

A MATHEMATICAL PICTURE LANGUAGE PROJECT

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ABSTRACT. The mathematical picture language project that we began in 2016 has already yielded some interesting results. We also point out some areas of mathematics and physics where we hope that it will prove useful in the future.

1. PICTURES

Pictures in mathematics flourished since the time of Euclid, who lived in Greece during the 4th-century BC. About the same time in China, Chuang-Tzu promoted the philosophy of replacing words by images. In modern mathematics pictures are pervasive, including Feynman diagrams, category theory, and planar algebra.

Category theory, introduced by Eilenberg and MacLane [7], developed into a picture language, with transformations that led to mathematical pictures. Feynman's use of diagrams to represent polynomial interactions in quantum many-body processes or quantum field theory gives an early illustration of these concepts in physics [8]. Extra structure on category pictures arose in Jones' theory of link invariants [18, 19] and planar algebras [20], and the general framework of topological quantum field theory [32, 1].

Although pictures in mathematics are far from new, we believe we have gained new insights into mathematics by using pictures somewhat differently from in the past. This led to our current "mathematical picture language project." We have been able to prove some interesting theoretical results about pictures, and apply them to different situations.

One aspect of our approach which we find important is the mathematical analysis of pictures, and how one might formulate a theory complementing the study of the use of pictures in topology and geometry. The analytic aspect of pictures in topological quantum field theory

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(TQFT) is a less-developed area than its topological and algebraic aspects, and has great potential for future advances.

Another notion we emphasize is proof through pictures. We wish to distinguish focusing on general mathematical properties of pictures, from using pictures in a particular concrete mathematical model. In other words, we aim to distinguish between the notion of the properties of a picture language \mathbf{L} on the one hand, from its use through a simulation \mathbf{S} to model a particular reality \mathbf{R} by mathematics. This distinction between \mathbf{L} and \mathbf{R} parallels the distinction in linguistics between *syntax* and *semantics*.

We propose that it is interesting to prove a result about the language \mathbf{L} , and thereby through simulation ensures results in \mathbf{R} . One can use a single picture language \mathbf{L} to simulate several different mathematical areas. In fact a theorem in \mathbf{L} can ensure different theorems in different mathematical subjects $\mathbf{R}_1, \mathbf{R}_2$, etc., as a consequence of different simulations $\mathbf{S}_1, \mathbf{S}_2$, etc. Different configurations of the hands of the clock reveal the interrelation between picture proofs for seemingly unrelated mathematical results. It is also important to distinguish between two types of concepts in \mathbf{R} that we simulate by a given \mathbf{S} . These may be *real* concepts, or they may be *virtual*. This distinction is not absolute, but depends on what language and simulation one considers.

1.1. Virtual Processes. The mathematical interpretation of Feynman diagrams arises from the pictorial representation of Gaussian moments. These occur naturally in quantum theory with a quadratic potential, through analytic continuation: one replaces the Schrödinger equation with the parabolic heat equation, the solution to which is described by a Gaussian measure on the space of random fields. For real, classical scalar random fields (contrasted with fermionic fields), the moments of this measure of degree $2n$ are the sums of $(2n - 1)!!$ terms, each represented as a Feynman diagram.

Perturbation of the Gaussian by a cubic potential leads to the physical interpretation by Feynman and Wheeler, and describes each diagram as a trajectory of particles that interact at the vertices of the diagram. An elementary Feynman diagram for the scattering of two particles of mass m with energy-momentum p, q colliding with and exchanging a third particle with energy-momentum k is illustrated in Figure 1:

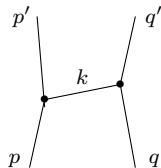


FIGURE 1. Virtual particle with energy-momentum k in a Feynman diagram arising from a cubic perturbation.

After inverse analytic continuation to physical time, the energy-momentum 4-vector $p = (E, \vec{p})$ of a real, physical particle with mass- m particle lies on the energy-momentum hyperboloid $E^2 - \vec{p}^2 = m^2$ with $E \geq 0$. One denotes this identity as the Minkowski inner product $\langle p, p \rangle_M = m^2$, and one identifies the particle with its energy-momentum p .

If energy and momentum are conserved at each collision vertex in Figure 1, then $k = p - p'$. Evaluating for p, p' at rest means $k = (0, \vec{0})$, and therefore $\langle k, k \rangle_M = 0$. Hence if k designates a real particle, it must be massless. In turn, this means that with p, p' not at rest, $0 = \langle p - p', p - p' \rangle_M = 2(m^2 - \langle p, p' \rangle_M)$. But it is impossible that $\langle p, p' \rangle_M = m^2$, unless $p = p'$ and $k = 0$. We conclude that the particle k , described by Figure 1, cannot be real; it must be a “virtual” particle, without a direct physics interpretation as a physical particle. However it serves as an excellent way to think about the mediation of a force—namely by exchanging a virtual particle. Other corrections to this elementary mediation involve the exchange of many virtual particles.

While some physicists would prefer to suppress summing over such virtual effects, these concepts are very useful in many problems—especially when they illustrate symmetries that otherwise would remain hidden. Virtual effects run through much of modern mathematics and also physics—from Fourier duality, to algebraic identities, to classical or quantum error correction.

1.2. Start of our Project. Our collaboration on pictures began in 2015 with our study of planar algebras and statistical physics. We showed at that time that there is a geometric proof of the reflection positivity (RP) property for certain statistical mechanics models [10]. To do this we combined insights of author ZL on subfactor theory and planar algebras with insights of author AJ on parafermions, to yield a new type of planar algebra, well-suited for the description of quantum spin systems with d individual states of a single particle; we call this parafermion planar para algebra (PAPPA) of order d , and we

interpreted this parameter as a *charge* in the abelian group \mathbb{Z}_d . This is a special example of a family of generalizations of planar algebras that ZL had considered.

We found that PAPPa gives a topological interpretation of protocols in quantum information, and in fact we gave a dictionary to translate in both directions between algebraic operations with Pauli matrices, Fourier transform, and measurements (on the one hand) and pictorial representations by PAPPa (on the other). Ultimately we were able to use PAPPa to give a method of topological design for new protocols. At that point it started to become clear to us, that central to our methods were certain mathematical languages. The PAPPa gave a language of “braided charged strings” in a plane.

We later discovered that it is useful to introduce a redundancy by replacing two strings with four. Then the real physics of a single spin lies in a d -dimensional physically-relevant subspace of a d^2 -dimensional Hilbert space that is mostly virtual.¹ Implementing this correctly for multi-particle states led to the three-dimensional quon language.

One can view quon as a topological quantum field theory having certain defects, given by a special set of charged strings. This language had immediate application both in quantum information and in algebra, and eventually for other things as well.

What is perhaps more surprising is that one can develop a branch of Fourier analysis based on transformations of pictures. The pictorial Fourier transform \mathfrak{F}_s , that we call the string Fourier transform (SFT) plays a central role in many applications of our picture languages. Over the course of time, we shifted focus to thinking of the languages as primary, and their application to various problems in different mathematical areas as secondary. We have found interesting things in algebra, topology, representation theory, as well as in mathematical physics.

2. THE TWO STRING LANGUAGE

In this section we describe how our pictures, that evolved from planar algebra diagrams, became useful in quantum information. We began to study picture languages in an effort to combine our work on planar algebras with the notion of parafermions [10]. Following a suggestion

¹Physical subspaces in larger virtual spaces arise in many other physics contexts, of which we mention three: One recovers a reflection-positive subspace by pairing random fields with their reflection. One recovers a single-photon Hilbert space of states by considering the Gupta-Bleuler analysis of defining a positive inner product on a gauge-invariant subspace of single photon configurations. In non-abelian gauge theory one has other methods related to Becchi, Rouet, Stora, and Tyutin to single out physical states with non-negative norm.

ensuring $\zeta^{k^2} = \zeta^{(k+d)^2}$. We also use the following relation to move a charge across a string with fixed endpoints:

$$k \frown = \zeta^{k^2} \frown k, \quad \text{so} \quad \smile^{-k} = \zeta^{k^2} \smile^{-k}. \quad (8)$$

2.3. Basic Transformations. The basic transformations in quantum information protocols involve the identity, the Pauli matrices X, Y, Z , the Fourier transform (or Hademard) matrix F , and the phase (Gaussian) transformation G that implements a braid. Pictorially,

$$I = \left| \begin{array}{c} | \\ | \end{array} \right\rangle, \quad X = \left| \begin{array}{c} | \\ 1 \end{array} \right\rangle, \quad Y = \left| \begin{array}{c} -1 \\ | \end{array} \right\rangle, \quad Z = \left| \begin{array}{c} 1 \\ -1 \end{array} \right\rangle. \quad (9)$$

2.4. The Analytic Fourier Transform. Consider the projection P_k onto the k^{th} basis vector (3). The ordinary Fourier transform F acting on this picture P_k as a function of k is

$$FP_k = \frac{1}{\sqrt{d}} \sum_{\ell \in \mathbb{Z}_d} q^{k\ell} P_\ell,$$

or

$$F \begin{array}{c} \smile^{-k} \\ \frown k \end{array} = \frac{1}{\sqrt{d}} \sum_{\ell \in \mathbb{Z}_d} q^{k\ell} \begin{array}{c} \smile^{-\ell} \\ \frown \ell \end{array}. \quad (10)$$

2.5. The pictorial Fourier transform. The pictorial Fourier transform \mathfrak{F}_s , or string Fourier transform (SFT), rotates the one-qudit transformation picture by 90° . In other words,

$$\mathfrak{F}_s \begin{array}{c} | \\ \boxed{T} \\ | \end{array} = \begin{array}{c} | \\ \boxed{\mathfrak{F}_s T} \\ | \end{array} = \begin{array}{c} \smile \\ \boxed{T} \\ \smile \end{array}.$$

This transform \mathfrak{F}_s applied to a neutral transformation T generalizes the Fourier transform F .

Theorem 2.1. *On diagonal $d \times d$ matrices, the transformations \mathfrak{F}_s and F agree $\mathfrak{F}_s = F$. The picture theorem is, compare with (10),*

$$\boxed{\mathfrak{F}_s \begin{array}{c} \smile^{-k} \\ \frown k \end{array} = \left| \begin{array}{c} k \\ -k \end{array} \right\rangle = F \begin{array}{c} \smile^{-k} \\ \frown k \end{array}}. \quad (11)$$

Proof. The proof takes only a couple of lines. We use the definition of \mathfrak{F}_s and the properties (3) and (10) to write

$$\begin{aligned}
\mathfrak{F}_s \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} &= \zeta^{k^2} \mathfrak{F}_s \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} = \zeta^{k^2} \left| \begin{array}{c} -k \\ k \end{array} \right| = \frac{1}{\delta} \sum_{\ell \in \mathbb{Z}_d} \zeta^{k^2} \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} \\
&= \frac{1}{\delta} \sum_{\ell \in \mathbb{Z}_d} \zeta^{k^2} q^{k\ell} \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} = \frac{1}{\delta} \sum_{\ell \in \mathbb{Z}_d} q^{k^2+k\ell} \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} \\
&= \frac{1}{\delta} \sum_{\ell \in \mathbb{Z}_d} q^{k\ell} \begin{array}{c} \text{---}(-\ell) \\ \text{---} \\ \text{---}\ell \end{array} = \frac{\sqrt{d}}{\delta} F \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array} = F \begin{array}{c} \text{---}(-k) \\ \text{---} \\ \text{---}k \end{array}.
\end{aligned}$$

In the first equality we use the relation (8), while the second is the definition of the SFT. In the third identity we insert the resolution of the identity relation (4). Next we use the para-isotopy relation (5), and after that we again apply (8). In the next-to-last equality we translate the sum, replacing $\ell \in \mathbb{Z}_d$ with $\ell - k$. Thus $\mathfrak{F}_s = F$ on these projections. Use the definition of para-isotopy interpolation to identify the middle term in the statement of the picture theorem, as the third expression in the calculation above. \square

3. THE REFLECTION-POSITIVITY PROPERTY

The reflection-positivity property has had wide application in mathematical physics (quantum field theory, statistical physics). It is central in the mathematical study of phase transitions and also the relations between the two subjects. It is also important when one considers abstract Fourier analysis.

In this section we use a single string (tensor network) notation for a vector $W \in \mathcal{H}$, representing it by the picture \boxed{W} , in place of (2).

3.1. Reflection. In the definition of the string Fourier transform \mathfrak{F}_s , we considered a finite-dimensional Hilbert space \mathcal{H} and its dual $\widehat{\mathcal{H}} = \theta\mathcal{H}$. The map θ denotes the Riesz representation map $\mathcal{H} \rightarrow \widehat{\mathcal{H}}$. Then duality θ and the adjoint $*$ are related;

$$\boxed{\theta(W)} = \boxed{W^*} \quad , \quad \text{and} \quad \boxed{\theta(W)^*} = \boxed{W} \quad . \quad (12)$$

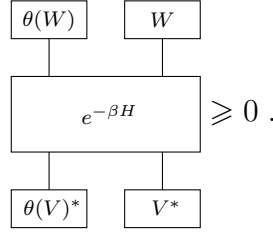
3.2. Reflection Positivity. We formulate the reflection-positivity property that we considered in [10] in a recent work [12] in terms of \mathfrak{F}_s acting as a map

$$\mathfrak{F}_s : \text{hom}(\widehat{\mathcal{H}} \otimes \mathcal{H}) \rightarrow \text{hom}(\mathcal{H} \otimes \widehat{\mathcal{H}}) .$$

Definition 3.1. A map $H \in \text{hom}(\widehat{\mathcal{H}} \otimes \mathcal{H})$ has the RP property if for every $V, W \in \mathcal{H}$,

$$\langle \theta(V) \otimes V, e^{-\beta H} \theta(W) \otimes W \rangle_{\widehat{\mathcal{H}} \otimes \mathcal{H}} \geq 0 .$$

Pictorially, the RP property means that for all $V, W \in \mathcal{H}$,

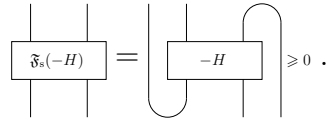


Here vertical reflection symmetry of the picture is not apparent. However there appears to be some horizontal symmetry. We relate the Fourier transform of the terms in the power series of $e^{-\beta H}$ to the convolution powers of the Fourier transform on $-H$.

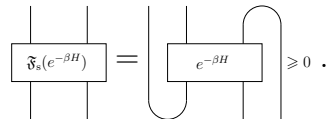
Remark . Considering the tensor product $\widehat{\mathcal{H}} \otimes \mathcal{H}$ means we consider here the special “bosonic” case. The construction can be modified to cover possible twisting. We now outline the pictorial proof the following result.

Theorem 3.2. If $\mathfrak{F}_s(-H) \geq 0$, then $\mathfrak{F}_s(e^{-\beta H}) \geq 0$, and H has the RP property.

Proof. Pictorially the assumption that $\mathfrak{F}_s(-H) \geq 0$, means that



We prove in reference [12] that positivity of $\mathfrak{F}_s(-H)$ entails positivity of $\mathfrak{F}_s(e^{-\beta H})$ for $\beta \geq 0$. Thus,



The picture proof of the RP property is evident, since \mathcal{H} is a Hilbert space. Using (12), one sees that

$$\begin{array}{c}
 \boxed{\theta(W)} \quad \boxed{W} \quad \boxed{V} \quad \boxed{\theta(W)} \\
 | \quad | \quad | \quad | \\
 \boxed{e^{-\beta H}} = \boxed{\mathfrak{F}_s(e^{-\beta H})} \geq 0 . \\
 | \quad | \quad | \quad | \\
 \boxed{\theta(V)^*} \quad \boxed{V^*} \quad \boxed{V^*} \quad \boxed{\theta(W)^*}
 \end{array} \quad (13)$$

To justify the positivity statement, look at the picture on the right of the equal sign. The vector on the bottom is the vertical reflection of the vector on top. Algebraically, the relation (13) between pictures means

$$\begin{aligned}
 & \langle \theta(V) \otimes V, e^{-\beta H}(\theta(W) \otimes W) \rangle_{\widehat{\mathcal{H}} \otimes \mathcal{H}} \\
 & = \langle V \otimes \theta(W), \mathfrak{F}_s(e^{-\beta H})(V \otimes \theta(W)) \rangle_{\mathcal{H} \otimes \widehat{\mathcal{H}}} .
 \end{aligned}$$

Thus $\mathfrak{F}_s(e^{-\beta H}) \geq 0$ for all $\beta \geq 0$ guarantees the RP property, though this is a stronger condition than required. \square

One can carry the picture proof further. Extend θ to $\widehat{\mathcal{H}} \otimes \mathcal{H}$, so that $\theta(V \otimes W) = \theta(W) \otimes \theta(V)$. Let $\theta(H) = \theta H \theta$.

Theorem 3.3. *In case $H = H_- + H_0 + H_+$ where $H_+ = \theta(H_-) \in \text{hom}(\mathcal{H})$, and $\mathfrak{F}_s(-H_0) \geq 0$, then H has the RP property.*

We also have used a generalization of this method to establish the RP property in Levin-Wen models [12], now a fashionable model to describe topological insulators.

4. THE QUON LANGUAGE AND VIRTUAL STATES

The three dimensional quon language can work even better than the two-dimensional pictures discussed above, and this brings to the fore the concept of virtual states. The 2D picture language could be considered as the local boundary theory of the 3D quon language. For a single-particle Hilbert space we take two copies of the two-string language in §2. Then one can represent the Pauli matrices for a single spin by neutral transformations:

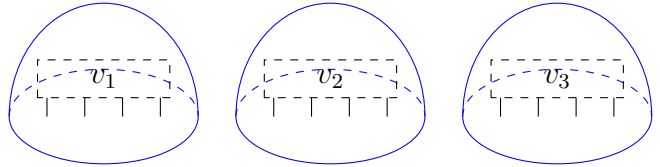
$$\text{I} = \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array}, \quad \text{X} = \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline -1 \\ \hline \end{array}, \quad \text{Y} = \begin{array}{|c|} \hline -1 \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array}, \quad \text{Z} = \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline -1 \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array}. \quad (14)$$

Each of these transformations is unitary; each equals the identity when raised to the power d . Also each of these Pauli matrices is neutral, and each commutes with the grading transformation

$$\gamma = \begin{array}{|c|} \hline -1 \\ \hline \end{array} \begin{array}{|c|} \hline -1 \\ \hline \end{array} \begin{array}{|c|} \hline | \\ \hline \end{array} \begin{array}{|c|} \hline -1 \\ \hline \end{array} = \zeta^{-1} X Y Z, \quad (15)$$

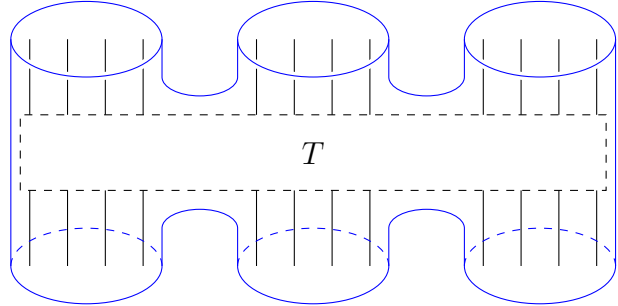
where ζ is given in (7). Therefore each Pauli matrix acts on an eigenspace of γ . On the neutral eigenspace for which $\gamma = 1$, these matrices satisfy the desired relation $XYZ = \zeta$. We call the $\gamma = 1$ space, the d -dimensional *physical* or *real* subspace of the d^2 -dimensional vector space of eigenstates of Z .

This description of a single particle excitation as a neutral pair led to our realization that particles are best described by pictures in one higher dimension. A three-particle state is given by a cap enclosing the four output lines with internal connections denoting vectors v_1, v_2, v_3 ,



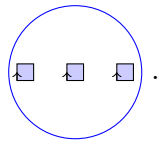
(16)

A transformation can be represented by a 3D-TQFT,




(17)

In fact it is more symmetric to arrange the output lines on the corners of a square, with an arrow indicating the numbering of the strings. For clarity we suppress the three-dimensional TQFT, except for the outline (here a circle) of a cross-section. Looking down on the three-particle outputs, and suppressing the output lines, one can represent the three-particle state in (16) as



By defining the internal connections in two different ways, one obtains the three-particle Greenberger-Horne-Zeilinger resource state $|\text{GHZ}\rangle_3$ [9], or its SFT the state $|\text{Max}\rangle_3$ [13], as



(18)

This language illustrates these two states as related by rotation, given by the pictorial Fourier transform \mathfrak{F}_s . For further details, see [13, 24]. Both GHZ and Max are maximally entangled resource vector states [9, 13], so the string Fourier transform produces maximal entanglement in one operation.

5. QUANTUM FOURIER ANALYSIS

One theme that runs through much of the work in the picture language program comes from analytic properties of the pictorial Fourier transform \mathfrak{F}_s . ZL with Chunlan Jiang and Jinsong Wu began to analyze analytic properties of the SFT. They found that standard inequalities such as the Young and Hausdorff-Young estimates, as well as classical uncertainty principles such as those of Heisenberg, Hirschmann or Beckner, have generalizations in which the analytic Fourier transform F is replaced by the pictorial Fourier transform \mathfrak{F}_s [15, 27, 16]. The study of the analytic properties of \mathfrak{F}_s has become known as *quantum Fourier analysis*. This is a new subject, with only a few of many potentially interesting things already discovered; it has become a central part of our picture language project.

The pictorial proof of inequalities allowed them to determine the extremizers for the inequalities, namely to solve the inverse problem. Now these methods have been extended to analysis on quantum groups and other frameworks, and this leads to an entire realm of analytical questions that should be investigated.

For example, one has shown that with the trace norm, defined by pairing input and output strings in a non-overlapping fashion,

$$\mathrm{Tr}(A^*B) = \begin{array}{c} \dots \\ \boxed{B} \\ \dots \\ \boxed{A^*} \\ \dots \end{array} \cdot \quad (19)$$

Assuming that the trace is positive, the GNS representation allows one to define the corresponding norms $\|A\|_p = (\mathrm{Tr}((A^*A)^{p/2}))^{1/p}$. One finds inequalities that generalize inequalities of classical Fourier analysis. This is what we call *quantum Fourier analysis*. We state here only a few examples. We are preparing a lengthy summary paper on this new subject.

Theorem 5.1 (Quantum Hausdorff-Young Inequality [15]). *Let $0 \neq A$ be a two-box in an irreducible, subfactor planar algebra with*

quantum dimension δ . Then

$$\|\mathfrak{F}_s(A)\|_q \leq \delta^{1-\frac{2}{p}} \|A\|_p, \quad 1 \leq p \leq 2, \quad p^{-1} + q^{-1} = 1. \quad (20)$$

The equality holds in (20) for some $1 < p < 2$, iff the equality holds for all $1 \leq p \leq 2$, iff A is a bi-shift of a biprojection.

Not only does this yield the optimal constant in the quantum inequality, but its extremizers are characterized by interesting algebraic properties. This has been applied by ZL in the classification of subfactors in [22]. This gave rise to a quantum analogue of the fundamental theorem of finite abelian groups: An irreducible subfactor planar algebra generated by 2-boxes is a free product of finite abelian groups and Temperley-Lieb-Jones planar algebras, if and only if its 2-boxes are commutative, and its 3-boxes (modulo the basic construction ideal) are commutative.

There is a rich spectrum of analytic inequalities for quantum symmetries. There are also uncertainty principles that arise in case for certain q, p one has equality, such as in the inequality (20) which holds as an identity for $p = q = 2$. A corresponding quantum Young's inequality also holds, with convolution defined by horizontal, rather than vertical, multiplication.

Another interesting result that relates to quantum information is an entropic uncertainty principle for quantum entanglement. There are several new inequalities involving Fourier duality, entropy, or related uncertainty principles. We give an example here, that gained insights from picture language: The Rényi entropy of order p for a two-box A in a subfactor planar algebra is

$$h_p(A) = \frac{p}{1-p} \log \|A\|_p, \quad \text{for } 0 < p \neq 1. \quad (21)$$

This tends to the von Neumann entropy as $p \searrow 1+$:

$$h(A) = \text{Tr}(-|A| \log |A|). \quad (22)$$

Theorem 5.2 (Quantum Entropic Uncertainty Principle [15]).

Let A be a two-box in an irreducible, subfactor planar algebra with quantum dimension δ . Then

$$h(|A|^2) + h(|\mathfrak{F}_s(A)|^2) \geq 2\|A\|_2^2 \log \delta - 2\|A\|_2^2 \log \|A\|_2^2. \quad (23)$$

Quantum Rényi entropic uncertainty Principles were proved in [29], and the above result appeared as a special limit.

The von-Neumann entropy could be interpreted as an entanglement entropy between the domain and the range of an operator. The above result could be transformed into an uncertainty principle for entanglement entropy.

For example, the zero state and the Bell state are Fourier dual to each other. The above uncertainty principle yields a lower bound for the sum of their entanglement entropy. As the zero state is a product state with no entanglement, the Bell state achieves maximal entanglement. Using the entropic uncertainty principle of the general SFT on n -boxes [15], similar arguments apply to other states which are Fourier dual to the zero state, such as the GHZ or Max state mentioned above. From this point of view, we believe that the Fourier duality produces a *minimal-maximal entanglement pair*.

Following the pictorial intuition, ZL, Jiang and Wu introduced block maps for subfactors and proved that any 2-box converges to a multiple of a biprojection under the dynamic action of the block map [17]. This can be considered as a quantum analogue of the central limit theorem, with biprojections playing the role of Gaussians. The block map could also be considered as a 2D renormalization flow from the physics point of view.

6. ALGEBRAIC IDENTITIES

Many other results have already come from the analysis of these pictures. Some of the greatest insights seem to be associated with the three-dimensional quon language [24]. This appears to be a new and very interesting field. ZL has used the quon language to establish a duality relation leading to the proof of an interesting identity for $6j$ -symbol self-duality. This identity had been discovered for quantum $SU(2)$ in the context of the study of quantum gravity[2]. But it was only conjectured in more general form [31]. Here the S matrix plays the role of \mathfrak{F}_s [30].

Theorem 6.1 (See §6 of Liu [23]). *With \bar{X} denoting the dual object to X in a modular tensor category,*

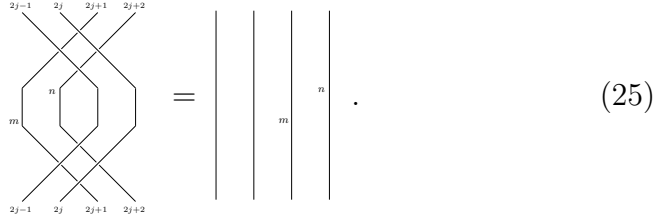
$$\left| \begin{pmatrix} X_6 & X_5 & X_4 \\ \bar{X}_3 & \bar{X}_2 & \bar{X}_1 \end{pmatrix} \right|^2 = \sum_{\bar{Y}} \left(\prod_{k=1}^6 S_{X_k}^{Y_k} \right) \left| \begin{pmatrix} Y_1 Y_2 Y_3 \\ Y_4 Y_5 Y_6 \end{pmatrix} \right|^2. \quad (24)$$

Usually the S matrix is known, but the $6j$ -symbols are hard to solve. Even for quantum $SU(N)$, the closed form formula of $6j$ symbols remains unknown. From the view of Fourier analysis, we suggest another scheme to compute the $6j$ symbols. It is possible to compute parts of the $6j$ -symbols and then to solve the modulus of $6j$ -symbols from partially known data in Equation (24), similar to signal recovery [4, 5]. One may further solve for the phases using the fact that the $6j$ -symbols form a biunitary.

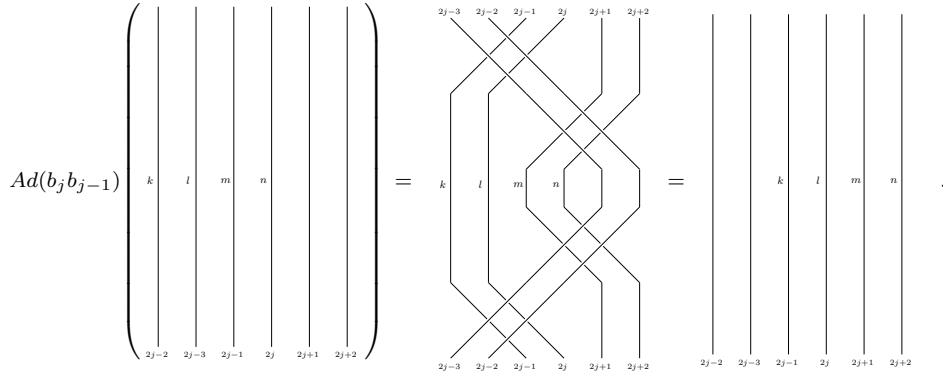
7. A DE FINETTI THEOREM FOR BRAIDS

The famous de Finetti theorem in classical probability theory clarifies the relationship between permutation symmetry and the independence of a sequence of random variables [6]. Consequently an infinite sequence of symmetric random variables can be written as a convex combination (or integration) of independent identically distributed (i.i.d.) random variables.

Motivated by pictures, we have established a de Finetti theorem for states on parafermion algebras of order d . In particular, we use the fact that a pair of parafermions of order d generate the $d \times d$ matrix algebra $\mathbb{M}_d(\mathbb{C})$, that we denote by PF_2 . Thus it is natural to consider pairs of parafermions as a unit, and to study double braids that exchange these pairs. The adjoint action of a double braid b_j acting on strands $2j - 1, \dots, 2j + 1$ implements the second Reidemeister move in the form,



Similarly, the adjoint action of $b_j b_{j-1}$ moves the charges on a four-strings,



The infinite parafermion algebra PF_∞ is a \mathbb{Z}_d -graded, tensor product of algebras PF_2 of parafermion pairs. Here we consider the braid group \mathbb{B}_∞ , acting on pairs of parafermions. Let $S_{\mathbb{B}_\infty}$ denote the states on PF_∞ that are invariant under the action of \mathbb{B}_∞ . Here are two results:

Theorem 7.1 (First de Finetti for braided parafermions [3]). *Let $\varphi \in S_{\mathbb{B}_\infty}$ be a braid-invariant state on PF_∞ . Then the following are equivalent:*

- (1) *The state φ is extremal in the set of states $S_{\mathbb{B}_\infty}$ on PF_∞ .*

- (2) The state $\varphi = \rho^{\otimes\infty}$ is the infinite tensor product of a state ρ on PF_2 .

As a consequence, any \mathbb{B}_∞ -invariant state on PF_∞ is in the closure of the convex hull of the product states. Let $\overline{PF}_\infty^\varphi$ denote the von Neumann algebra generated by PF_∞ in the Gelfand-Naimark-Segal (GNS) construction with respect to the state $\varphi \in S_{\mathbb{B}_\infty}$. Also let $(\overline{PF}_\infty^\varphi)^{\mathbb{B}_\infty}$ be the fixed point algebra under the action of the braid group \mathbb{B}_∞ . The neutral subalgebra of $(\overline{PF}_\infty^\varphi)^{\mathbb{B}_\infty}$ is the subalgebra generated by monomials in parafermions of degree zero mod d .

It is interesting that a distinction arises in this characterization, according to whether or not the order of the parafermion algebra is square free. (This means that $d = \prod_i p_i$, where the primes p_i are distinct.) Let us now suppose that the degree d of the parafermion algebra is square free. In this case one finds that extremal, braid-invariant states are neutral and that they give rise to a factor. One can refine Theorem 7.1 as follows:

Theorem 7.2 (Second de Finetti for square-free parafermion degree [3]). *Let $\varphi \in S_{\mathbb{B}_\infty}$ be a braid-invariant state on a parafermion algebra PF_∞ of square-free degree d . The following are equivalent:*

- (1) The state φ is extremal in $S_{\mathbb{B}_\infty}$.
- (2) The state $\varphi = \rho^{\otimes\infty}$, where ρ is a neutral state on PF_2 .
- (3) The neutral subalgebra of $(\overline{PF}_\infty^\varphi)^{\mathbb{B}_\infty} = \mathbb{C}$.
- (4) The algebra $(\overline{PF}_\infty^\varphi)^{\mathbb{B}_\infty} = \mathbb{C}$.
- (5) The von Neumann algebra $\overline{PF}_\infty^\varphi$ is a factor.

We have also given a characterization in case d is not square free, see [3].

8. SOME OUTLOOKS

We believe that we have only scratched the surface of an enormous new area of pictorial mathematics. We are excited by the fact that these advances have tentacles into many different subfields of mathematics. They not only have already made contact with algebra and topology. But we also have found new and deep directions for analysis appearing in our early results.

In our paper [11] we gave a long list of open questions. Here are some complementary remarks on directions that we and our collaborators are investigating:

- I. **Quantum Fourier Analysis.** There are many new inequalities to be found, including topological Brascamp-Lieb inequalities. There are new uncertainty principles still to be discovered. The analysis

of the period- $2n$ string Fourier transform on n -box transformations presumably has a rich and interesting structure.

- II. **Quantum Information.** While we have already investigated the topological design of protocols, little work has been done on algorithms. This will be a major focus in the future. The quon language seems beautifully suited to describe stabilizer quantum error correction codes. How to identify optimal codes is an open problem. Also the use of quon language to understand questions of *topological complexity* appears very ripe.

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