2n+1} = \tau^{\mathrm{JLO}}_{2n+1}(a_0, a_1, ..., a_k, \dot{q}(\lambda), ..., a_{2n}; g), \quad \text{(VII.109)}$$

one can rewrite (VII.106) in the form

$$(\partial h)_{2n}(a_0, ..., a_{2n}; g) = \sum_{j=1}^{2n} \langle a_0, d_\lambda a_1, ..., [\dot{q}(\lambda), a_j], ..., d_\lambda a_{2n}; g\rangle_{2n} - \sum_{j=0}^{2n} \tau^{\mathrm{JLO}}_{2n+1}(a_0, a_1, ..., a_k, \dot{q}(\lambda), ..., a_{2n}; g). \quad \text{(VII.110)}$$

The right side of (VII.110) is $L_{2n}(\lambda)$, defined in (VII.82). So this completes the proof of Proposition VII.14.

VII.6. *Equivalence of Parallel, Radon Hyperplanes*

The basic ingredient in the definition of the cocycle τ^{JLO} is $\hat{X}(\beta)$. This operator is the Radon transform of the heat kernel regularization $X(s)$ of $(n+1)$ vertices, evaluated on the hyperplane $\beta = s_0 + \cdots + s_n$. Until now, we restricted our attention to the hyperplane $\beta = 1$. In this section, we consider a hyperplane parallel to this one and defined by $\beta > 0$. We define the corresponding $\tau^{\mathrm{JLO},\beta}$ and show that both $\tau^{\mathrm{JLO},\beta}$ and $\tau^{\mathrm{JLO}} = \tau^{\mathrm{JLO},1}$ belong to the same equivalence class. This justifies our restriction earlier to the case $\beta = 1$.

Let $\{\mathscr{H}, Q, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ denote a Θ-summable, fractionally-differentiable structure. Let $X = \{a_0, da_1, ..., da_n\}$ be an $(n+1)$-vertex set, where $a_j \in \mathfrak{A}$. Define the expectation $\tau^{\mathrm{JLO},\beta}$ with components

$$\tau^{\mathrm{JLO}}_n(a_0, ..., a_n; g) = \beta^{-n}\,\mathrm{Tr}(\gamma U(g)\,\hat{X}(\beta)), \quad \text{(VII.111)}$$

where $\hat{X}(\beta)$ denotes the Radon transform of the heat kernel regularization of X evaluated on the hyperplane $\sum_{j=0}^n s_j = \beta$.

PROPOSITION VII.15. *The expectation $\tau^{\mathrm{JLO},\beta} \in \mathscr{C}(\mathfrak{A})$. Furthermore $\tau^{\mathrm{JLO},\beta} \in [\tau^{\mathrm{JLO},1}]$. In particular, $\tau^{\mathrm{JLO},\beta}$ is a cocycle for each $\beta > 0$.*

Proof. The statement $\tau^{\mathrm{JLO},\beta} \in \mathscr{C}(\mathfrak{A})$ follows immediately from the estimates already proved for $X^{\mathrm{JLO}}(s)$, at least in the case $0 < \beta \leqslant 1$. For $\beta > 1$ we require minor modification of the constants in certain bounds; we leave this to the reader.

Note that the scaling properties of the Radon transform ensure that $\tau_n^{\mathrm{JLO},\beta}$ defined for Q is identical with $\tau_n^{\mathrm{JLO},1}$ defined for $\beta^{1/2}Q$. Hence $\tau_n^{\mathrm{JLO},\beta}$ is a cocycle. In order to demonstrate that $\tau^{\mathrm{JLO},\beta}$ belongs to the same equivalence class as τ^{JLO}, we need only study the family $\tau^{\mathrm{JLO}}(\beta^{1/2})$ given by $\{\mathscr{H}, \beta^{1/2}Q, \gamma, U(g), \mathfrak{A}\}$. As a function of $\beta^{1/2}$, we have a linear perturbation of Q. Define $Q(\beta^{1/2}) = Q + q(\beta^{1/2})$, where $q(\beta^{1/2}) = (\beta^{1/2} - 1)\, Q$. Since $Q \in \mathscr{T}(0, 1)$, and it has norm 1, we infer that $Q(\beta^{1/2})$ is a regular Q-family. Thus Theorem VII.1 shows that $\tau^{\mathrm{JLO},\beta} = \tau^{\mathrm{JLO}} + \partial H$, as long as for $|\beta^{1/2} - 1| < 1$. With redefinition of the starting point of the homotopy from $\beta = 1$ to $\beta = 2$, etc., we show that $\tau^{\mathrm{JLO},\beta}$ is cohomologous for all $\beta > 0$. This completes the proof.

VIII. END POINTS

We often encounter a family $\{\mathscr{H}, Q(\lambda), \gamma, U(g), \mathfrak{A}\}$ which is a regular family for λ in the interior of a set Λ, but for which we lack some relevant information as λ tends to the boundary. In fact, this often arises in the case that Λ is an interval and λ tends to one endpoint of Λ. We consider here a case of such a phenomenon. For simplicity, let us assume $\lambda \in \Lambda = (0, 1]$, with $\lambda = 0$ the singular endpoint.

VIII.1. *End Point Regularization*

As an example, we replace the energy operator $Q(\lambda)^2$ by the regularized energy operator $H(\varepsilon, \lambda)$,

$$H(\varepsilon, \lambda) = Q(\lambda)^2 + \varepsilon^2 Z^* Z. \tag{VIII.1}$$

Here ε is a real, non-zero parameter and $Z^*Z \geqslant 0$ is an operator chosen so that it regularizes $Q(\lambda)^2$. This means that if $\beta > 0$ and $\varepsilon > 0$ are fixed, then

$$\mathrm{Tr}(e^{-\beta H(\varepsilon, \lambda)}) \tag{VIII.2}$$

is bounded uniformly in λ as $\lambda \to 0$.

Suppose that Z^*Z commutes with the representation of $\mathfrak{G}$, namely for all $g \in \mathfrak{G}$,

$$U(g)\, Z^*Z = Z^*Z U(g),$$

and also suppose that $\gamma Z^*Z = Z^*Z\gamma$. Then in defining the heat kernel regularizations $\hat{X}$ of sets of vertices, or in defining expectations, we can replace $\exp(-s_j Q(\lambda)^2)$ with $\exp(-s_j H(\varepsilon, \lambda))$. We obtain

$$\langle \hat{X}; g \rangle_n (\varepsilon, \lambda) = \langle x_0, x_1, ..., x_n; g \rangle_n (\varepsilon, \lambda). \tag{VIII.3}$$

Similarly, we arrive at a regularized JLO cochain $\tau^{\mathrm{JLO}}(\varepsilon, \lambda)$, by using regularized expectations in place of expectations (IV.16) or (V.62). In particular

$$\tau_n^{\mathrm{JLO}}(a_0, ..., a_n; g)(\varepsilon, \lambda) = \mathrm{Tr}(\gamma U(g)\, \hat{X}^{\mathrm{JLO}}(\varepsilon, \lambda)), \tag{VIII.4}$$

where $X^{\mathrm{JLO}} = \{a_0, d_\lambda a_1, ..., d_\lambda a_n\}$. In general, for $X = \{x_0, ..., x_n\}$, one can define

$$\hat{X}(\varepsilon, \lambda) = \int x_0 e^{-s_0 H(\varepsilon, \lambda)} x_1 e^{-s_1 H(\varepsilon, \lambda)} \cdots x_n e^{-s_n H(\varepsilon, \lambda)} \times \delta(s_0 + \cdots + s_{n-1})\, ds_0 \cdots ds_n. \tag{VIII.5}$$

Given $a \in \mathfrak{A}^{\mathfrak{G}}$ and $a^2 = I$, the regularized cochain $\tau^{\mathrm{JLO}}(\varepsilon, \lambda) = \{\tau_n^{\mathrm{JLO}}(\varepsilon, \lambda)\}$ yields the regularized pairing defined as

$$\mathfrak{Z}^{H(\varepsilon, \lambda)}(a; g) = \frac{1}{\sqrt{\pi}} \int e^{-t^2}\, \mathrm{Tr}(\gamma U(g)\, a e^{-H(\varepsilon, \lambda) + it\, d_\lambda a})\, dt. \tag{VIII.6}$$

This pairing $\mathfrak{Z}^{H(\varepsilon, \lambda)}(a; g)$ converges as $\varepsilon \to 0$ to $\mathfrak{Z}^{Q(\lambda)}(a; g)$ of (I.28). However, $\tau^{\mathrm{JLO}}(\varepsilon, \lambda)$ is not a cocycle. Nor is the pairing (VII.6) an invariant function of λ or of ε. For $\varepsilon \neq 0$, the pairing $\mathfrak{Z}^{H(\varepsilon, \lambda)}(a; g)$ depends on both ε and λ.

VIII.2. *Exchange of Limits*

In certain examples, we have studied the dependence of $\mathfrak{Z}^{H(\varepsilon, \lambda)}$ on ε and λ in detail, at least in a neighborhood of $\varepsilon = \lambda = 0$, see [15]. In these examples we have shown that $\mathfrak{Z}^{Q(\lambda)}$ can be recovered from $\mathfrak{Z}^{H(\varepsilon, 0)}$. Also we choose the regularizing factor Z^*Z to be sufficiently simple so that we can evaluate $\mathfrak{Z}^{H(\varepsilon, 0)}$ in closed form. On the other hand, we are interested in knowing $\mathfrak{Z}^{Q(\lambda)} = \mathfrak{Z}^{H(0, \lambda)}$ for $\lambda > 0$, where it is constant. The important fact is that the function $\mathfrak{Z}^{H(\varepsilon, \lambda)}$ is *not* jointly continuous in (ε, λ) in the unit square $0 \leqslant \varepsilon \leqslant 1$, $0 \leqslant \lambda \leqslant 1$ at the point $(0, 0)$.

However, another fact saves the day; it is our ability to prove that while $\mathfrak{Z}^{H(\varepsilon, \lambda)}(a, g)$ is not jointly continuous in (ε, λ) in the unit square and for every value of $g \in \mathfrak{G}$, this function *is* jointly continuous in (ε, λ) for *almost every* value of $g \in \mathfrak{G}$. We establish this continuity in the examples, with the aid of a new expansion that we name the *holonomy expansion.* As a consequence of the expansion, we obtain bounds on

$$\frac{d}{d\varepsilon} \mathfrak{Z}^{H(\varepsilon, \lambda)} \qquad \text{and} \qquad \frac{d}{d\lambda} \mathfrak{Z}^{H(\varepsilon, \lambda)} \tag{VIII.7}$$

of the form

$$\left|\frac{d}{d\varepsilon}\,\mathfrak{Z}^{H(\varepsilon,\lambda)}\right| \leqslant M\varepsilon, \qquad \left|\frac{d}{d\lambda}\,\mathfrak{Z}^{H(\varepsilon,\lambda)}\right| \leqslant M\varepsilon^2, \tag{VIII.8}$$

for $0 \leqslant \varepsilon, \lambda, 0 < \varepsilon + \lambda$, and sufficiently close to $(\varepsilon, \lambda) = (0, 0)$. Thus we obtain,

$$|\mathfrak{Z}^{H(\varepsilon,0)} - \mathfrak{Z}^{H(0,\lambda)}| \leqslant O(\varepsilon^2), \tag{VIII.9}$$

as $\varepsilon \to 0$. We combine this information with two other facts: (i) in the examples, the explicit form of $\mathfrak{Z}^{H(\varepsilon,0)}$ has a pointwise limit as $\varepsilon \to 0$ for almost all $g \in \mathfrak{G}$. (ii) In Corollary VII.2, we established the *a priori* continuity of $\mathfrak{Z}^{H(0,\lambda)}$ as a function of $g \in \mathfrak{G}$. These two pieces of information ensure that for almost all $g \in \mathfrak{G}$,

$$\begin{aligned} \mathfrak{Z}^{Q(\lambda)}(a, g) &= \lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \mathfrak{Z}^{H(\varepsilon,\lambda)}(a, g) \\ &= \lim_{\varepsilon \to 0} \lim_{\lambda \to 0} \mathfrak{Z}^{H(\varepsilon,\lambda)} \\ &= \lim_{\varepsilon \to 0} \mathfrak{Z}^{Q(0)^2 + \varepsilon^2 Z^* Z}(a, g). \end{aligned} \tag{VIII.10}$$

The right side of (VIII.10) can be evaluated, and extends by continuity to all $g \in \mathfrak{G}$.

IX. SPLIT STRUCTURES

We define *splitting* of Q as a decomposition

$$Q = \frac{1}{\sqrt{2}}(Q_1 + Q_2), \tag{IX.1}$$

such that also

$$Q^2 = \tfrac{1}{2}(Q_1^2 + Q_2^2).$$

A splitting is associated with corresponding derivatives on $\mathfrak{A}$ given by

$$da = \frac{1}{\sqrt{2}}(d_1 a + d_2 a), \qquad d_j a = [Q_j, a]. \tag{IX.2}$$

In Section IX.1 we specify this assumption in more detail. Clearly a splitting into a sum of many parts is possible, but we concentrate here on a splitting in two.

As in earlier sections, we assume

$$\operatorname{Tr}(e^{-\beta Q^2}) < \infty, \qquad \text{for} \quad \beta > 0. \tag{IX.3}$$

However, we do not assume that $\operatorname{Tr}(e^{-\beta Q_j^2})$ exists for the individual Q_1 or Q_2. In addition, while we assume that the group $U(g)$ of unitary symmetries commutes with Q_1, we do not assume that Q or Q_2 are necessarily invariant. For these reasons, the resulting framework will be different from the equivariant framework studied earlier.

Within this revised setting we generalize the cochain τ^{JLO} to a cochain $\tau^{\{Q_j\}}$ defined on a suitable algebra $\mathfrak{A}$. Letting $\mathfrak{A}^{\mathfrak{G}}$ denote the pointwise $\mathfrak{G}$-invariant part of $\mathfrak{A}$, we obtain for $a \in \mathfrak{A}^{\mathfrak{G}}$ and $a^2 = I$ the following expression for a pairing:

$$\mathfrak{Z}^{\{Q_j\}}(a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-q^2 + it\, d_1 a})\, dt. \tag{IX.4}$$

While this formula bears a close resemblance to (VI.4), the resulting pairing $\mathfrak{Z}^{\{Q_j(\lambda)\}}$ in general is not an invariant. If $Q_j(\lambda)$ depends on a parameter λ, then $\mathfrak{Z}^{\{Q_j\}}(a; g)$ remains a function of λ. However, we describe a special family of variations $Q_j(\lambda)$ and algebra $\mathfrak{A}$, such that (IX.4) *is* an invariant. Within this class of variations, we find that, as previously, there is a cochain $h(\lambda)$ such that

$$\frac{d}{d\lambda} \tau^{\{Q_j(\lambda)\}} = \partial h^{\{Q_j(\lambda)\}}. \tag{IX.5}$$

IX.1. *A Q_1-Invariant Splitting*

We split the self-adjoint operator Q into a sum of self-adjoint operators Q_1 and Q_2, as in (IX.1). Each operator Q_1, Q_2, and Q is odd under γ,

$$Q_j \gamma + \gamma Q_j = 0, \qquad j = 1, 2. \tag{IX.6}$$

Furthermore, the decomposition (IX.3) of Q^2 has the interpretation that Q_1 and Q_2 generate *independent* translations. Algebraically,

$$Q_1 Q_2 + Q_2 Q_1 = 0. \tag{IX.7}$$

Recall that the Q is assumed to be essentially self-adjoint on the domain $\mathscr{D}$; such a domain is called a core for a symmetric operator. The core is invariant if $Q\mathscr{D} \subset \mathscr{D}$. If $\mathscr{D}$ is a common, invariant core for Q_1 and Q_2, the products are defined on $\mathscr{D}$ and on this domain

$$[Q_1, Q_2^2] = 0 = [Q_2, Q_1^2]. \tag{IX.8}$$

DEFINITION IX.1. The self-adjoint operator Q splits into the sum of two independent parts, if there is a common, invariant core for Q_1, Q_2, and Q, such that the bounded functions of Q_1 and Q_2 commute with the bounded functions of Q_1^2 and of Q_2^2.

In the definition that Q splits into a sum of two independent parts, we assume that the spectral projections of Q_1 commute with those of Q_2^2. Hence the unbounded, self-adjoint operators Q_1 and Q_2^2 commute.

Complementing H defined in (IX.3), we define the self-adjoint operator[8]

$$P = \tfrac{1}{2}(Q_1^2 - Q_2^2). \tag{IX.9}$$

Note that P and H commute. Furthermore as a consequence of (IX.3, 9), we infer

$$\pm P \leqslant H, \tag{IX.10}$$

and the joint spectrum of H and P lies in a cone.

Let us define the Sobolev spaces $\mathscr{H}_\alpha = \mathscr{D}((H+I)^{\alpha/2})$, which are Hilbert spaces with inner product defining a norm $\|f\|_{\mathscr{H}_\alpha} = \|(H+K)^{\alpha/2} f\|$. The space $\mathscr{T}(\beta, \alpha)$ of bounded linear transformations from $\mathscr{H}_\alpha$ to $\mathscr{H}_\beta$ is a Banach space with norm $\|T\|_{(\beta, \alpha)} = \|(H+I)^{\beta/2}\, T(H/I)^{-\alpha/2}\|$. Thus (IX.9) can be interpreted as saying $P \in \mathscr{T}(-1, 1)$ with

$$\|P\|_{(-1, 1)} \leqslant 1. \tag{IX.11}$$

In the equivariant case, we also are interested in the unitary group of symmetries $u(g)$ acting on $\mathscr{H}$. We assume as in earlier sections that

$$\gamma U(g) = U(g)\gamma \qquad \text{and} \qquad Q^2 U(g) = U(g)\, Q^2 \tag{IX.12}$$

for all $g \in \mathfrak{G}$.

[8] The notation H and P is suggestive of energy and momentum. In fact, this is no accident, as such examples arise as examples of supersymmetry in two-dimensional space-time. In that case, Q_1 and Q_2 are generators of symmetries arising from two space-time directions, and $Q_1^2 = H + P$ and $Q_2^2 = H - P$, with H and P being the energy and momentum operators. In addition Q_1 and Q_2 are assumed independent. The condition (IX.9–10), supplemented by $0 \leqslant H = Q^2$, can be interpreted as a restriction of special relativity for the energy-momentum to lie in (or on) the positive cone.

Definition IX.2. A splitting (IX.1) is Q_1-invariant, if (IX.9) holds and also for all $g \in \mathfrak{G}$,

$$Q_1 U(g) = U(g) Q_1. \tag{IX.13}$$

Note that a Q_1-invariant splitting has the property

$$Q_2^2 U(g) = U(g) Q_2^2. \tag{IX.14}$$

Let us define the two-parameter, abelian representation

$$V(t, x) = e^{-itH + ixP}. \tag{IX.15}$$

The relation (IX.6) ensures that for all $(t, x) \in \mathbb{R}^2$,

$$\gamma V(t, x) = V(t, x) \gamma. \tag{IX.16}$$

Definition IX.3. The representation $V(t, x)$ is $U(g)$-*invariant* if $V(t, x)\, U(g) = U(g)\, V(t, x)$ for all $g \in \mathfrak{G}$ and all $(x, t) \in \mathbb{R}^2$.

IX.2. *Observables*

We define a new algebra $\mathfrak{A}$ of observables, suitable for a Q_1-invariant splitting of Q. First we define an interpolation space $\mathscr{T}^{(1)}_{\beta,\alpha}$ based on the d_1 derivative. In particular, let

$$\mathscr{T}^{(1)}_{\beta,\alpha} = \{b\colon b \in \mathscr{B}(\mathscr{H}), \text{ and } R_1^{\beta}(d_1 b)\, R_1^{\alpha} \in \mathscr{B}(\mathscr{H})\}, \tag{IX.17}$$

where $R_1 = (Q_1^2 + I)^{-1/2}$. We give $\mathscr{T}^{(1)}_{\beta,\alpha}$ the norm

$$\|b\|_{\mathscr{T}^{(1)}_{\beta,\alpha}} = \|b\| + c_{\alpha+\beta}\, \|R_1^{\beta}(d_1 b)\, R_1^{\alpha}\|. \tag{IX.18}$$

Here $c_{\alpha+\beta}$ is defined in (V.77). Clearly $\mathscr{T}^{(1)}_{\beta,\alpha}$ is invariant under the action of $\mathfrak{G}$ defined by conjugation with $U(g)$, as a consequence of (IX.11–12). Define the (spatial) translate $b(x)$ of an element $b \in \mathscr{B}(\mathscr{H})$ by

$$b(x) = e^{ixP} b e^{-ixP} = V(0, x)\, b V(0, x)^*. \tag{IX.19}$$

Definition IX.4. The zero-momentum subalgebra $\mathscr{B}(\mathscr{H})_0$ of $\mathscr{B}(\mathscr{H})$ consists of all elements $b \in \mathscr{B}(\mathscr{H})$ such that $b(x) = b$ for all $x \in \mathbb{R}$.

We remark that $b \in \mathscr{B}(\mathscr{H})$ is an element of $\mathscr{B}(\mathscr{H})_0$ if and only if

$$Pb = bP. \tag{IX.20}$$

We assume that $\mathfrak{A}$ is a Banach-subalgebra of the interpolation space

$$\mathscr{B}(\mathscr{H})_0 \cap \mathscr{T}^{(1)}_{\beta, \alpha}, \tag{IX.21}$$

which we call the zero-momentum subalgebra of $\mathscr{T}^{(1)}_{\beta, \alpha}$. Here α, β satisfy $0 \leqslant \alpha, \beta$, with $\alpha + \beta < 1$. Assume the pointwise, γ-invariance of $\mathfrak{A}$, namely $a = a^{\gamma}$ for $a \in \mathfrak{A}$. Furthermore assume

$$\|a\|_{\mathscr{T}^{(1)}_{\beta, \alpha}} \leqslant |||a|||. \tag{IX.22}$$

Also let $\mathfrak{A}^{\mathfrak{G}} \subset \mathfrak{A}$ denote the pointwise $\mathfrak{G}$-invariant subalgebra of $\mathfrak{A}$.

The analysis of the interpolation spaces $\mathscr{B}(\mathscr{H})_0 \cap \mathscr{T}^{(1)}_{\beta, \alpha}$ follows step by step the analysis of the interpolation spaces $\mathscr{T}_{\beta, \alpha}$ in Section V. In order to carry this out, we note the following:

Lemma IX.4. *Let $b \in \mathscr{B}(\mathscr{H})_0$. Then as a bilinear form on $\mathscr{D}(Q^2) \times \mathscr{D}(Q^2) \subset \mathscr{D}(Q_1^2) \times \mathscr{D}(Q_1^2)$,*

$$d^2 b = d_1^2 b. \tag{IX.23}$$

The proof of this lemma is an elementary consequence of $[P, b] = 0$. Thus estimates on $d^2 b$ can be reduced to estimates on $d_1^2 b$. Also

$$R = (Q^2 + I)^{-1/2} = (Q_1^2 + Q_2^2 + I)^{-1/2} \leqslant (Q_1^2 + I)^{-1/2} = R_1. \tag{IX.24}$$

As a result, we can reduce all estimates in the proof of this statement to estimates of the same form as those of Section V. In particular, $\mathscr{B}(\mathscr{H})_0 \cap \mathscr{T}^{(1)}_{\beta, \alpha}$ is a Banach algebra, and we can prove analogs of Proposition V.5 (with R_1 replacing R and d_1 replacing d), as well as Corollary V.16, Proposition V.7, and Corollary V.8.

IX.3. *The Cochain $\tau^{\{Q_i\}}$ and Invariants*

Let Q in (IX.1) denote a Q_1-invariant splitting of Q. Let

$$\mathfrak{A} \subset \mathscr{B}(\mathscr{H})_0 \cap \mathscr{T}^{(1)}_{\beta, \alpha}, \qquad 0 \leqslant \alpha, \beta, \quad \alpha + \beta < 1,$$

denote the zero-momentum algebra of observables, with fractional Q_1-derivatives, as in Section IX.2. Let

$$\{\mathscr{H}, Q, Q_i, \gamma, \mathfrak{G}, U(g), \mathfrak{A}\} \tag{IX.25}$$

denote a Q_1-invariant, split, fractionally-differentiable structure, generalizing Definition VI.1 to the Q_1-invariant, split case. Here we also assume the hypothesis of Θ-summability for $\exp(-\beta Q^2)$, $\beta > 0$.

Define a cochain $\tau\{Q_i\}$ on $\mathfrak{A}$ with components,

$$\tau_n^{\{Q_i\}}(a_0, ..., a_n; g) = \langle a_0, d_1 a_1, ..., d_1 a_n; g\rangle_n, \tag{IX.26}$$

where the $(n+1)$-linear expectation $\langle \cdot, \cdot, ..., \cdot; g\rangle_n$ in (IX.26) is defined in (IV.16). The results of Section IX.2 allow us to establish the following Propositions.

PROPOSITION IX.5. a. *Let* $\{\mathscr{H}, Q, Q_i, \gamma, U(g), \mathfrak{A}\}$ *be as in* (IX.25), *and let* $\tau\{Q_i\}$ *be as in* (IX.26). *Then* $\tau^{\{Q_i\}} \in \mathscr{C}(\mathfrak{A})$. *There exists a constant* $m < \infty$ *such that*

$$|||\tau_n^{\{Q_i\}}||| \leqslant m^{n+1} \left(\frac{1}{n!}\right)^{(1/2)+((1-\alpha-\beta)/2)} \operatorname{Tr}(e^{-Q^2/2}). \tag{IX.27}$$

b. *Let* $a \in \mathrm{Mat}_n(\mathfrak{A}^{\mathfrak{G}})$ *satisfy* $a^2 = I$. *Then*

$$\mathfrak{Z}^{\{Q_i\}}(a; g) = \langle \tau^{\{Q_i\}}, a\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(g)\, a e^{-Q^2 + it\, d_1 a})\, dt \tag{IX.28}$$

is the natural pairing of cochains in $\mathscr{C}(\mathfrak{A})$ *with elements* $a \in \mathrm{Mat}_n(\mathfrak{A}^{\mathfrak{G}})$ *satisfying* $a^2 = I$. *Here* Tr *denotes both the trace in* $\mathscr{H}$ *and the trace in* $\mathrm{Mat}_n(\mathfrak{A})$.

c. *The cochain* $\tau^{\{Q_i\}}$ *is a cocycle,*

$$\partial \tau^{\{Q_i\}} = 0. \tag{IX.29}$$

The proof of Proposition IX.5 again follows the proofs in Section VI. We can also generalize the results of Sections VII–VIII. In order to understand the parameter dependence of $\tau^{\{Q_i(\lambda)\}}$ on a parameter λ, we consider in particular the following common case, where the parameter λ is called a *coupling constant*:

DEFINITION IX.6. We say that the splitting $Q(\lambda) = (1/\sqrt{2})(Q_1(\lambda) + Q_2(\lambda))$ depends parametrically on a coupling constant λ, if $Q(\lambda)$ depends on λ, but $P = \frac{1}{2}(Q_1(\lambda)^2 - Q_2(\lambda)^2)$ is independent of λ.

As far as the analytic bounds are concerned, for a regular, linear deformation we retain the assumptions formulated in Section VII.2.b. We also suppose that, as in Section VII.2.c–d, the symmetry group $U(g)$ and the algebra $\mathfrak{A}$ are both independent of λ. Alternatively, in the case of a regular deformation, we retain Definition VII.7 of Section VII.3. We can then extend Theorem VII.1 and its corollary to the case of Q_1-invariant, split structures, following our preceding work. We summarize this result:

THEOREM IX.7. *Let* $\{\mathcal{H}, Q(\lambda), Q_i(\lambda), \gamma, \mathfrak{G}, U(g), \mathfrak{A}\}$ *be a regular family* (*in the sense explained above*) *of* Θ*-summable, split,* Q*-invariant, fractionally-differentiable structures, depending on a coupling constant* λ *in the sense of Definition* VII.6. *Then the family of cocycles* $\{\tau^{\{Q_i(\lambda)\}}\} \subset \mathcal{C}(\mathfrak{A})$ *is continuously differentiable in* λ *as a function* $\Lambda \to \mathcal{C}(\mathfrak{A})$. *There is a continuous family of cochains* $h(\lambda)$ *in* $\mathcal{C}(\mathfrak{A})$ *such that for all* $\lambda \in \Lambda$,

$$\frac{d}{d\lambda}\,\tau^{\{Q_i(\lambda)\}} = \partial h(\lambda).$$

Furthermore the pairing of $\tau^{\{Q_i(\lambda)\}}$ *with* $a \in \mathrm{Mat}_n\{\mathfrak{A}^{\mathfrak{G}}\}$ *satisfying* $a^2 = I$ *and given by* (*IX*.27), *namely*

$$\mathfrak{Z}^{\{Q_i\}}(a, g) = \langle \tau^{\{Q_i(\lambda)\}}, a \rangle, \tag{IX.30}$$

is independent of $\lambda \in \Lambda$.

IX.4. *An Example of a Split Structure*

We mention here an example which we analyze elsewhere by these methods [16]. This example arises as an equivariant index in the Wess–Zumino quantum field theory on a cylindrical, two dimensional space-time. This field theory is also called the Landau–Ginsburg field theory, and it is specified by a holomorphic, quasi-homogeneous polynomial $\lambda V(z)$ on $\mathcal{C}^n$, where λ is a real parameter. In order to carry out our investigation, we require that the polynomial $V(z)$ be further restricted, so that the magnitude of its gradient grows at least linearly in $|z|$ as $z \to \infty$.

In our example there are four self-adjoint operators $Q_1 = Q_1(\lambda)$, $Q_2 = Q_2(\lambda)$, $\tilde{Q}_1 = \tilde{Q}_1(\lambda)$, and $\tilde{Q}_2 = \tilde{Q}_2(\lambda)$, such that for a given $H = h(\lambda)$ and P (independent of λ),

$$Q_1^2 = \tilde{Q}_1^2 = H + P, \qquad Q_2^2 = \tilde{Q}_2^2 = H - P. \tag{IX.31}$$

We fix $V(z)$ but let λ vary in the interval $\lambda \in (0, 1]$. Each of the six pairs of distinct Q_1, Q_2, $\tilde{Q}_1$, and $\tilde{Q}_2$'s anticommutes, namely $Q_1 Q_2 + Q_2 Q_1 = 0$, etc. As a consequence, they are said to be mutually-independent.

The symmetry group $\mathfrak{G}$ that we study is $u(1) \times U(1)$. The first factor $U(1)$ has the form $e^{i\theta J}$, where the generator J implements the symmetry of H due to the quasi-homogeneity of the potential $V(z)$. The operator J commutes with γ, with H, and with P. Furthermore J commutes both with Q_1 and with $\tilde{Q}_1$. On the other hand, the group generated by J performs a rotation on the pair of operators Q_2 and $\tilde{Q}_2$, namely

$$e^{i\theta J} Q_2 e^{-i\theta J} = Q_2 \cos\theta + \tilde{Q}_2 \sin\theta. \tag{IX.32}$$

While $e^{i\theta J}$ does not commute with Q_2,

$$e^{i\theta J} Q_2^2 e^{-i\theta J} = (e^{i\theta J} Q_2 e^{-i\theta J})^2 = H - P = Q_2^2,$$

so $e^{i\theta J}$ commutes with Q_2^2. Hence this group implements a symmetry of H.

The second $U(1)$ arises as the translation group of the circle, our one-dimensional space-coordinate. This $U(1)$ group is generated by the momentum operator P, and has the form $e^{i\tau P}$, which is unitary for a real parameter τ. Each operator Q_j and $\tilde{Q}_j$ is defined in translation invariant manner, so the group $e^{i\tau P}$ commutes with both Q_1 and Q_2 and therefore with H, as well as with γ and with J. We take the full $U(1) \times U(1)$ symmetry group $U(g)$ to have the form

$$U(\sigma, \theta) = e^{i\sigma P + i\theta J}, \tag{IX.33}$$

where $g = (\sigma, \theta)$. Even though the operators Q_j depend on the parameter λ, the operators J and P are both independent of λ, and therefore so is the group $U(\sigma, \theta)$. This group commutes with $H(\lambda)$.

We study the equivariant index $\mathfrak{Z}^{\{Q_i\}}(I; g)$ of the form (IX.28), that we also denote as

$$\mathfrak{Z}^{Q_j(\lambda)}(\tau, \theta) = \operatorname{Tr}(\gamma U(\sigma, \theta)\, e^{-H(\lambda)}), \tag{IX.34}$$

where $H = H(\lambda) = \frac{1}{2}(Q_1(\lambda)^2 + Q_2(\lambda)^2)$ does depend on λ. Here $\tau = \frac{1}{2}(\sigma - i)$. We also write this as

$$\mathfrak{Z}^{Q_j(\lambda)}(\tau, \theta) = \mathfrak{Z}^{Q_j(\lambda)}(\bar{\tau}, \tau, \theta), \tag{IX.35}$$

where

$$\mathfrak{Z}^{Q_j(\lambda)}(\bar{\tau}, \tau, \theta) = \operatorname{Tr}(\gamma e^{i\theta J} e^{i\tau_1 Q_1(\lambda)^2 - i\tau_2 Q_2(\lambda)^2} e^{-H(\lambda)}). \tag{IX.36}$$

By the methods used in this paper, $\mathfrak{Z}(\tau_1, \tau_2, \theta)$ is independent of λ and also independent of τ_1. Thus

$$\begin{aligned}\mathfrak{Z}^{Q_j(\lambda)}(\tau, \theta) &= \mathfrak{Z}^{Q_j(\lambda)}(\bar{\tau}, \tau, \theta) = \mathfrak{Z}^{Q_j(\lambda)}(-\tau, \tau, \theta)\\ &= \operatorname{Tr}(\gamma e^{i\theta J} e^{-i(\sigma+1)\, H(\lambda)}).\end{aligned} \tag{IX.37}$$

This index has been considered in the physics literature [21, 33]. Witten suggests that it can be evaluated at the singular endpoint $\lambda = 0$ of the interval of the coupling constant parameter $\lambda \in (0, 1]$. At this endpoint the representation (IX.37) is no longer valid, as the operator $\gamma e^{i\theta J} e^{i(\sigma+i)\, H(0)}$ has continuous spectrum. However, with some work, we prove that the index can be evaluated as a limit of a regularized index. Our method is an extension to quantum field theory of the holonomy expansion method introduced in [15] to study the constant Fourier mode of the quantum field. This limit is a product of modular forms, and the dependence of $\mathfrak{Z}$ on the variable τ has an analytic extension into a half place. The modular symmetry of $\mathfrak{Z}$ as a function of the variables (τ, θ) is a hidden symmetry of the original example.

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