Annals of Physics **279**, 161–262 (2000)
doi:10.1006/aphy.1999.5920, available online at http://www.idealibrary.com on IDEAL®

The Holonomy Expansion: Invariants and Approximate Supersymmetry[1]

Arthur Jaffe

Harvard University, Cambridge, Massachusetts 02138

Received October 27, 1998

In this paper we give a new expansion, based on cyclicity of the trace, to study regularity properties of twisted expectations $\langle X(s)\rangle = \mathrm{Tr}_{\mathscr{H}}(\gamma U(\theta)\, X(s))$. Here $X(s) = X_0 e^{-s_0 Q^2} X_1 e^{-s_1 Q^2} \cdots X_k e^{-s_k Q^2}$ is a product of operators X_j, regularized by heat kernels $e^{-s_j Q^2}$ with $s_j > 0$. The twist groups $\gamma \in \mathbb{Z}_2$ and $U(\theta) \in U(1)$ are commuting symmetries of Q^2. The name "holonomy expansion" arises from picturing $\langle X(s)\rangle$ as a circular graph, with vertices in the graph representing the operators X_j, in the order that they appear in the product, and the line-segment following X_j representing the heat kernel $e^{-s_j Q^2}$. The trace functional is cyclic, so the graph is circular. We generate our expansion by "transporting" a vertex X_k around the circle, ending in its original position. We choose an X_k that transforms under a one-dimensional representation of $\mathbb{Z}_2 \times U(1)$. For θ in the complement of the discrete set Υ_{sing} (where the group $\mathbb{Z}_2 \times U(1)$ acts trivially on X_k) we obtain an identity between the original expectation and some new expectations. We study an example from supersymmetric quantum mechanics, with a Dirac operator $Q(\lambda)$ depending on a parameter λ and with a $U(1)$ group of symmetries $U(\theta)$. We apply our expansion to invariants $\mathfrak{Z}(\lambda; \theta) = \mathfrak{Z}(Q(\lambda); \theta)$ suggested by non-commutative geometry. These invariants are sums of expectations of the form above. We investigate this example as a first step toward developing an expansion to evaluate related invariants arising in supersymmetric quantum field theory. We establish differentiability of $\mathfrak{Z}(\lambda; , \theta)$ in λ for $\lambda \in (0, 1]$ and show $\mathfrak{Z}(\lambda; \theta)$ is independent of λ. We wish to evaluate $\mathfrak{Z}(\lambda; \theta)$ at the endpoint $\lambda = 0$, but $\mathfrak{Z}(0; \theta)$ is ill-defined. We regularize the endpoint, while preserving the $U(\theta)$-symmetry, by replacing $Q(\lambda)^2$ with $H(\varepsilon, \lambda) = Q(\lambda)^2 + \varepsilon^2 |z|^2$. The regularized function $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ depends on all three variables ε, λ, θ; for fixed θ, it is differentiable in the unit (ε, λ) square, except at the origin. Using the holonomy expansion, we prove for fixed $\theta \notin \Upsilon_{\mathrm{sing}}$ that $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ is also jointly continuous in (ε, λ), at the origin. As a consequence, if $\theta \notin \Upsilon_{\mathrm{sing}}$, then we can interchange limits and $\mathfrak{Z}(\lambda; \theta) = \lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon, 0; \theta)$. We observe that the joint continuity of $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ in (ε, λ) is *not* uniform in θ, and $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ is not jointly continuous for $\theta \in \Upsilon_{\mathrm{sing}}$. But the limiting function $\mathfrak{Z}(\lambda; \theta)$ *is* continuous in θ; so the ε-limit also determines $\mathfrak{Z}(\lambda; \theta)$ for all θ, including for $\theta \in \Upsilon_{\mathrm{sing}}$. We use these facts to calculate $\mathfrak{Z}(\lambda; \theta)$. Our regularization destroys supersymmetry, but the holonomy expansion gives quantitative bounds on the error terms. © 2000 Academic Press

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[1] Supported in part by the Department of Energy under Grant DE-FG02-94ER-25228 and by the National Science Foundation under Grant DMS-94-24344. A portion of this work was done while visiting the E.T.H., Zürich; I warmly thank J. Fröhlich for his hospitality there. I am further grateful to Sarah for typing, TeX advice, and her sense of humor.

0003-4916/00 $35.00

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I. INTRODUCTION

A major theme in establishing convergent expansions is to build these expansions from repeated application of simple identities. In mathematical physics problems, cluster expansions, Peierls' expansions, phase cell localization, etc., all arise in this manner. In the present paper we explore a new expansion called the *holonomy expansion*, based on cyclicity of the trace. The trace of a product of operators on a Hilbert space possesses a natural graphical representation, with vertices representing the operators placed on a circle in the same order as the operators in the product. Cyclicity leads to the circular graph. The name of our expansion arises from the fact that we generate the expansion by the elementary step of translating a vertex around the circle.

We apply this expansion to study and to compute certain geometric invariants. Our examples here arise from supersymmetric quantum mechanics. They provide a first step toward developing an expansion to evaluate related invariants arising in supersymmetric quantum field theory, and in particular the character valued index or elliptic genus [W], as well as other invariants arising in the same models. This paper is divided into two parts. In the first part, we study the character-valued index of a Dirac operator, see Sections II–IX. In the second part, we generalize our method to evaluate other invariants arising from the pairing of a JLO cocycle in non-commutative geometry, see Sections X–XI. Both examples arise from the study of a first-order Dirac operator $D(\lambda)$ depending on a parameter $\lambda > 0$, and the associated self-adjoint operator

$$Q(\lambda) = D(\lambda) + D(\lambda)^*.$$

These operators both act on a Hilbert space $\mathscr{H}$. The operator $D(\lambda)$ is a function of a holomorphic polynomial $\lambda V(z)$, called a *superpotential.* Actually $D(\lambda)$ is an affine function of the gradient $\lambda \overline{\partial V(z)}$ of $\lambda \overline{V(z)}$, where $z \in \mathbb{C}^n$. We make two sorts of assumptions about $V(z)$. First we assume that $V(z)$ satisfies an elliptic estimate. This estimate ensures the growth of the absolute value of the gradient $|\partial V(z)|$ as $|z| \to \infty$, so $|\partial V(z)|$ has no "flat directions." As a consequence, we prove that $Q(\lambda)$ is essentially self-adjoint on its domain of definition $\mathscr{D}$; also the spectrum of $Q(\lambda)$ is discrete, and the magnitude of the eigenvalues grows sufficiently fast. See Subsection II.4 for details. Second, we assume that $V(z)$ is a quasi-homogeneous polynomial. Quasi-homogeneity is a property describing different homogeneous behavior in different coordinate directions. A class of quasi-homogeneous polynomials V is defined by a set of n rational numbers $\omega = \{\omega_j\}$, called the *weights*, lying in the interval $(0, \frac{1}{2}]$. See Subsection II.5. There is a $\mathbb{Z}_2$-grading γ that satisfies

$$\gamma Q(\lambda) = -Q(\lambda)\, \gamma.$$

There also exists a symmetry group $U(\theta)$ that implements quasi-homogeneity on $\mathscr{H}$. This group commutes with both $Q(\lambda)$ and also $H(\varepsilon, \lambda)$, and that depends on $V(z)$ only through its weights ω. In particular, for fixed V the symmetry $U(\theta)$ does not depend on the parameter λ. The Hamiltonian $H(0, \lambda) = Q(\lambda)^2$ has a symmetry known as "supersymmetry." Our regularization of the heat kernel destroys this natural symmetry, but the holonomy expansion gives us quantitative control of the error terms in traces of $e^{-H(\varepsilon, \lambda)}$ and how supersymmetry is recovered as $\varepsilon \to 0$.

With these assumptions, we develop a method to evaluate the character-valued index,

$$\mathfrak{Z}(\lambda; \theta) = \mathrm{Tr}_{\mathscr{H}}(\gamma U(\theta)\, e^{-Q(\lambda)^2}), \qquad \text{for} \quad \lambda > 0. \tag{I.1}$$

In later sections of the paper we generalize this method to the following class of invariants arising from non-commutative geometry. Consider a bounded operator a on $\mathscr{H}$ which is a square root of one, $a^2 = I$, and invariant under the groups Z_2 and $U(1)$ implemented by γ and $U(\theta)$. We assume that the derivative

$$d_\lambda a = [\, Q(\lambda), a\,]$$

is a bilinear form in an appropriate space. Then

$$\mathfrak{Z}(\lambda; a; \theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \mathrm{Tr}(\gamma U(\theta)\, a e^{-Q(\lambda)^2 + it\, d_\lambda a})\, dt \tag{I.2}$$

is such an invariant, and for $a = I$ it reduces to the equivariant index (I.1). See [QHA] for the details of the construction, the fact that (I.2) is an invariant, and its relation to entire cyclic cohomology. In [QI] a simple, direct proof of the invariance of (I.2) is given by finding a representation of (I.2) for which it is clear that

$$\frac{\partial \mathfrak{Z}(\lambda; a; \theta)}{\partial \lambda} = 0.$$

In particular examples, the existence of the exponential, of the trace, and the integral in this representation all need to be established. Furthermore, we require differentiability of $\mathfrak{Z}(\lambda; \theta)$ with respect to λ, and the exchange of the order of differentiation with the trace and the integral in (I.2). The existence and interchangeability of all these limits are a consequence of *a priori* estimates. In [QHA] we reduce these regularity properties to a set of elementary assumptions on the regularity of $Q(\lambda)$ as a function of λ, and to a bound on $[\, Q(\lambda), a\,]$; we verify these assumptions in our example studied here. The invariant (I.2), up to a constant, arises from pairing the cochain introduced by Lesniewski, Osterwalder, and the author in [JLO]. We denote this cochain τ^{JLO}.

I.1. *Some Details about the Character-Valued Index*

In Section II we define a certain Dirac operator $Q(\lambda)$. In Section III we introduce the symmetry operator $U(\theta)$, and in Section VI we show that $Q(\lambda)$ is essentially self-adjoint. In Section VII, we show that the heat kernel $\exp(-Q(\lambda)^2)$ is trace class on $\mathscr{H}$ for $0<\lambda$. In Section VIII, we show that it is differentiable with respect to λ in a suitably strong sense. The operator $Q_+(\lambda)$ is the part of $Q(\lambda)$ mapping the positive eigenspace of γ to the negative eigenspace. The index (I.1) is the character-valued index of $Q_+(\lambda)$ with respect to the group $U(\theta)$ and the grading γ. For the first nine sections of this paper we study this index for $\lambda\in(0,1]$, and we develop a method to evaluate the index at the singular endpoint $\lambda=0$.

We work on the Hilbert space $\mathscr{H}=L^2(\mathbb{C}^n)\otimes\bigwedge\mathbb{C}^{2n}$, see Section II. The first factor is the bosonic Hilbert space. With z the coordinate on $\mathbb{C}^n$, the multiplication operator V on $L^2(\mathbb{C}^n)$ acts on $\mathscr{H}$ as $V(z)\otimes I$. The fermionic Hilbert space is the factor $\bigwedge\mathbb{C}^{2n}$ in $\mathscr{H}$, where j denotes the fermion number. This space is the exterior algebra over $\mathbb{C}^{2n}$, namely

$$\bigwedge\mathbb{C}^{2n}=\bigoplus_{j=0}^{2n}\bigwedge^{j}\mathbb{C}^{2n},$$

and it has dimension 4^n. Define γ^f on $\bigwedge\mathbb{C}^{2n}$ as $(-1)^j$ on $\bigwedge^j\mathbb{C}^{2n}$, and let γ act on $\mathscr{H}$ as $\gamma=I\otimes\gamma^f$. Thus $\gamma=(-I)^{N^f}$ is a $\mathbb{Z}_2$-grading, where N^f denotes the fermion number operator with eigenspaces $\bigwedge^j\mathbb{C}^{2n}$ with eigenvalues j. Also

$$\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_- \tag{I.3}$$

is the decomposition of $\mathscr{H}$ into the $(+1)$-eigenspace $\mathscr{H}_+$ and (-1)-eigenspace $\mathscr{H}_-$ of γ.

The operator $D(\lambda)$ will be defined so that on its domain,

$$D(\lambda)\colon L^2(\mathbb{C})\otimes\bigwedge^{j}\mathbb{C}^{2n}\to L^2(\mathbb{C})\otimes\left(\bigwedge^{j+1}\mathbb{C}^{2n}\oplus\bigwedge^{j-1}\mathbb{C}^{2n}\right). \tag{I.4}$$

Here $\bigwedge^{j-1}\mathbb{C}^{2n}=0$ if $j=0$, and $\bigwedge^{j+1}\mathbb{C}^{2n}=0$ if $j=2n$. As $D(\lambda)$ changes the fermion number by 1, it is odd with respect to γ, and

$$\gamma D(\lambda)=-D(\lambda)\,\gamma,\qquad \gamma Q(\lambda)=-Q(\lambda)\,\gamma. \tag{I.5}$$

The explicit form of $D(\lambda)$, and its dependence on both the parameter λ and the potential function $V(z)$, is given in (II.20). If we wish to emphasize the dependence of D on V, we write $D(V)$ in place of $D(\lambda)$.

We study $\mathfrak{Z}(\lambda;\theta)$ defined by (I.1), which for $\theta=0$ reduces to the integer-valued index

$$\dim\operatorname{Null}(Q\restriction\mathscr{H}_+)-\dim\operatorname{Null}(Q\restriction\mathscr{H}_-). \tag{I.6}$$

As a constant function of λ, it has the limit

$$\mathfrak{Z}(0;\theta)=\lim_{\lambda\to 0}\mathfrak{Z}(\lambda;\theta). \tag{I.7}$$

However, $Q(0)^2=-\partial\bar{\partial}$, and $e^{-Q(0)^2}=e^{\partial\bar{\partial}}$ is the heat kernel of the Laplacian—which does not have a trace. Therefore

$$\operatorname{Tr}(\gamma U(\theta)\,e^{\partial\bar{\partial}}) \tag{I.8}$$

is basis dependent, and the trace needs to be regularized. On the other hand, the presence of the symmetry operator $U(\theta)$ partially regularizes the behavior of the trace near $(\varepsilon,\lambda)=(0,0)$, even though it is not evident at this point for $\lambda>0$.

In order to regularize $\mathfrak{Z}(\lambda;\theta)$, we replace $H(\lambda)=Q(\lambda)^2$ with an approximate Hamiltonian

$$H(\varepsilon,\lambda)=Q(\lambda)^2+\varepsilon^2\,|z|^2. \tag{I.9}$$

Correspondingly, one can replace the function $\mathfrak{Z}(\lambda;\theta)$ with

$$\mathfrak{Z}(\varepsilon,\lambda;\theta)=\operatorname{Tr}(\gamma U(\theta)\,e^{-H(\varepsilon,\lambda)}). \tag{I.10}$$

The regularizing term $\varepsilon^2\,|z|^2$ commutes with both γ and $U(\theta)$. It is easy to diagonalize simultaneously the three mutually-commuting operators $U(\theta)$, γ, and $H(\varepsilon,0)$. Unfortunately, the trace $\mathfrak{Z}(\varepsilon,\lambda;\theta)$ with $\varepsilon\neq 0$ is no longer an index.

It is natural to ask whether

$$\lim_{\varepsilon\to 0}\mathfrak{Z}(\varepsilon,0;\theta)=\lim_{\varepsilon\to 0}\,\lim_{\lambda\to 0}\mathfrak{Z}(\varepsilon,\lambda;\theta), \tag{I.11}$$

actually equals the desired index, which is

$$\lim_{\lambda\to 0}\,\lim_{\varepsilon\to 0}\mathfrak{Z}(\varepsilon,\lambda;\theta)=\operatorname{Tr}(\gamma U(\theta)\,e^{-Q(\lambda)^2}).$$

The main result in this paper is to develop a method to show that one can interchange these limits, as long as θ is in the complement of a discrete set of singular points Υ_{sing}. In Corollary IV.7 we show that for $\theta\notin\Upsilon_{\mathrm{sing}}$,

$$\lim_{\varepsilon\to 0}\,\lim_{\lambda\to 0}\mathfrak{Z}(\varepsilon,\lambda;\theta)=\lim_{\lambda\to 0}\,\lim_{\varepsilon\to 0}\mathfrak{Z}(\varepsilon,\lambda;0). \tag{I.12}$$

The introduction of ε, and the resulting analysis of the joint continuity of $\mathfrak{Z}(\varepsilon,\lambda;\theta)$ as a function of ε and λ, is also convenient for calculations. In fact the $\varepsilon>0$, $\lambda=0$ problem can be calculated in closed form, and this fact gives a convenient way to evaluate $\mathfrak{Z}(0,\lambda;\theta)$. The quantity on the left side of (I.12) can be computed, while the quantity on the right of (I.12) is the desired equivariant index.

We show in Proposition VIII.6 that $\mathfrak{Z}(\varepsilon,\lambda;\theta)$ is continuous in θ for each fixed (ε,λ) with $\varepsilon+\lambda>0$. We also know that on the $\varepsilon=0$ axis, $\mathfrak{Z}(0,\lambda;\theta)$ is independent of λ. We therefore infer that $\mathfrak{Z}(0, \lambda; \theta)$, whose values we know for almost all θ (by the method explained above) extends by continuity to *all* values of θ. Once we have obtained the value of $\mathfrak{Z}(\lambda; \theta)$, we can pass to the limit $\lambda \to 0$, with λ fixed. Then we can perform a perturbation replacing $V(z)$ by $V(z)+\mu V_1(z)$ that satisfies the elliptic estimate Assumption E uniformly in μ, but that is no longer quasi-homogeneous, except for $\mu = 0$.

In the physics literature, interchanges of limits are normally taken for granted, but a ratio of quadratic polynomials shows this is not necessarily the case. Consider, for example, the ratio $f(\varepsilon, \lambda)$ of the geometric mean to the arithmetic mean of ε^2 and $(\varepsilon+\lambda)^2$. In the "unit square" $0<\varepsilon, \lambda \leqslant 1$, the function $f(\varepsilon, \lambda) = 2\varepsilon(\varepsilon+\lambda)/(\varepsilon^2+(\varepsilon+\lambda)^2)$ varies between 0 and 1. In the interior f is real analytic in each variable (ε, λ), and on the two axes, f is constant. But $f=0$ on the λ-axis, while $f=1$ on the ε-axis, and any limit of f between 0 and 1 may be obtained as $(\varepsilon, \lambda) \to (0, 0)$ in the unit square. We are concerned here with this limit for the function $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ with θ fixed, but as we know even less about the behavior of $\mathfrak{Z}$ than about f, we must establish some additional information about $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ in order to justify the interchange of limits (I.12). To answer this question we bound the components of the gradient of f in the unit square, showing that for $\theta \notin \Upsilon_{\mathrm{sing}}$,

$$|\mathfrak{Z}_\varepsilon(\varepsilon, \lambda; \theta)| = |\partial \mathfrak{Z}(\varepsilon, \lambda; \theta)/\partial\varepsilon| \leqslant M\varepsilon, \tag{I.13}$$

and

$$|\mathfrak{Z}_\lambda(\varepsilon, \lambda; \theta)| = |\partial \mathfrak{Z}(\varepsilon, \lambda; \theta)/\partial\lambda| \leqslant M\varepsilon^2. \tag{I.14}$$

Here the constant M is independent of ε and λ, but depends on V and on $\theta \notin \Upsilon_{\mathrm{sing}}$. As a consequence,

$$|\mathfrak{Z}(\varepsilon, 0; \theta) - \mathfrak{Z}(0, \lambda; \theta)| \leqslant 2M\varepsilon^2, \tag{I.15}$$

so the limits (I.12) agree.

THEOREM I.1. *Assume that $V(z)$ is a holomorphic polynomial satisfying both the elliptic bound* (*Assumption* E *in Subsection* II.4) *and the quasi-homogeneity property with weights $\omega = \{\omega_1, ..., \omega_n\}$* (*namely Assumption* Q *in Subsection* II.5). *Define γ, $Q(\lambda)$, and $H(\varepsilon, \lambda)$ by* (I.3), (I.9) *and* (II.20)–(II.23), *and let $0 \leqslant \varepsilon, \lambda$ and $0 < \varepsilon + \lambda$.*

(a) *The operators $Q(\lambda)$ and $H(\varepsilon, \lambda)$ are essentially self-adjoint, the heat kernel $e^{-H(\varepsilon, \lambda)}$ is trace class, and* (I.12) *is valid. Explicitly, we find that*

$$\mathfrak{Z}(\lambda; \theta) = \prod_{j=1}^{n} \frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j \theta/2)}. \tag{I.16}$$

(b) *The $\theta \to 0$ limit of* (I.16) *is the* (*integer-valued*) *index,*

$$\mathfrak{Z}(\lambda; 0) = \prod_{j=1}^{n} (\omega_j^{-1} - 1). \tag{I.17}$$

(c) *Let $V_1(z)$ be a holomorphic polynomial* (*not necessarily quasi-homogeneous*) *and let $M = M(V_1, V)$ be a constant such that V_1 is relatively bounded with respect to V in the sense that*

$$|\partial V_1| \leqslant |\partial V| + 1, \qquad \text{and} \qquad |\partial^j V_1| \leqslant M(|\partial V| + 1) \qquad \text{for} \quad 2 \leqslant |j|.$$

Then for $-1 \leqslant \mu \leqslant 1$, the index of $Q(V+\mu V_1)$ also equals (I.17). *In particular,*

$$\mathfrak{Z}(V+\mu V_1; \theta=0) = \operatorname{Tr}(\gamma e^{-Q(V+\mu V_1)^2}) = \prod_{j=1}^{n} (\omega_j^{-1} - 1). \tag{I.18}$$

Remark. We demonstrate in Subsection IX.3 how the holonomy expansion can be used to give leading asymptotics for $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ in the variables (ε, λ), with fixed $\theta \notin \Upsilon_{\text{sing}}$. For example, up to order four in (ε, λ),

$$\mathfrak{Z}(\varepsilon, \lambda; \theta) = \left(\prod_{j=1}^{n} \frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j \theta/2)} \right) \left(1 + \sum_{1 \leqslant i+j \leqslant 4} a_{ij}(\theta)\, \varepsilon^i \lambda^j + O(\varepsilon^2 \lambda^3) \right).$$

To this order, only a_{20}, a_{22}, and a_{40} are non-zero, and they are given in (IX.51). The terms a_{j0} could also be obtained from the closed form expression (IX.3), (IX.4) for $\mathfrak{Z}(\varepsilon, 0; \theta)$. However, only by using the holonomy expansion can we establish that the expansion in ε and λ is asymptotic.

I.2. *A General Class of Invariants*

We extend the method of the holonomy expansion to certain invariants of the form $\mathfrak{Z}(\lambda; a; \theta)$ of the form (I.2) above. We show that for $\lambda > 0$, our problem lies in the framework of [QHA]. We establish bounds on expectations of the form

$$\tau_k^{\text{JLO}}(a_0, a_1, ..., a_k; \lambda; \theta) = \int_{s_j > 0} \operatorname{Tr}(\gamma U(\theta)\, a_0 e^{-s_0 Q(\lambda)^2}\, d_\lambda a_1\, e^{-s_1 Q(\lambda)^2} \cdots d_\lambda a_k\, e^{-s_k Q(\lambda)^2})$$
$$\times \delta\left(1 - \sum_{j=1}^{k} s_j\right) ds_0\, ds_1 \cdots ds_k, \tag{I.19}$$

where the a_j belong to a certain algebra $\mathfrak{A}$. We build the invariant $\mathfrak{Z}(\lambda; a; \theta)$ as a convergent series of these expectations, and we establish properties of the invariant based on properties of the expectations. In particular, we establish in this paper that for the algebras $\mathfrak{A}$ which we choose

$$|||\tau_k^{\text{JLO}}||| \leqslant m^{k+1} \operatorname{Tr}(e^{-Q(\lambda)^2/2}) \frac{1}{k!}, \tag{I.20}$$

$m < \infty$ is a constant. This bound allows us to define $\mathfrak{Z}(\lambda; a; \theta)$ as a convergent series in the τ_k^{JLO}. We also show that $\mathfrak{Z}(\lambda; a; \theta)$ is appropriately differentiable in λ, from which we infer that $\mathfrak{Z}(\lambda; a; \theta)$ is an invariant. Let us introduce the class $\mathfrak{V}^\omega$ of potentials with the two properties:

(i) Every $V(z) \in \mathfrak{V}^{\omega}$ is a holomorphic polynomial which satisfies the elliptic Assumption E of Subsection II.4.

(ii) Every $V(z) \in \mathfrak{V}^{\omega}$ is quasi-homogeneous with given weight ω, as described in Assumption Q of Subsection II.5.

We say that a subset $\mathfrak{V}_0^{\omega} \subset \mathfrak{V}^{\omega}$ is *uniformly bounded* if the estimates (II.37)–(II.38) hold uniformly for all $V(z) \in \mathfrak{V}_0^{\omega}$ with a given constant M, and the coefficients of the polynomial $V(z)$ are uniformly bounded. We say that $\mathfrak{V}_0^{\omega}$ is *connected* if it is path connected. We also define $\mathfrak{A}^{\theta} \subset \mathfrak{A}$ as the pointwise, $\gamma U(\theta)$-invariant subset of $\mathfrak{A}$. For

$$D(V) = \psi_1 \partial + \psi_2 \overline{\partial V}, \qquad \text{and} \qquad Q = Q(V) = D(V) + D(V)^*, \tag{I.21}$$

see (II.20) for details, define $\mathfrak{Z}(V; a; \theta)$ as in (I.2).

THEOREM I.2. *Let $V \in \mathfrak{V}_0^{\omega}$, where $\mathfrak{V}_0^{\omega}$ is a uniformly bounded, connected subset of $\mathfrak{V}^{\omega}$. Let $a \in \mathfrak{A}^{\theta}$, satisfy $a^2 = I$. Then the τ_k^{JLO} exist and satisfy the bounds* (I.20). *Furthermore, $\mathfrak{Z}(V; a; \theta)$ exists and it is independent of V.*

We return to study a particular potential of the form $\lambda V(z)$, and in the case of $\mathfrak{Z}(\lambda; \theta)$ above, $\mathfrak{Z}(\lambda; a; \theta)$ looks especially simple at the singular endpoint $\lambda = 0$ of the interval $(0, 1]$. To evaluate $\mathfrak{Z}(0; a; \theta) = \lim_{\lambda \to 0} \mathfrak{Z}(\lambda; a; \theta)$, as in the case of $\mathfrak{Z}(\lambda; \theta)$ discussed above, we must regularize the singularity at $\lambda = 0$. Introduce $H(\varepsilon, \lambda)$ defined in (I.9) in place of $Q(\lambda)^2$, yielding the regularized expression $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$, which is an invariant only on the λ axis in the (ε, λ) plane. In principle, the holonomy expansion allows us to evaluate these invariants by a method similar to the one above, namely by calculating $\mathfrak{Z}(\varepsilon, 0; a; \theta)$. We begin by studying

$$\mathfrak{Z}(\varepsilon, \lambda; a; \theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \operatorname{Tr}(\gamma U(\theta)\, a e^{-H(\varepsilon, \lambda) + i t d_{\lambda} a})\, dt, \tag{I.22}$$

for $\varepsilon + \lambda > 0$. The method explained above of exchanging limits can also be used for certain invariants $\mathfrak{Z}(\lambda; a; \theta)$, but we need to choose the algebra $\mathfrak{A}$ as a (small) subalgebra of the algebras considered in [QHA]. We give the details in Sections X and XI, where we restrict $\mathfrak{A}$ to be one of two quite concrete examples, $\mathfrak{A}_1$ or $\mathfrak{A}_2$. These algebras are unital, Banach-subalgebras of $\mathscr{B}(\mathscr{H})$ on which the holonomy expansion can be performed. The algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ have somewhat complicated norms $|||\cdot|||_1$ and $|||\cdot|||_2$ which we generically denote $|||\cdot|||$. We postpone a discussion of details to Section X. In Section XI we establish the required estimates on expectations using a holonomy expansion. These estimates are not quite as precise as the previous estimates as far as the small (ε, λ)-asymptotics are concerned. However, we do prove sufficiently strong bounds so that we may conclude, for each fixed $\theta \notin \Upsilon_{\mathrm{sing}}$, that the limits $\varepsilon \to 0$ and $\lambda \to 0$ of the partition function $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$ defined in (I.22) exist, and that they can be interchanged. We find that there is a constant $M = M(V, \theta)$, such that the functionals τ_k^{JLO} on the diagonal, namely $\tau_k^{\mathrm{JLO}}(a, a, ..., a; \varepsilon, \lambda; \theta)$ satisfy the bound

$$|\tau_k^{\mathrm{JLO}}| \leqslant M^{k+1} \frac{1}{k!} |||a|||^{k+1}. \tag{I.23}$$

Furthermore, the derivatives with respect to ε and to λ are similarly bounded,

$$\left|\frac{\partial \tau_k^{\mathrm{JLO}}}{\partial \varepsilon}\right| + \left|\frac{\partial \tau_k^{\mathrm{JLO}}}{\partial \lambda}\right| \leqslant M^{k+1} \frac{1}{k!} |||a|||^{k+1}. \tag{I.24}$$

As a consequence, we have the following result:

Theorem I.3. *Consider* $V \in \mathfrak{V}^{\omega}$ *for some* ω. *Assume* $a \in \mathfrak{A}$, *for* $\mathfrak{A} = \mathfrak{A}_1$ *or* $\mathfrak{A} = \mathfrak{A}_2$ *as defined in Section* X. *Then* $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$ *is continuous in* θ *for each* (ε, λ). *Furthermore, for each fixed* $\theta \notin \Upsilon_{\mathrm{sing}}$, *the bounds* (I.23)–(I.24) *hold in* $0 \leqslant \varepsilon, \lambda \leqslant 1$ *uniformly in* (ε, λ), *with a constant* $M = M(V, \theta)$. *Hence*

$$\mathfrak{Z}(0, \lambda; a; \theta) = \lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon, 0; a; \theta). \tag{I.25}$$

Since we show that $\mathfrak{Z}(0, \lambda; a; \theta)$ is continuous in θ, our determination of the invariant $\mathfrak{Z}(0, \lambda; a; \theta)$ for almost all values of θ uniquely specifies $\mathfrak{Z}(0, \lambda; a; \theta)$ for all θ.

I.3. *Expectations*

In the course of our study, we encounter traces of the general form $\mathrm{Tr}(T_0 T_1 \cdots T_k)$, where each T_j, $0 \leqslant j \leqslant k$, is a bounded, trace class operator of the form

$$T_j = X_j e^{-s_j H}, \qquad 0 < s_j. \tag{I.26}$$

If $X_j \in \mathscr{B}(\mathscr{H})$, then as a consequence of the fact that $e^{-s_j H}$ is trace class, it follows that each T_j is trace class. If $X_0 = \gamma U(\theta)\, a_0 \in \mathscr{B}(\mathscr{H})$ and $X_j = da_j \in \mathscr{B}(\mathscr{H})$, for each $1 \leqslant j \leqslant k$, then the integrand in (I.19) is a special case. The s_j's are assumed periodic in the index j, so $s_{-1} = s_k$. They also lie in the hyperplane

$$\sum_{j=0}^{k} s_j = 1. \tag{I.27}$$

The $\{s_j\}$'s can therefore be interpreted as positive displacements (or time differences) along a unit time interval $[0, 1]$.

Cyclicity of the trace is the relation

$$\mathrm{Tr}(T_0 T_1 \cdots T_k) = \mathrm{Tr}(T_k T_0 T_1 \cdots T_{k-1}), \tag{I.28}$$

so it is natural to replace the time interval $[0, 1]$ by a periodic time on a unit circle. We represent the trace (I.28) as a graph, with the operator X_0 denoted by a vertex at time $t=0$, while each X_j is denoted by a vertex at time $t_j=\sum_{l=0}^{j-1} s_l$, for $j=1, 2, ..., n$.

Furthermore, a Hölder inequality shows that the absolute value of (I.28) is bounded independently of the $\{s_j\}$, by

$$|\mathrm{Tr}(T_0 \cdots T_k)| \leqslant \mathrm{Tr}(e^{-H}) \prod_{j=0}^{k} \|T_j\|.$$

Thus the integral over this hyperplane exists, and

$$\int_{s_j>0} \mathrm{Tr}(T_0 \cdots T_k)\, \delta\left(1-\sum_{j=0}^{k} \delta_j\right) ds_0\, ds_1 \cdots ds_k, \tag{I.29}$$

represents an integral of a $(k+1)$-vertex circular graph. The vertices in the graph denote the location of the operators X_j at the ordered set of times $0<t_0<t_1<\cdots<t_k<1$. This defines a $(k+1)$-multilinear functional on $\mathscr{B}(\mathscr{H})$. It is convenient to define a sequence of θ-dependent expectations, given by the twisted trace. The twist arises from the unitary operator $\gamma U(\theta)$. Given $k \in \mathbb{Z}_+$, let

$$\begin{aligned}\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle_k = &\int_{s_j>0} \mathrm{Tr}(\gamma U(\theta)\, X_0 e^{-s_0 H} X_1 e^{-s_1 H} \cdots X_k e^{-s_k H}) \\ &\times \delta\left(1-\sum_{j=0}^{k} \delta_j\right) ds_0\, ds_1 \cdots ds_k.\end{aligned} \tag{I.30}$$

The expectation (I.30) extends by continuity from bounded operators $X_j \in \mathscr{B}(\mathscr{H})$ to a class X_j of operator-valued distributions. Define the Sobolev space $\mathscr{H}_\alpha$, for $0 \leqslant \alpha$ as the domain of $H(\varepsilon, \lambda)^{\alpha/2}$, with inner product

$$\langle f, g\rangle_{\mathscr{H}_\alpha} = \langle (I+H(\varepsilon, \lambda))^{\alpha/2} f, (I+H(\varepsilon, \lambda))^{\alpha/2} g\rangle_{\mathscr{H}}. \tag{I.31}$$

For $\alpha \geqslant 0$ the space $\mathscr{H}_{-\alpha}$ is the completion of $\mathscr{H}$ in the norm given by the inner product

$$\langle f, g\rangle_{\mathscr{H}_{-\alpha}} = \langle (I+H(\varepsilon, \lambda))^{-\alpha/2} f, (I+H(\varepsilon, \lambda))^{-\alpha/2} g\rangle_{\mathscr{H}}.$$

The spaces $\mathscr{H}_\alpha$ and $\mathscr{H}_{-\alpha}$ are canonically dual to each other. Then define the class $\mathscr{T}(-\beta, \alpha)$ of operator-valued distributions as the linear space of continuous linear transformations between two Sobolev spaces $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$. Here $0 \leqslant \alpha, \beta$. A natural norm on the space $\mathscr{T}(-\beta, \alpha)$ is

$$\|X\|_{\mathscr{T}(-\beta, \alpha)} = \|(I+H(\varepsilon, \lambda))^{-\beta/2}\, X (I+H(\varepsilon, \lambda))^{-\alpha/2}\|. \tag{I.32}$$

The space of all bounded operators $\mathscr{B}(\mathscr{H})$ on $\mathscr{H}$ is naturally imbedded in $\mathscr{T}(-\beta, \alpha)$ and is a dense subspace of $\mathscr{T}(-\beta, \alpha)$ in this norm.

Note that both $\mathscr{H}_\alpha$ and $\mathscr{T}(-\beta, \alpha)$ have an implicit (ε, λ)-dependence. When there may be a source of confusion, we denote $\mathscr{H}_\alpha$ by $\mathscr{H}_\alpha(\varepsilon, \lambda)$. We let $\mathscr{T}_{\varepsilon, \lambda}(-\beta, \alpha)$ denote the space of bounded linear transformations from $\mathscr{H}_\alpha(\varepsilon, \lambda)$ to $\mathscr{H}_{-\beta}(\varepsilon, \lambda)$. We only need consider the same (ε, λ) in $\mathscr{H}_\alpha(\varepsilon, \lambda)$ and in $\mathscr{H}_{-\beta}(\varepsilon, \lambda)$. With $R = R(\varepsilon, \lambda) = (H(\varepsilon, \lambda) + I)^{-1/2}$, we then write

$$\|X\|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta, \alpha)} = \|R^\beta X R^\alpha\|. \tag{I.33}$$

Consider the set vertices, or the ordered n-tuple of operator-valued distributions

$$\{X_0, X_1, ..., X_n\} \in \mathscr{T}_{\varepsilon, \lambda}(-\beta_0, \alpha_0) \times \mathscr{T}_{\varepsilon, \lambda}(-\beta_1, \alpha_1) \times \cdots \times \mathscr{T}_{\varepsilon, \lambda}(-\beta_n, \alpha_n).$$

Here $0 \leqslant \alpha_j, \beta_j$, for $0 \leqslant j \leqslant k$, $\beta_{k+1} = \beta_0$, and

$$\eta_j = \tfrac{1}{2}(2 - \alpha_j - \beta_{j+1}).$$

We require that

$$0 < \eta_j, \qquad \text{for each} \quad j = 0, 1, ..., k. \tag{I.34}$$

Let

$$\eta_{\min} = \min_j \{\eta_j\}, \qquad \text{and} \qquad \eta_{\text{tot}} = \sum_{j=0}^{k} \eta_j. \tag{I.35}$$

Definition I.4. A set of vertices $\{X_0, X_1, ..., X_k\}$ for which $0 < \eta_j$ for all j is called a *regular set* of (operator-valued distribution) vertices with respect to $H(\varepsilon, \lambda)$.

Remark 1. An operator-valued distribution vertex X_j may actually be an unbounded operator on $\mathscr{H}$. This class of vertices arises throughout our expansion.

Remark 2. Although all the $\alpha_j, \beta_j \in [0, 2)$ as a consequence of (I.34), it is possible that certain of the $\alpha_j + \beta_j$ are close to 4. For example, a Q^3 vertex may occur as the jth-vertex in a regular set of vertices, if the vertices to the left and to the right have $\alpha_{j-1} + \beta_{j+1} < 1$.

Remark 3. As λ varies, the exponents α_j, β_j, and η_j could also depend on λ; but we do not consider this case.

We establish the following result as an application of Proposition VIII.6:

Theorem I.5. *Let $V(z)$ be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *In the domain $0 \leqslant \varepsilon, \lambda \leqslant 1$, with also $0 < \varepsilon + \lambda$, we have the following estimates*:

(a) *The expectation* (I.30) *extends by continuity from sets of bounded, operator-valued vertices*

$$\{X_0, X_1, ..., X_k\} \in \mathscr{B}(\mathscr{H})^{k+1}$$

to regular sets of operator-valued distribution vertices

$$\{X_0, X_1, ..., X_k\} \in \mathscr{T}_{\varepsilon,\lambda}(\beta_0, \alpha_0) \times \mathscr{T}_{\varepsilon,\lambda}(-\beta_1, \alpha_1) \times \cdots \times \mathscr{T}_{\varepsilon,\lambda}(-\beta_k, \alpha_k).$$

We denote this extended expectation also by

$$\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle_k. \tag{I.36}$$

(b) *For each* (ε, λ), *the expectation* (I.36) *is continuous in* θ.

(c) *For each* θ, *the expectation* (I.36) *is jointly continuous in* (ε, λ).

(d) *The expectation* (I.36) *is bounded by*

$$|\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle_k| \leqslant \frac{(4\Gamma(\eta_{\min}))^{k+1}}{\Gamma(\eta_{\text{tot}})} \operatorname{Tr}(e^{-H(\varepsilon,\lambda)/2}) \left(\prod_{j=0}^{k} \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}\right). \tag{I.37}$$

In order to study invariants, we need some additional analytic assumptions on those vertices $a \in \mathfrak{A}$ which arise in quantities such as (I.19). For $\varepsilon + \lambda > 0$ we use

$$a \in \mathscr{B}(\mathscr{H}) \qquad \text{with} \quad da \in \mathscr{T}(-\beta, \alpha) \quad \text{for some } \alpha + \beta < 1. \tag{I.38}$$

In fact the space of a's with these restrictions is exactly the interpolation space $\mathfrak{J}_{\beta,\alpha} = \mathfrak{J}_{\beta,\alpha}(\lambda)$. We give $\mathfrak{J}_{\beta,\alpha}$ the norm

$$\|a\|_{\mathfrak{J}_{\beta,\alpha}} = \|a\| + c_{\alpha+\beta} \|da\|_{\mathscr{T}(-\beta,\alpha)}, \tag{I.39}$$

with the constant $c_{\alpha+\beta}$ sufficiently large. As in Corollary V.6 of [QHA], we have the following:

Theorem I.6. *The space* $\mathfrak{J}_{\beta,\alpha}$ *is a Banach algebra with unit, and furthermore for* $a, b \in \mathfrak{J}_{\beta,\alpha}$, *the Leibniz rule for differentiation holds,*

$$d(ab) = (da)\, b + a^{\gamma}(db), \tag{I.40}$$

where each term belongs to $\mathscr{T}(-\beta, \alpha)$, *and where* $a^{\gamma} = \gamma a \gamma$.

As previously remarked, the spaces $\mathfrak{J}_{\beta,\alpha}$ are too large to carry out the holonomy expansion for $\mathfrak{Z}(\lambda; a; \theta)$ or for the τ_j^{JLO}. In order to study $\lambda \to 0$, we introduce in Section X certain Banach sub-algebras $\mathfrak{A}_j$ of $\mathfrak{J}_{\beta,\alpha}$ that are suitable for the expansion, and we choose $\mathfrak{A}$ to be one of these sub-algebras. The operators $a \in \mathfrak{A}$ which occur at the start of the expansion belong to $\mathfrak{A}$, but as the expansion progresses, it gives rise to "new vertices," namely vertices of a sort that cannot be present at the start of the expansion as these vertices are not necessarily in the algebra $\mathfrak{A}$. However, these new vertices X_j *do* need to belong to some $\mathcal{T}_{\varepsilon,\lambda}(-\beta_j, \alpha_j)$, for it is on such a space that we define the expectations of the form (I.36). Furthermore, the set of all vertices present in any $(k+1)$-vertex term of the expansion must be a regular set that satisfies the conditions (I.34).

The proof that a vertex is an element of some $\mathcal{T}_{\varepsilon,\lambda}(-\beta, \alpha)$ is an *a priori* estimate. We need to prove such an estimate for *each* vertex that we generate during the course of the expansion. Vertices in $\mathcal{T}_{\varepsilon,\lambda}(-\beta, \alpha)$ may in fact be unbounded operators on $\mathcal{H}$. For example, we may obtain vertices such as $\varepsilon\,|z_j|^2$, λW, $\lambda\partial_j V$, or other derivatives of λV in this way. The estimates that we establish as a consequence of the elliptic bound on the potential V actually show that the vertices which arise in the expansion yield convergence factors ε or λ. The longer we carry out the expansion, the more such factors accumulate. It is these factors that give the desired regularity at the origin, and that dominate the *a priori* divergent factors in (I.18) as $(\varepsilon, \lambda) \to (0, 0)$.

I.4. *The Holonomy Expansion*

The holonomy expansion is a succession of elementary *moves*. Each elementary move replaces a $(k+1)$-vertex expectation of the form (I.30) by a finite sum of similar expectations, each of which have $(k+1)$ or $(k+2)$ vertices. The elementary moves will be combined into elementary *steps*, that consist of a coherent collection of a finite number of related moves. We formulate in Section XI a set of rules that we use to define the expansion of a given term. The expansion terminates after a finite number of steps. The expansion is made up from two sorts of elementary moves.

I.4.1. *Holonomy Moves*

The first of these moves are called *holonomy moves*, after which we name the expansion. A holonomy move for an expectation (I.30) has the following structure. Suppose we are expanding $\langle X_0, X_1, ..., X_k; \lambda, \theta\rangle$. We consider one vertex, say X_j, and generally isolate a particular factor in X_j, say X, on which we perform the holonomy. We generate the X-holonomy move by translating X in one time direction around the circle representing the trace, finally arriving back at its original position. We choose the operator X so that the result after the translation around the circle is the sum of two parts. The first part is exactly equal to the original term, multiplied by a phase σ, namely $|\sigma| = 1$. This phase arises from the action of the groups $\mathbb{Z}_2$ and G on X, as given below. It is essential that $\sigma \neq 1$.

The second part of the outcome of the X-holonomy move, which we denote $\langle\ \rangle_{\text{Smoother}}$, will generally have some additional regularity as $\varepsilon + \lambda \to 0$. The terms "smoother" will be composed of a finite sum of expectations.

In fact, for at least a given, fixed fraction of the holonomy moves, the expectations denoted "smoother" will actually be so; each will have some additional factor proportional to ε or λ which is small as $\varepsilon + \lambda \to 0$. In the fraction of cases that the "smoother" expectations do not actually have additional powers of ε or of λ, the estimates on the "smoother" terms have the same behavior as $\varepsilon+\lambda\to 0$ as the original term being expanded—i.e., they are no worse. Thus after several moves we definitely obtain a given degree of improvement in the small $\varepsilon+\lambda$ behavior. The reason that we group expansion moves together into steps is that by doing so we may obtain steps each of which improve the small $\varepsilon+\lambda\to 0$ behavior.

The general form of the X-holonomy move will be

$$\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle = \sigma\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle + \langle\ \rangle_{\text{Smoother}};$$

which we rewrite as

$$\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle = (1-\sigma)^{-1}\langle\ \rangle_{\text{Smoother}}. \tag{I.41}$$

The best way to explain the expansion in detail is to illustrate a particular simple example of a holonomy move, which we do shortly.

I.4.2. *Perturbation Moves*

The second type of move which occurs in the holonomy expansion is a perturbation expansion of the expectation in the parameter ε or in the parameter λ. This move is an application of the fundamental theorem of calculus. The ε-perturbation move produces a leading term of the form $\langle X_0, X_1, ..., X_k; 1, \lambda; \theta\rangle$ plus error terms which are integrals of the derivative of $\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle$ with respect to ε. In other words, we write the factors such as $e^{-H(\varepsilon,\lambda)}$ as the integral of a derivative with respect to ε or to λ. We use these perturbation moves when we wish to generate new vertices on which to perform holonomy moves. The rules for the expansion are not unique; rather, different combinations of moves serve different purposes. We explain the basic moves below and also in Section IX. Other combinations of basic expansion steps can be useful for other problems.

I.4.3. *The Expansion*

Let us now look at an X-holonomy move in more detail. We require that the operator X being holonomied must transform under the groups $\mathbb{Z}_2$ and under $U(\theta)$ according to one-dimensional representations. In particular we suppose that for $U(\theta)$,

$$U(\theta)\, XU(\theta)^* = e^{i\mu\theta}X, \tag{I.42}$$

where μ belongs to a fixed, finite set of rational numbers. In fact, in the examples we give here, $\mu\in\{0, \pm\omega_j, \pm(1-\omega_j), \pm 1\}$. Likewise, for the group $\mathbb{Z}_2$, we have

$$\gamma X\gamma = (-1)^f X, \tag{I.43}$$

where $f=0$ for bosonic vertices X, and $f=1$ for fermionic X.

We illustrate the idea of an X-holonomy with the elementary choice, namely a z_1-holonomy. We focus on a vertex X_j that contains the coordinate z_1. For simplicity, suppose that the vertex is $X_1 = |z_1|^2$. As z_1 is even under γ, the representation (I.43) is the identity. As for (I.42), we have the defining relation for $U(\theta)$, namely

$$U(\theta)\, z_1 = e^{i\omega_1\theta} z_1\, U(\theta). \tag{I.44}$$

The idea of the z_1-holonomy is to translate z_1 in a given direction around the circle graph, back to its original position. In terms of operators, we move z_1 past each other operator in the trace, always in one particular time direction, generating a commutator error term from passing each other factor. A phase arises in this case only from z_1 passing through $U(\theta)$. The "smoother" terms in (I.41) are generated by commuting the z_1 with each of the other factors in the trace, namely the other vertices and the heat kernels.

The z_1-holonomy gives rise to an identity which is useful for θ outside the set on which $e^{i\omega_1\theta}=1$. As the weights ω_j are rational, this phase is periodic and in a period it equals 1 only at a finite number of points in each period (including $\theta=0$ and its translates by the period). We obtain the z_1-holonomy identity,

$$\begin{aligned}
&\langle |z_1|^2, X_1, ..., X_k, |z_1|^2; \varepsilon, \lambda; \theta\rangle_k \\
&\quad = e^{i\omega_1\theta}(1-e^{i\omega_1\theta})^{-1} \sum_{l=1}^{k} \langle \overline{z_1}, ..., [X_l, z_1], ..., X_k; \varepsilon, \lambda; \theta\rangle_k \\
&\qquad + e^{i\omega_1\theta}(1-e^{i\omega_1\theta})^{-1} \sum_{l=1}^{k} \langle \overline{z_1}, ..., [z_1, H(\varepsilon, \lambda)], X_l, ..., X_k; \varepsilon, \lambda; \theta\rangle_{k+1} \\
&\qquad + e^{i\omega_1\theta}(1-e^{i\omega_1\theta})^{-1} \langle \overline{z_1}, ..., X_k, [z_1, H(\varepsilon, \lambda)]; \varepsilon, \lambda; \theta\rangle_{k+1}.
\end{aligned} \tag{I.45}$$

The overall constant $e^{i\omega_1\theta}(1-e^{i\omega_1\theta})^{-1}$ is fixed for fixed θ, so it does not cause a problem on the complement of the discrete set of $\theta \in \Upsilon_{\text{sing}}$, which includes those θ at which $e^{i\omega_1\theta}=1$, as well as those values of θ for which other holonomies give similar vanishing factors.

So let us understand how the expansion improves the small $\varepsilon+\lambda$ estimates. Inspect the second sum and last term on the right side of (I.45). Note that the commutator $[z_1, H(\varepsilon, \lambda)]$ for our Hamiltonians can be evaluated and equals $\overline{\partial_1}$, namely the derivative operator $\partial/\partial\overline{z_1}$. This operator is independent of ε and λ, and it has a norm in $\mathscr{T}(0, 1)$ which is also bounded uniformly as $\varepsilon+\lambda \to 0$. Thus $[z_1, H(\varepsilon, \lambda)] = \overline{\partial_1}$ remains neutral as far as the $\varepsilon+\lambda \to 0$ behavior of the expectation is concerned.

However, allowing the replacement of the $|z_1|^2$-vertex in the kth-position, with a $\overline{z_1}$-vertex in the kth position, actually gives the improvement. The reason is that $|z_1|^2$ is bounded in $\mathscr{T}(-1, 1)$ with a norm $O(\varepsilon^{-2})$ as $\varepsilon+\lambda \to 0$. On the other hand, $\overline{z_1}$ is bounded in $\mathscr{T}(0, 1)$ with a norm that is $O(\varepsilon^{-1})$ as $\varepsilon+\lambda \to 0$. Both these estimates are consequences of Theorem VII.1. Thus we actually gain a relative factor of $O(\varepsilon)$ from each of the terms in the second sum on the right of (I.45), and from the final term of the expansion.

Now we return to the terms in the first sum on the right side of (I.45). These terms are problematic, as they involve commutators $[X_l, z_1]$ about which we may have no knowledge. This indicates that in the overall design of the expansion, we can use a z_1-holonomy only if every vertex X_l which is present at the start of the holonomy is of the form that its commutator with z_1 can be estimated. Thus the history of which vertices are introduced during the course of the expansion requires a detailed analysis. This is also true for all holonomy moves and perturbation moves that we employ. In fact we have been able to achieve the goal of designing an expansion with the possibility to estimate each term. But we leave the detailed explanation to Sections IX and XI where we give the various precise rules for generating the expansion.

I.5. *The Main Result*

Here we consider the invariant (I.2). In Subsection XI.4 we establish the following theorem that represents our main abstract result.

THEOREM I.7. *Assume that* $V \in \mathfrak{V}_0^\omega$ *is a holomorphic polynomial in a uniformly bounded set of potentials that satisfy both Assumptions* E *and Assumption* Q *of Subsections* II.4 *and* II.5. *Let* $a \in \mathfrak{A}_1$ *or* $a \in \mathfrak{A}_2$ *satisfy both* $a^2 = I$ *and* $U(\theta)\, a U(\theta)^* = a$ *for all* θ. *Then for* $\theta \notin \Upsilon_{\rm sing}$, *the invariant*

$$\begin{aligned}\mathfrak{Z}(a;\theta) &= \langle \tau^{\rm JLO}(\lambda;\theta), a\rangle \\ &= \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \mathrm{Tr}_{\mathscr{H}}\big(\gamma U(\theta)\, a e^{\partial\bar\partial + it\psi_1(\partial a) - it\psi_1^*(\bar\partial a) - \varepsilon^2 |z|^2}\big)\, dt. \qquad \text{(I.46)}\end{aligned}$$

As a function of θ, *the invariant* $\mathfrak{Z}(a;\theta)$ *is continuous, so the evaluation* (I.46) *established for* $\theta \notin \Upsilon_{\rm sing}$ *extends by continuity to all values of* θ.

Note in particular, that $\mathfrak{Z}(a;\theta)$ depends on V only through its quasi-homogeneous class ω. In fact the right-hand side of (I.46) (even before taking the $\varepsilon \to 0$ limit) depends on V only through $U(\theta)$, and $U(\theta)$ depends on V only through its quasi-homogeneous weights ω. On the other hand, we have *not* proved that the JLO-cocycle $\tau^{\rm JLO}(\varepsilon, \lambda; \theta)$ is defined for $(\varepsilon, \lambda) = (0, 0)$ as a cochain within the framework of [QHA]. Only the *invariant data* $\mathfrak{Z}$ obtained by pairing $\tau^{\rm JLO}$ with an appropriate a have been shown to have a limit that is regular in θ.

II. DEFINITIONS AND ASSUMPTIONS

II.1. *The Fock Space*

All operators act on the quantum mechanical Hilbert space

$$\mathscr{H} = \mathscr{H}^b \otimes \mathscr{H}^f, \qquad \text{where} \quad \mathscr{H}^b = L^2(\mathbb{C}^n), \quad \mathscr{H}^f = \bigwedge \mathbb{C}^{2n}. \qquad \text{(II.1)}$$

The spaces $\mathscr{H}^b$ and $\mathscr{H}^b$ are the bosonic and fermionic Hilbert spaces, respectively. We give the basic definitions.

II.1.1. *The Bosonic Subspace*

The bosonic variable $z \in \mathbb{C}^n$ acts as multiplication by the coordinate on $\mathbb{C}^n$. We normalize the Lebesque measure on $\mathbb{C}^n$ so that the Gaussian function

$$\Omega_0^b = e^{-z\bar{z}} = \prod_{j=1}^{n} e^{-z_j \bar{z}_j} \tag{II.2}$$

has norm 1. We express the coordinate z_j in terms of real and imaginary parts x_j and y_j, respectively, as $z_j = (1/\sqrt{2})(x_j + iy_j)$, $j = 1, 2, ..., n$. Thus $dz\, d\bar{z} = \prod_{j=1}^{n} dz_j\, d\bar{z}_j$, where $dz_j\, d\bar{z}_j = (1/\pi)\, dx_j\, dy_j$. Define annihilation operators $a_\alpha^{(j)}$ as

$$a_1^{(j)} = \frac{1}{\sqrt{2}}\left(x_j + \frac{\partial}{\partial x_j}\right), \qquad a_2^{(j)} = \frac{1}{\sqrt{2}}\left(y_j + \frac{\partial}{\partial y_j}\right). \tag{II.3}$$

With their adjoint creation operators, they satisfy the usual canonical commutation relations,

$$[a_\alpha^{(j)}, a_{\alpha'}^{(j')}] = 0, \qquad [a_\alpha^{(j)}, a_{\alpha'}^{(j')\,*}] = \delta_{j,\,j'} \delta_{\alpha,\,\alpha'} I.$$

The unitarily equivalent set of operators defined by

$$a_+^{(j)} = \frac{1}{\sqrt{2}}(a_1^{(j)} - ia_2^{(j)}), \qquad a_-^{(j)} = \frac{1}{\sqrt{2}}(a_1^{(j)} + ia_2^{(j)}) \tag{II.4}$$

satisfy

$$[a_+^{(j)}, a_+^{(k)\,*}] = \delta_{jk} I = [a_-^{(j)}, a_-^{(k)\,*}], \qquad [a_+^{(j)}, a_-^{(k)\,*}] = 0, \tag{II.5}$$

with in addition all commutators vanishing between pairs of annihilation operators $a_\pm^{(j)}$. Complex coordinates can be expressed in terms of these operators as

$$z_j = \frac{1}{\sqrt{2}}(a_+^{(j)\,*} + a_-^{(j)}), \qquad \text{and} \qquad \partial_j = \frac{\partial}{\partial z_j} = \frac{1}{\sqrt{2}}(a_+^{(j)} - a_-^{(j)\,*}) \tag{II.6}$$

or

$$a_+^{(j)\,*} = \frac{1}{\sqrt{2}}(z_j - \bar{\partial}_j), \qquad a_-^{(j)\,*} = \frac{1}{\sqrt{2}}(\bar{z}_j - \partial_j), \tag{II.7}$$

and

$$a_+^{(j)} = \frac{1}{\sqrt{2}}(\bar{z}_j + \partial_j), \qquad a_-^{(j)} = \frac{1}{\sqrt{2}}(z_j + \bar{\partial}_j). \tag{II.8}$$

The Fock representation of $\mathscr{H}^b$ is unitarily equivalent to

$$\mathscr{H}^b = L^2(\mathbb{C}^n) = \bigotimes_{j=1}^{n} L^2(\mathbb{C}) \tag{II.9}$$

and is spanned by the orthonormal basis of the jth copy of $L^2(\mathbb{C})$ as follows: let $\Omega_0^{b(j)}$ denote the function $\exp(-|z_j|^2)$. For $n_+, n_- \in \mathbb{Z}_+$, an orthonormal set of basis vectors for the jth factor is

$$\Omega^{b(j)}_{n_+, n_-} = \frac{1}{(n_+!\, n_-!)^{1/2}}\, a_+^{(j)*n_+} a_-^{(j)*n_-} \Omega_0^{b(j)}. \tag{II.10}$$

II.1.2. *The Fermionic Subspace*

Let us now give the explicit form of the Fock representation on the fermionic space $\mathscr{H}^f$. We introduce $2n$ independent creation and annihilation operators acting on $\mathscr{H}^f$. The annihilation operators $b_+^{(j)}$ and $b_-^{(j)}$, $j = 1, 2, ..., n$, all mutually anticommute,

$$b_+^{(j)} b_+^{(k)} + b_+^{(k)} b_+^{(j)} = 0 = \{b_+^{(j)}, b_+^{(k)}\}, \qquad \text{etc.} \tag{II.11}$$

Also

$$\{b_+^{(j)}, b_-^{(j)*}\} = 0 \tag{II.12}$$

and

$$\{b_+^{(j)}, b_+^{(k)*}\} = \delta_{jk} I = \{b_-^{(j)}, b_-^{(k)*}\}. \tag{II.13}$$

The zero-particle fermion state Ω_0^f is a unit vector which satisfies

$$b_\pm^{(j)} \Omega_0^f = 0. \tag{II.14}$$

These operators, like the $a_\pm^{(j)}$, could be expressed in terms of $2n$, independent, real components $b_1^{(j)}$, $b_2^{(j)}$ as

$$b_\mp^{(j)} = \frac{1}{\sqrt{2}} (b_1^{(j)} \pm i b_2^{(j)}). \tag{II.15}$$

We also define fermionic variables ψ as linear combinations of the b's and their adjoints. These are fermionic coordinates which play the role of the bosonic coordinates z. For $j = 1, 2, ..., n$, let

$$\psi_1^{(j)} = \frac{1}{\sqrt{2}} (b_+^{(j)*} + b_-^{(j)}), \qquad \psi_2^{(j)} = \frac{i}{\sqrt{2}} (b_+^{(j)*} - b_-^{(j)}). \tag{II.16}$$

Then we infer the canonical anticommutation relations

$$\{\psi_{\alpha}^{(j)}, \psi_{\beta}^{(k)}\} = 0, \qquad \{\psi_{\alpha}^{(j)}, \psi_{\beta}^{(k)*}\} = \delta_{jk}\delta_{\alpha\beta} I. \tag{II.17}$$

The operators $b_{\alpha}^{(j)}$, $b_{\pm}^{(j)}$, and $\psi_{\alpha}^{(j)}$ all have norms equal to 1.

Let us specify a standard orthonormal basis for $\mathscr{H}^{f}$. We can write $\mathscr{H}^{f}$ as canonically isomorphic to

$$\mathscr{H}^{f} \cong \mathscr{H}_{1}^{f} \wedge \mathscr{H}_{2}^{f} \wedge \cdots \wedge \mathscr{H}_{n}^{f}, \tag{II.18}$$

where $\mathscr{H}_{j}^{f}$ is the 4-dimensional space $\bigwedge \mathbb{C}^{2}$ with the orthonormal basis

$$\begin{aligned} f_{+}^{(j)} &= b_{+}^{(j)*}\,\Omega_{0}^{f}, & f_{-}^{(j)} &= b_{-}^{(j)*}\,\Omega_{0}^{f}, \\ e_{+}^{(j)} &= \frac{1}{\sqrt{2}}\,(\Omega_{0}^{f} + b_{+}^{(j)*}\,b_{-}^{(j)*}\,\Omega_{0}^{f}), & e_{-}^{(j)} &= \frac{1}{\sqrt{2}}\,(\Omega_{0}^{f} - b_{+}^{(j)*}\,b_{-}^{(j)*}\,\Omega_{0}^{f}). \end{aligned} \tag{II.19}$$

II.1.3. *The Canonical Basis for the Tensor Product*

We take skew tensor products over j of the vectors (II.19) to span the 4^{n}-dimensional space $\mathscr{H}^{f}$. We define the canonical orthonormal basis for $\mathscr{H} = \mathscr{H}^{b} \otimes \mathscr{H}^{f}$ as the tensor products of basis vectors (II.8) in $\mathscr{H}^{b}$ with the basis elements in $\mathscr{H}^{f}$. Expressed in terms of the coordinates z on $\mathscr{H}^{b}$, these basis vectors are just polynomials in z and $\bar{z}$ times the Gaussian function $\exp(-z\bar{z})$, tensored with elements of the finite-dimensional space $\mathscr{H}^{f}$. Let Ω denote the vacuum vector $\Omega_{0}^{b} \otimes \Omega_{0}^{f}$.

Definition II.1. Let $\mathscr{D}$ denote the finite linear span of basis vectors defined above.

We also denote $\mathscr{D}$ by $\mathscr{D}^{b} \otimes \mathscr{D}^{f}$ when we wish to work with the individual spaces $\mathscr{H}^{b}$ or $\mathscr{H}^{f}$. We use the domain $\mathscr{D}$ to serve as the initial domain of definition for all unbounded transformations we study here. Many basic operators, such as z_{j}, $\psi^{(j)_{\alpha}}$, ∂_{j}, $D(\lambda)$, $Q(\lambda)$, or $H(\varepsilon, \lambda)$, not only are defined on $\mathscr{D}$, but also map $\mathscr{D}$ into itself. For these operators, $\mathscr{D}$ is called an *invariant domain*, and all products of such operators are defined on $\mathscr{D}$.

II.2. *The Holomorphic Potential and a Dirac Operator*

Let $V(z)$ be a holomorphic polynomial on $\mathbb{C}^{n}$ and let λ be a positive parameter. Define a Dirac operator

$$D(\lambda) = \sum_{j=1}^{n} \left(\psi_{1}^{(j)}\partial_{j} + \lambda\psi_{2}^{(j)}\,\overline{\partial_{j} V}\right) = \psi_{1}\partial + \lambda\psi_{2}\,\overline{\partial V}. \tag{II.20}$$

Here $(\partial_j V)(z) = \partial V(z)/\partial z_j$ denotes the multiplication operator on $L^2(\mathbb{C}^n)$ by the components of the gradient of V. Also $\overline{\partial_j V}$ denotes the complex conjugate of $\partial_j V$. We use the shorthand $\psi_1 \partial = \sum_{j=1}^n \psi_1^{(j)} \partial_j$, etc.

As $V(z)$ is holomorphic, the anticommutation relations for ψ yield

$$D(\lambda)^2 = 0. \tag{II.21}$$

Also $D(\lambda)^*$ is defined with $\mathscr{D} \subset \operatorname{Dom}(D(\lambda)^*)$, and on this domain

$$D(\lambda)^* = -\psi_1^* \bar{\partial} + \lambda \psi_2^* \, \partial V. \tag{II.22}$$

The supercharge $Q(\lambda)$ with domain $\mathscr{D}$ is $Q(\lambda) = D(\lambda) + D(\lambda)^*$. The supersymmetric Hamiltonian is also defined on $\mathscr{D}$ and equals

$$\begin{aligned} H(\lambda) = Q(\lambda)^2 &= D(\lambda)\, D(\lambda)^* + D(\lambda)^*\, D(\lambda) \\ &= -\partial\bar{\partial} + \lambda^2\, |\partial V|^2 + \lambda(W + W^*), \end{aligned} \tag{II.23}$$

where $\partial\bar{\partial}$ denotes $\sum_{j=1}^n \partial_j \bar{\partial}_j$, where $|\partial V|^2$ denotes $\sum_j |\partial_j V|^2$, and where

$$W = \psi_1 \psi_2^* \, \partial^2 V = \sum_{j,k=1}^n \psi_1^{(j)} \psi_2^{(k)*} \frac{\partial^2 V}{\partial z_j\, \partial z_k}. \tag{II.24}$$

II.3. *Independent Square Roots of $H(\lambda)$*

Just as $Q(\lambda)$ is the real part of $D(\lambda)$, the imaginary part of $D(\lambda)$ also provides a square root of $H(\lambda)$. In fact let us define

$$Q_1(\lambda) = D(\lambda) + D(\lambda)^* = Q(\lambda), \qquad \text{and} \qquad Q_2(\lambda) = -i(D(\lambda) - D(\lambda)^*). \tag{II.25}$$

Thus $\gamma Q_2(\lambda) + Q_2(\lambda)\, \gamma = 0$, and also

$$Q_2(\lambda)^2 = H(\lambda). \tag{II.26}$$

These square roots $Q_1(\lambda)$ and $Q_2(\lambda)$ are independent in the sense that

$$\{ Q_1(\lambda), Q_2(\lambda) \} = 0. \tag{II.27}$$

Actually there are two other mutually independent, square roots of $H(\lambda)$. These arise from another Dirac operator $\tilde{D}(\lambda)$. Let us define

$$\tilde{D}(\lambda) = \psi_2^* \partial - \lambda \psi_1^* \, \overline{\partial V(z)}. \tag{II.28}$$

As a consequence of the assumption that $V(z)$ is holomorphic, and the anticommutation relations for ψ_j, we have

$$\tilde{D}(\lambda)^2 = 0. \tag{II.29}$$

This operator $\tilde{D}(\lambda)$ anticommutes with both $D(\lambda)$ and $D(\lambda)^*$,

$$\{\tilde{D}(\lambda), D(\lambda)\} = 0 = \{\tilde{D}(\lambda), D(\lambda)^*\}. \tag{II.30}$$

Thus $\tilde{D}(\lambda)^*$ also commutes with $D(\lambda)$ and $D(\lambda)^*$. The two symmetric operators defined by

$$\tilde{Q}_1(\lambda) = \tilde{D}(\lambda) + \tilde{D}(\lambda)^* \qquad \text{and} \qquad \tilde{Q}_2(\lambda) = -i(\tilde{D}(\lambda) - \tilde{D}(\lambda)^*) \tag{II.31}$$

are independent,

$$\{\tilde{Q}_1(\lambda), \tilde{Q}_2(\lambda)\} = 0, \tag{II.32}$$

and they are also square roots of $H(\lambda)$,

$$Q_1(\lambda)^2 = Q_2(\lambda)^2 = \tilde{Q}_1(\lambda)^2 = \tilde{Q}_2(\lambda)^2 = H(\lambda). \tag{II.33}$$

Furthermore, both $\tilde{Q}_1(\lambda)$ and $\tilde{Q}_2(\lambda)$ are independent of $Q_1(\lambda)$ and of $Q_2(\lambda)$. In other words, we have four, pair-wise anti-commuting (i.e., mutually independent) operators

$$Q_1(\lambda), \qquad Q_2(\lambda), \qquad \tilde{Q}_1(\lambda), \qquad \text{and} \qquad \tilde{Q}_2(\lambda), \tag{II.34}$$

each of which is a square root of $H(0, \lambda)$. We still refer to this system as $N = 2$ supersymmetry.[2]

II.4. *Analytic Assumptions*: *Ellipticity*

We require that the holomorphic potential function $V(z)$ satisfies certain elliptic estimates. These estimates require, among other things, that $\partial V(z)$ grow sufficiently fast at infinity. Let

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2}, \tag{II.35}$$

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[2] The reason for this nomenclature stems from the fact that in quantum field theory this degeneracy is removed. In field theory, $Q_1(\lambda)^2 = \tilde{Q}_1(\lambda)^2 = H(\lambda) + P$, while $Q_2(\lambda)^2 = \tilde{Q}_2(\lambda)^2 = H(\lambda) - P$. Here P is the momentum operator. Our present case is isomorphic to the constant Fourier modes in the field theory. These constant modes lie in the $P = 0$ subspace of $\mathscr{H}$. Hence in quantum mechanics the squares $H(\lambda) + P$ and $H(\lambda) - P$ coincide. In field theory, $(1/\sqrt{2})(Q_1(\lambda) + Q_2(\lambda))$ and $(1/\sqrt{2})(\tilde{Q}_1(\lambda) + \tilde{Q}_2(\lambda))$ are two independent square roots of $H(\lambda)$. The operator P has a symmetric spectrum, and hence no positive square root, but it is the difference of squares, $P = \frac{1}{2}(Q_1(\lambda)^2 - Q_2(\lambda)^2) = \frac{1}{2}(\tilde{Q}_1(\lambda)^2 - \tilde{Q}_2(\lambda)^2)$. In quantum field theory a zero-momentum subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ is characterized by $[P, \mathfrak{A}_0] = 0$. The Hilbert space $\mathscr{H}$ of our present quantum mechanics example is a subspace of the field theory eigenspace of P corresponding to eigenvalue 0. Thus $P = 0$ in our example, and the entire algebra $\mathfrak{A}$ automatically qualifies as a zero momentum algebra.

and

$$|\partial V(z)| = \left(\sum_{j=1}^{n} |\partial_j V|^2 \right)^{1/2}. \tag{II.36}$$

Furthermore, we let ∂^j denote a multi-derivative

$$\partial^j = \left(\frac{\partial}{\partial z_1} \right)^{j_1} \cdots \left(\frac{\partial}{\partial z_n} \right)^{j_n},$$

of total degree $|j| = j_1 + j_2 + \cdots + j_n$, and let $|\partial^j V|$ denote the magnitude of $\partial^j V$. We make the following assumption.

Assumption E (Elliptic Bound). There exists a constant $M = M(V) < \infty$ such that for any j with $2 \leqslant |j|$, we have

$$|z| \leqslant M(|\partial V| + 1), \qquad \text{and} \qquad |\partial^j V| \leqslant M(|\partial V| + 1). \tag{II.37}$$

Remark 1. If $n = 1$, then any polynomial $V(z)$ of degree 2 or greater satisfies the elliptic bound.

Remark 2. The lower bound by $|z|$ in (II.37) ensures that ∂V has no "flat directions" along which it does not grow as $|z| \to \infty$. We use the first bound (II.37) in Section VI in order to show that $H(\varepsilon, \lambda)$ can be bounded from below by const.$(h(\varepsilon + \lambda) + I)$. As a consequence, we will show that $e^{-H(\varepsilon, \lambda)}$ is trace class.

Remark 3. In certain situations, we wish to assume a stronger bound than (II.37) on derivatives of V. Since each derivative of V lowers the degree of the resulting polynomial, it is natural to expect that if the derivative ∂^j of V is of order $|j| \geqslant 2$, then $\partial^j V$ can be estimated by a fractional power of $|\partial V|$. This leads to the formulation of Assumption E′, in which we retain the bound (II.37) for $|z|$, but strengthen the bound for derivatives with $|j| \geqslant 2$ by assuming that there exists $\eta > 0$, and $M = M(\eta, V) < \infty$, such that

$$|z| \leqslant M(|\partial V| + 1),$$

and for all $|j| \geqslant 2$,

$$|\partial^j V(z)| \leqslant M(|\partial V(z)|^{1-(|j|-1)\eta} + 1). \tag{II.38}$$

Condition (II.38) ensures that each derivative of V of order 2 or more, gives a small improvement in estimates by a small, fractional power $\eta/2$ of $H(\varepsilon, \lambda)$, see Section VII.

II.5. *Geometric Assumption*: *Quasi-Homogeneity*

As explained in Section I, we require that $V(z)$ be a quasi-homogeneous polynomial.

Assumption Q (Quasi-Homogeneous Potential). There exist rational weights

$$\omega = \{\omega_j \colon \omega_j \in (0, \tfrac{1}{2}]\}, \tag{II.39}$$

such that

$$V(z) = e^{-i\theta} V(e^{i\omega_j \theta} z_j). \tag{II.40}$$

Remark that every monomial of the form $c_j z^j = c_j z_1^{j_1} z_2^{j_2} z_3^{j_3} \cdots z_n^{j_n}$ that occurs in a given quasi-homogeneous polynomial $V(z)$ must have exponents such that

$$\sum_{l=1}^{n} j_l \omega_l = 1. \tag{II.41}$$

Some Examples of Quasi-Homogeneous Polynomials. For dimension $n = 1$,

$$V(z) = z_1^k, \qquad \text{where} \quad \omega_1 = k^{-1}. \tag{II.42}$$

Any such monomial satisfies Assumption E.

For any n,

$$V(z) = \sum_j^n c_j z_j^{k_j}, \qquad \text{where} \quad \omega_j = k_j^{-1}. \tag{II.43}$$

Again, as long as every $c_j \neq 0$, any such polynomial satisfies Assumption E.

We now give several examples for $n = 2$,

$$V(z) = (z_1^2 - z_2^2)^2, \qquad \text{where} \quad \omega_1 = \omega_2 = \tfrac{1}{4}. \tag{II.44}$$

This polynomial does not satisfy Assumption E.

$$V(z) = z_1^4 + z_2^8 + c_1 z_1 z_2^6 + c_2 z_1^3 z_2^2, \qquad \text{where} \quad \omega_1 = \tfrac{1}{4}, \quad \text{and} \quad \omega_2 = \tfrac{1}{8}. \tag{II.45}$$

This polynomial satisfies Assumption E for $|c_1| + |c_2|$ sufficiently small, and it does not satisfy Assumption E for $|c_1| + |c_2|$ large.

$$V(z) = z_1^k + z_1 z_2^l, \qquad \text{where} \quad \omega_1 = \frac{1}{k}, \quad \omega_2 = \frac{k-1}{kl}, \tag{II.46}$$

and we require that $k \geqslant 2$ and $l \geqslant 1$. This polynomial satisfies Assumption E. Note that the individual inverse weight ω_2^{-1} in general is not integer. However, the $\theta \to 0$ limit (I.17) must be integer, and in this case it is $kl - k + 1$.

Note two simple properties of quasi-homogeneous polynomials. Either $V(z)$ has no zero except $z=0$, or else $V(z)$ has flat directions. In other words, if V vanishes at a point $z=w\neq 0$, then $V(z)$ vanishes on the curve $z(\lambda)=\{\lambda^{\omega_j}w_j\}$, parameterized by $\lambda\in[0,\infty)$. Thus either a quasi-homogeneous polynomial V vanishes only at $z=0$, or else the zeros of V are not confined to any bounded set of $\mathbb{C}^n$.

Second, if V is quasi-homogeneous with weights ω, then $\partial_l V$ is also quasi-homogeneous. The weight of coordinate z_j is $\omega_j/(1-\omega_l)$. Even though $\partial V/\partial z_j$ may have flat directions, $|\partial V|$ has flat directions only if the various components of the gradient have a common zero $w\neq 0$. We rule out this possibility with our ellipticity assumption.

II.6. *The Regularized Hamiltonian*

Introduce the regularized Hamiltonian

$$H(\varepsilon,\lambda)=Q(\lambda)^2+\varepsilon^2\,|z|^2, \tag{II.47}$$

where $Q(\lambda)=Q_1(\lambda)$, $Q_2(\lambda)$, $\tilde{Q}_1(\lambda)$, or $\tilde{Q}_2(\lambda)$. The form of $H(\varepsilon,\lambda)$ is

$$H(\varepsilon,\lambda)=H(\lambda)+\varepsilon^2\,|z|^2=-\partial\bar{\partial}+\lambda^2\,|\partial V|^2+\lambda(W+W^*)+\varepsilon^2\,|z|^2, \tag{II.48}$$

where $W=\psi_1\psi_2^*\,\partial^2 V$ is given in (II.24). For $\lambda=0$, we denote the harmonic oscillator Hamiltonian by the sum of n, independent Hamiltonians

$$h(\varepsilon)=H(\varepsilon,0)=-\partial\bar{\partial}+\varepsilon^2\,|z|^2=\sum_{j=1}^{n}h_j(\varepsilon), \tag{II.49}$$

with

$$h_j(\varepsilon)=-\partial_j\bar{\partial}_j+\varepsilon^2\,|z_j|^2. \tag{II.50}$$

We also denote the purely bosonic Hamiltonian by $h(\varepsilon,\lambda)$, namely

$$h(\varepsilon,\lambda)=H(\varepsilon,\lambda)-\lambda(W+W^*)=-\partial\bar{\partial}+\lambda^2\,|\partial V|^2+\varepsilon^2\,|z|^2. \tag{II.51}$$

III. SYMMETRIES

In this section we investigate the properties of a number of symmetries of $Q(\lambda)$ or $H(\varepsilon,\lambda)$. The first symmetry is that of quasi-homogeneity.

III.1. *Quasi-Homogeneity as a Symmetry*

There is a continuous representation of the $U(1)$ group on $\mathscr{H}$ that implements quasi-homogeneity of $V(z)$ as a unitary symmetry. Here $V(z)$ is assumed to belong to a particular quasi-homogeneous class determined by a particular set of weights $\omega = \{\omega_j\}$. For each such class, we define an operator $U^b(\theta)$ that implements this symmetry on $\mathscr{H}^b$ as $e^{iJ^b\theta}$, where the self-adjoint generator of this group is J^b. This generator J^b is given on the domain $\mathscr{D}$ as

$$J^b = \sum_{j=1}^{n} \omega_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right). \tag{III.1}$$

Clearly J^b is symmetric. Furthermore, each basis vector (II.10) is an eigenvector for J^b, as we infer from the relation $J^b \Omega_0^b = 0$ and from the commutation relations $[J^b, a_\pm^{(j)*}] = \pm a_\pm^{(j)*}$. Since J^b is diagonal on the dense set $\mathscr{D}$, it is essentially self-adjoint. The closure of J^b generates the unitary $U^b(\theta) = e^{iJ^b\theta}$ which implements the transformation of quasi-homogeneity on $\mathscr{H}^b$. This group has the property

$$U^b(\theta)\, z_j\, U^b(\theta)^* = e^{i\omega_j\theta} z_j. \tag{III.2}$$

As a consequence,

$$U^b(\theta)\; V(z)\; U^b(\theta)^* = e^{i\theta} V(z). \tag{III.3}$$

The symmetry defined in this fashion on $\mathscr{H} = \mathscr{H}^b \otimes \mathscr{H}^f$ is not a symmetry of $D(\lambda)$, of $Q(\lambda)$, or of $H(\lambda)$ on $\mathscr{H}$. However, we can find $U^f(\theta)$ acting on $\mathscr{H}^f$, so that $U(\theta) = U^b(\theta) \otimes U^f(\theta)$ is a symmetry of $D(\lambda)$. In this case, $U(\theta)$ is also a symmetry of $Q(\lambda)$ and of $H(\lambda)$. Up to an additive constant, there is one way to define J^f. It is the bounded, self-adjoint operator

$$J^f = \sum_{j=1}^{n} J_j^f = \sum_{j=1}^{n} \left(\psi_1^{(j)} \psi_1^{(j)*}\, \omega_j + \psi_2^{(j)} \psi_2^{(j)*} (1 - \omega_j) - \tfrac{1}{2} \right), \tag{III.4}$$

where J_j^f acts on the 4-dimensional space $\mathscr{H}_j^f$. The operators $\psi_\alpha^{(j)} \psi_\alpha^{(j)*} = P_\alpha^{(j)}$ are mutually commuting projections, each of which projects onto a one-dimensional subspace, and hence has eigenvalues 0, 1. Thus we see that J_j^f has spectrum $\pm\frac{1}{2}$, $\pm(\frac{1}{2} - \omega_j)$, from which we infer that the operators $\pm J^f$ are unitarily equivalent. It is clear that on $\mathscr{H}$, the operator $J = J^b \otimes I + I \otimes J^f$ is a bounded perturbation of an essentially self-adjoint operator, and therefore it is also essentially self-adjoint. It generates $U(\theta) = e^{iJ\theta}$ and implements quasi-homogeneity of $V(z)$ as a symmetry of $D(\lambda)$. In fact

$$U(\theta)\, \gamma = \gamma U(\theta), \qquad U(\theta)\, z_j\, U(\theta)^* = e^{i\omega_j\theta} z_j, \tag{III.5}$$

and

$$U(\theta)\, \psi_1^{(j)}\, U(\theta)^* = e^{i\omega_j\theta} \psi_1^{(j)}, \qquad U(\theta)\, \psi_2^{(j)}\, U(\theta)^* = e^{i(1-\omega_j)\theta} \psi_2^{(j)}. \tag{III.6}$$

Thus for $D(\lambda) = \psi_1 \partial + \lambda \psi_2\, \overline{\partial V(z)}$, we find that

$$U(\theta)\; D(\lambda)\; U(\theta)^* = D(\lambda).$$

Therefore also

$$U(\theta)\, Q_1(\lambda)\, U(\theta)^* = Q_1(\lambda), \qquad \text{and} \qquad U(\theta)\, Q_2(\lambda)\, U(\theta)^* = Q_2(\lambda). \tag{III.7}$$

Furthermore $U(\theta)\, |z|^2\, U(\theta)^* = |z|^2$, so

$$U(\theta)\, H(\varepsilon, \lambda)\, U(\theta)^* = H(\varepsilon, \lambda). \tag{III.8}$$

III.2. *A Second Implementation of Quasi-Homogeneity*

There is a second unitary group $\tilde{U}(\theta)$, which commutes with the group $U(\theta)$, and also is a symmetry of $H(\varepsilon, \lambda)$. This group implements quasi-homogeneity as a symmetry of $\tilde{D}(\lambda)$ rather than as a symmetry of $D(\lambda)$. This group $\tilde{U}(\theta)$ is even under γ, as is $U(\theta)$, and it is a symmetry group of $|z|^2$ as well as of $\tilde{D}(\lambda)$. As $H(\lambda) = \tilde{Q}_2(\lambda)^2$, we infer that $\tilde{U}(\theta)$ is a symmetry of $H(\lambda)$ and of $H(\varepsilon, \lambda)$. Together with $U(\theta)$, this gives a $U(1) \times U(1)$ symmetry group of $H(\varepsilon, \lambda)$.

The generator of $\tilde{U}(\theta)$ is just

$$\tilde{J} = J^b \otimes I + I \otimes \tilde{J}^f, \tag{III.9}$$

and $\tilde{U}(\theta) = e^{i\tilde{J}\theta}$. Note that J and $\tilde{J}$ have the same bosonic part, but

$$\tilde{J}^f = -\sum_{j=1}^{n} \left(\psi_1^{(j)} \psi_1^{(j)*} (1 - \omega_j) + \psi_2^{(j)} \psi_2^{(j)*}\, \omega_j - \tfrac{1}{2}\right). \tag{III.10}$$

It follows that

$$\begin{gathered} \tilde{U}(\theta)\, \gamma = \gamma \tilde{U}(\theta), \qquad \tilde{U}(\theta)\, U(\theta') = U(\theta')\, \tilde{U}(\theta), \\ \tilde{U}(\theta)\, z_j \tilde{U}(\theta)^* = e^{i\omega_j \theta} z_j. \end{gathered} \tag{III.11}$$

Also

$$\tilde{U}(\theta)\, \psi_1^{(j)}\, \tilde{U}(\theta)^* = e^{-i(1-\omega_j)\,\theta} \psi_1^{(j)}, \qquad \tilde{U}(\theta)\, \psi_2^{(j)}\, \tilde{U}(\theta)^* = e^{-i\omega_j \theta} \psi_2^{(j)}. \tag{III.12}$$

Thus

$$\tilde{U}(\theta)\, \tilde{D}(\lambda)\, \tilde{U}(\theta)^* = \tilde{D}(\lambda), \tag{III.13}$$

and hence

$$\tilde{U}(\theta)\, \tilde{Q}_1(\lambda)\, \tilde{U}(\theta)^* = \tilde{Q}_1(\lambda), \qquad \tilde{U}(\theta)\, \tilde{Q}_2(\lambda)\, \tilde{U}(\theta)^* = \tilde{Q}_2(\lambda). \tag{III.14}$$

As a consequence,

$$\tilde{U}(\theta)\, H(\varepsilon, \lambda)\, \tilde{U}(\theta)^* = H(\varepsilon, \lambda). \tag{III.15}$$

We remark that $\tilde{U}(\theta)$ is *not* a symmetry of $D(\lambda)$ or $Q(\lambda)$, and that $U(\theta)$ is *not* a symmetry of $\tilde{D}(\lambda)$ or $\tilde{Q}(\lambda)$. In fact

$$\tilde{U}(\theta)\, D(\lambda)\, \tilde{U}(\theta)^* = e^{-i\theta} D(\lambda), \qquad \text{and} \qquad U(\theta)\, \tilde{D}(\lambda)\, U(\theta)^* = e^{-i\theta} \tilde{D}(\lambda). \tag{III.16}$$

Thus $U(\theta)$ performs a rotation in $\{\tilde{Q}_j(\lambda)\}$ space, and $\tilde{U}(\theta)$ performs a similar rotation in $\{Q_j(\lambda)\}$ space,

$$\begin{aligned} U(\theta)\, \tilde{Q}_1(\lambda)\, U(\theta)^* &= \tilde{Q}_1(\lambda) \cos\theta + \tilde{Q}_2(\lambda) \sin\theta, \\ U(\theta)\, \tilde{Q}_2(\lambda)\, U(\theta)^* &= -\tilde{Q}_1(\lambda) \sin\theta + \tilde{Q}_2(\lambda) \cos\theta, \end{aligned} \tag{III.17}$$

and

$$\begin{aligned} \tilde{U}(\theta)\, Q_1(\lambda)\, \tilde{U}(\theta)^* &= Q_1(\lambda) \cos\theta + Q_2(\lambda) \sin\theta, \\ \tilde{U}(\theta)\, Q_2(\lambda)\, \tilde{U}(\theta)^* &= -Q_1(\lambda) \sin\theta + Q_2(\lambda) \cos\theta. \end{aligned} \tag{III.18}$$

Choosing $\theta = \pi/2$, remark that $Q_1(\lambda)$ is unitarily equivalent to $Q_2(\lambda)$. Also, $\tilde{Q}_1(\lambda)$ is unitarily equivalent to $\tilde{Q}_2(\lambda)$.

Furthermore, with these conventions,

$$\begin{aligned} J - \tilde{J} &= \sum_{j=1}^{n} \left(\psi_1^{(j)} \psi_1^{(j)*} + \psi_2^{(j)} \psi_2^{(j)*} - 1\right) \\ &= \sum_{j=1}^{n} \left(b_+^{(j)*} b_+^{(j)} - b_-^{(j)*} b_-^{(j)}\right) \\ &= N_+^f - N_-^f. \end{aligned} \tag{III.19}$$

Here we use the relation (II.16) between $\psi_\alpha^{(j)}$ and the fermionic creation and annihilation operators. Thus we find that

$$\gamma = (-I)^{N_+^f + N_-^f} = (-I)^{N_+^f - N_-^f} = (-I)^{(J - \tilde{J})} = e^{i\pi(J - \tilde{J})}, \tag{III.20}$$

or

$$\gamma = U(\pi)\, \tilde{U}(\pi)^*. \tag{III.21}$$

We define the partition functions

$$\mathfrak{Z}_\alpha(V; a; \theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, \mathrm{Tr}\left(\gamma \tilde{U}(t)\, a e^{-\tilde{Q}_\alpha(V)^2 + it\, \tilde{d}_\alpha a}\right) dt, \tag{III.22}$$

where $\tilde{Q}_\alpha(V)$ denotes $\tilde{Q}_1(V)$ or $\tilde{Q}_2(V)$, and $\tilde{d}_\alpha a = [\tilde{Q}_\alpha(V), a]$. Similarly define $\mathfrak{Z}_\alpha(\lambda; a; \theta)$ and $\mathfrak{Z}_\alpha(\varepsilon, \lambda; a; \theta)$. Likewise we have $\mathfrak{Z}_\alpha(V; a; \theta)$, $\mathfrak{Z}_\alpha(\lambda; a; \theta)$, and $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$.

III.3. *Additional Symmetries*

In this section we consider some additional symmetries. We construct operators corresponding to charge conjugation, exchange, and reversal. These operators also relate our various $D(V)$'s, $\mathfrak{Z}$'s, etc. We conclude among other things that the four operators $Q_j(\lambda)$ and $\tilde{Q}_j(\lambda)$ are all unitarily equivalent, and the corresponding $\mathfrak{Z}_j(V; a; \theta)$ and $\tilde{\mathfrak{Z}}_j(V; a; \theta)$ agree.

Before defining these operators, let us consider two sorts of transformation on $\mathscr{H}$, which we call the adjoint and phase rotation. For one pair of anticommuting fermionic operators $b_{\pm}$, we study the transformation

$$b_{\pm} \mapsto e^{i\varphi_{\pm}} b^*_{\pm} \tag{III.23}$$

of taking the adjoint with a phase rotation determined by the real pair of angles $\varphi = \{\varphi_+, \varphi_-\}$. This transformation can be implemented by the unitary, skew adjoint operator

$$T(\varphi) = (e^{i\varphi_+/2} b^*_+ - e^{-i\varphi_+/2} b_+)(e^{i\varphi_-/2} b^*_- e^{-i\varphi_-/2} b_-). \tag{III.24}$$

Each factor in (III.24) is unitary and skew adjoint, and the factors anti-commute. Then the relation

$$T(\varphi)\, b_{\pm}\, T(\varphi)^* = e^{i\varphi_{\pm}} b^*_{\pm} \tag{III.25}$$

follows, as we see by using the canonical anticommutation relations.

By composing two such T's with angles φ and φ', we obtain a unitary

$$T(\varphi')\, T(\varphi) = -T(\varphi)\, T(\varphi')$$

that implements the pure phase rotation

$$b_{\pm} \mapsto e^{i(\varphi_{\pm} - \varphi'_{\pm})} b_{\pm}. \tag{III.26}$$

This rotation is also implemented by the unitary

$$e^{(-i(\varphi_+ - \varphi'_+)\, N^f_+)} e^{-i(\varphi_- \varphi'_t)\, N^f_-}.$$

These two transformations agree up to a phase, and we see that

$$T(\varphi')\, T(\varphi) = e^{i(\varphi'_+ - \varphi_+)(N^f_+ - 1/2) + i(\varphi'_- \varphi)(N^f_- - 1/2)}. \tag{III.27}$$

With n independent pairs of fermions $b^{(j)}_{\pm}$, where $1 \leqslant j \leqslant n$, we have n, commuting operators $T^{(j)}(\varphi)$ depending on pairs of angles $\varphi_{\pm}$. For simplicity, we rotate every $b^{(j)}_{+}$, for $j = 1, 2, \ldots, n$, by the same angle φ_+, though we could choose independent angles $\varphi^{(j)}_{+}$. We define the unitary

$$T(\varphi^j) = \prod_{j=1}^{n} T^{(j)}(\varphi^f_{\pm}), \tag{III.28}$$

that implements the adjoint transformation with a phase rotation on $\mathscr{H}^f$, namely

$$T(\varphi)\, b_{\pm}^{(j)} T(\varphi)^* = e^{i\varphi_{\pm}} b_{\pm}^{(j)\,*}\,. \tag{III.29}$$

Likewise defining

$$S^f(\varphi - \varphi') = T(\varphi')\; T(\varphi), \tag{III.30}$$

we have the phase rotation

$$S^f(\varphi)\, b_{\pm}^{(j)} S^f(\varphi)^* = e^{i\varphi_{\pm}} b_{\pm}^{(j)}. \tag{III.31}$$

We could equally well apply the adjoint transformation to any set of $2n$ operators and their adjoints, as long as they satisfy the same anticommutation relations as the $b_{\pm}^{(j)}$. We study two other such cases. Consider $b_1^{(j)}$, $b_2^{(j)}$. The transformation

$$b_{\alpha}^{(j)} \to e^{i\varphi_{\alpha}} b_{\alpha}^{(j)\,*}\,, \qquad \alpha = 1, 2, \tag{III.32}$$

is implemented by $\tilde{T}(\varphi)$, where $\tilde{T}(\varphi)$ is obtained by replacing each factor $(e^{i\varphi_{\pm}/2} b_{\pm}^{(j)\,*} - e^{-i\varphi_{\pm}/2} b_{\pm}^{(j)})$ in $T(\varphi)$, with a corresponding factor

$$(e^{i\varphi_{\alpha}/2} b_{\alpha}^{(j)\,*} - e^{-i\varphi_{\alpha}/2} b_{\alpha}^{(j)})$$

in $\tilde{T}(\varphi)$. This gives rise to the phase rotation operator

$$\tilde{S}^f(\varphi - \varphi') = \tilde{T}(\varphi')\; \tilde{T}(\varphi), \tag{III.33}$$

for which

$$\tilde{S}^f(\varphi)\, b_{\alpha}^{(j)} \tilde{S}^f(\varphi)^* = e^{i\varphi_{\alpha}} b_{\alpha}^{(j)}. \tag{III.34}$$

The operators $\psi_1^{(j)}$, $\psi_2^{(j)}$ with their adjoints also satisfy the same anticommutation relations as those above. Thus we can implement the transformation

$$\psi_{\alpha}^{(j)} \to e^{i\varphi_{\alpha}} \psi_{\alpha}^{(j)\,*}\,, \qquad \alpha = 1, 2, \tag{III.35}$$

with $T_{\psi}(\varphi)$, by using $(e^{i\varphi_{\alpha}/2} \psi_{\alpha}^{(j)\,*} - e^{-i\varphi_{\alpha}/2} \psi_{\alpha}^{(j)})$ in building $T_{\psi}(\varphi)$. The phase rotation $S_{\psi}(\varphi) = T_{\psi}(\varphi')\; T_{\psi}(\varphi + \varphi')$ acts to give

$$S_{\psi}(\varphi)\, \psi_{\alpha}^{(j)} S_{\psi}(\varphi)^* = e^{i\varphi_{\alpha}} \psi_{\alpha}^{(j)}. \tag{III.36}$$

The phase rotation transformation can also be performed on $\mathscr{H}^b$. (The adjoint transformation cannot be implemented on $\mathscr{H}^b$; in fact, for a denoting a boson annihilation operator, a^*a and aa^* have a different spectrum.)

Again we have a choice of whether we rotate $a_1^{(j)}$ and $a_2^{(j)}$ or $a_+^{(j)}$ and $a_-^{(j)}$. For example, let

$$a_{\pm}^{(j)} \to e^{i\varphi^b_{\pm}}\, a_{\pm}^{(j)} \tag{III.37}$$

be implemented by the unitary

$$S^b(\varphi^b) = e^{-i(\varphi_+^b N_+^b + \varphi_-^b N_-^b)}, \tag{III.38}$$

where

$$N_\pm^b = \sum_{j=1}^{n} a_\pm^{(j)*} a_\pm^{(j)}. \tag{III.39}$$

Likewise, for $\alpha = 1, 2$, we implement

$$a_\alpha^{(j)} \to e^{i\varphi_\alpha^b} a_\alpha^{(j)} \tag{III.40}$$

by the unitary

$$\tilde{S}^b(\varphi^b) = e^{-i(\varphi_1^b N_1^b + \varphi_2^b N_2^b)}, \tag{III.41}$$

where

$$N_\alpha^b = \sum_{j=1}^{n} a_\alpha^{(j)*} a_\alpha^{(j)}. \tag{III.42}$$

Now we consider some special cases of $S(\varphi)$ and $T(\varphi)$. These examples give rise to the additional symmetries.

III.3.1. *Rotation or Inversion Symmetry of $V(z)$*

Let us define $U_r(\varphi)$ by the rotation

$$U_r(\varphi) = S_\psi(\varphi), \tag{III.43}$$

where $\varphi_1 = 0$ and $\varphi_2 = \varphi$. Then

$$U_r(\varphi)\,(V)\,U_r(\varphi)^* = D(e^{-i\varphi}V), \qquad \text{and} \qquad U_r(\varphi)\,J U_r(\varphi)^* = J. \tag{III.44}$$

Thus $Q_1(e^{-i\varphi}V)$, $Q_2(e^{-i\varphi}V)$ are unitarily equivalent for all phases. Furthermore J is invariant under a change of phase. Thus for a of the form $a \otimes I$ on $\mathscr{H}^b \otimes \mathscr{H}^f$, we have

$$\mathfrak{Z}_\alpha(V, a, \theta) = \mathfrak{Z}_\alpha(e^{-i\varphi}V, a, \theta). \tag{III.45}$$

In particular, the $\mathfrak{Z}(\pm V, a, \theta)$ are equal. We call the transformation $V \to -V$ the *inversion* of V.

III.3.2. *Charge Conjugation Symmetry*

Here we consider the phase rotation given by

$$U_c = \tilde{S}^b(\varphi)\,\tilde{S}^f(\varphi), \tag{III.46}$$

where on both bosonic and fermionic spaces

$$\varphi_1 = 0, \qquad \varphi_2 = \pi. \tag{III.47}$$

This rotation in (a_α, b_α)-space is an adjoint transformation in z, ψ space. We find that

$$U_c z_j U_c^* = \bar{z}_j, \qquad U_c \psi_\alpha U_c^* = \psi_\alpha^*. \tag{III.48}$$

For $V(z)$ of the form

$$V(z) = \sum_j c_j z^j, \tag{III.49}$$

define

$$V_c(z) = \overline{V(\bar{z})} = \sum_j \overline{c_j} z^j. \tag{III.50}$$

Then

$$U_c D(V)\, U_c^* = D(-V_c)^*. \tag{III.51}$$

Furthermore, under this unitary equivalence $J \mapsto -J = U_c J U_c^*$. Thus combining this transformation with the inversion of V and the phase rotation $\psi_1 \mapsto -\psi_1$, $\psi_2 \mapsto -\psi_2$, we conclude that $Q(V)$ is unitarily equivalent to $Q_j(V_c)$ for $j = 1, 2$. Let $a_c = U_c a U_c^*$, where $a = a \otimes I$. Then

$$\mathfrak{Z}_j(V, a, \theta) = \mathfrak{Z}_j(V_c, a_c, -\theta). \tag{III.52}$$

III.3.3. *Exchange Symmetry*

We inspect the adjoint transformation

$$U_e = T_\psi(\varphi), \tag{III.53}$$

where $\varphi_1 = 0$, $\varphi_2 = \pi$. Then $\psi_1 \to \psi_1^*$, $\psi_2 \to -\psi_2^*$, and

$$U_e D(V)\, U_e^* = \tilde{D}(V), \qquad U_e J U_e^* = \tilde{J}. \tag{III.54}$$

Thus the exchange symmetry shows that $Q_j(V)$ is unitarily equivalent to $\tilde{Q}_j(V)$,

$$U_e Q_j(V)\, U_e^* = \tilde{Q}_j(V). \tag{III.55}$$

Along with (III.54), we have for $a = a \otimes I$, $U_e a U_e^* = a$, and

$$\mathfrak{Z}_\alpha(V; a; \theta) = \tilde{\mathfrak{Z}}_\alpha(V; a; \theta). \tag{III.56}$$

III.3.4. *Scaling*

It is often useful to implement the scaling transformation $z_j \to \varepsilon z_j$ by a group of unitary transformations $U_s(\varepsilon)$ on $\mathscr{H}^b$. (There is no corresponding transformation on $\mathscr{H}^f$.) The operator $U_s(\varepsilon)$ is generated by the self-adjoint transformation

$$J_s = -i \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \bar{z}_j \right), \tag{III.57}$$

and $U_s(\varepsilon) = e^{i\varepsilon J_s}$. The action of $U_s(\varepsilon)$ on the coordinates is $U_s(\varepsilon)\, z_j U_s(\varepsilon)^* = \varepsilon z_j$.

Remark that the operator J_s is essentially self-adjoint on $\mathscr{D}$. This can be established in two steps. In the first step, we show that the domain of the closure $J_s \restriction \mathscr{D}$ contains all elements of the Schwartz space $\mathscr{S}$. The domain $\mathscr{D}$ is dense in $\mathscr{S}$ in the Schwartz space topology and J_s is a continuous transformation on $\mathscr{S}$. Hence if $f_n \in \mathscr{D}$ converges in this topology to $f \in \mathscr{S}$, also $J_s f_n \to J_s f$ both in $\mathscr{S}$ and also in L^2. Thus it is sufficient to show that $J_s \restriction \mathscr{S}$ is essentially self-adjoint.

Second, let $\mathscr{S}$ denote a dense domain for a symmetric operator, say J_s, and suppose that $\mathscr{S}$ is invariant under a continuous, unitary group whose generator equals J_s on $\mathscr{S}$. Then it is well known that the generator is essentially self-adjoint on $\mathscr{S}$. Note that the domain $\mathscr{S}$ is invariant under the group U_s. Thus J_s is essentially self-adjoint on $\mathscr{S}$, and therefore it is essentially self-adjoint on $\mathscr{D}$.

The scaling transformation U_s is not a symmetry of $H(\varepsilon, \lambda)$. However, the oscillator Hamiltonian scales with a simple transformation law,

$$U_s(\varepsilon^{1/2})\, h(1)\, U_s(\varepsilon^{1/2})^* = \varepsilon^{-1} h(\varepsilon). \tag{III.58}$$

The group $U_s(\varepsilon)$ is a symmetry both of J and of $\tilde{J}$, namely

$$U_s(\varepsilon)\, J U_s(\varepsilon)^* = J \qquad \text{and} \qquad U_s(\varepsilon)\, \tilde{J} U_s(\varepsilon)^* = \tilde{J}. \tag{III.59}$$

IV. THE FUNCTION $\mathfrak{Z}(\varepsilon, 0; \theta)$

In this section we consider

$$\mathfrak{Z}(\varepsilon, 0; \theta) = \operatorname{Tr}(\gamma U(\theta)\, e^{-h(\varepsilon)}) = \operatorname{Tr}(\gamma U(\theta)\, e^{\partial\bar{\partial} - \varepsilon^2 |z|^2}). \tag{IV.1}$$

Let us begin with the definition of the singular set Υ_{sing}. As each ω_j is rational, and there are at most n different ω_j, we infer that there is a common period $\theta_{\max}$ for every eigenvalue of $U(\theta)$, namely for $\prod_{j=1}^{n} e^{i\omega_j k_j \theta}$ and $\prod_{j=1}^{n} e^{i(1-\omega_j)\, l_j \theta}$, where $k_j, l_j \in \mathbb{Z}$. This angle $\theta_{\max}$ is also a $U(1)$ period of $U(\theta)$; we restrict θ to the interval $[0, \theta_{\max}]$. We define the singular set Υ_{sing} so it not only is the singular set for

$\mathfrak{Z}(\varepsilon, 0; \theta)$, but we take a somewhat larger set so that it also serves as the singular set for other holonomy expansion moves we consider in this paper. Let us define

$$\Upsilon_{\text{sing}} = \{\theta\colon \text{for any } j,\ e^{i\omega_j \theta} = \pm 1, \text{ or } e^{i(1-\omega_j)\theta} = \pm 1\}. \tag{IV.2}$$

Clearly the set $\Upsilon_{\text{sing}} \cap [0, \theta_{\max}]$ is finite, and we design the holonomy expansion so that for $\theta \notin \Upsilon_{\text{sing}}$, it will converge.

Proposition IV.1. *For* $\varepsilon > 0$, *the operator* $h(\varepsilon)$ *is essentially self-adjoint, the operator* $\exp(-h(\varepsilon))$ *is trace class, and*

$$\mathfrak{Z}(\varepsilon, 0; \theta) = \prod_{j=1}^{n} \mathfrak{Z}^{(j)}(\varepsilon, 0; \theta), \tag{IV.3}$$

where

$$\mathfrak{Z}^{(j)}(\varepsilon, 0; \theta) = \frac{\sin(\omega_j \theta/2) \sin((1-\omega_j)\,\theta/2)}{|\sinh((\varepsilon + i\omega_j)\,\theta/2)|^2}. \tag{IV.4}$$

Furthermore, for $\theta \notin \Upsilon_{\text{sing}}$, $\mathfrak{Z}(\varepsilon, 0; \theta)$ *is real analytic in a neighborhood of* $\varepsilon = 0$ *and*

$$\lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon, 0; \theta) = \prod_{j=1}^{n} \left(\frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j \theta/2)} \right). \tag{IV.5}$$

Proof. The operator $h(\varepsilon)$ of (II.49) can be explicitly diagonalized. Not only is $h(\varepsilon)$ unitarily equivalent to a multiple of $h(1)$, but the eigenfunctions clearly lie in the closure of $h(\varepsilon) \restriction \mathscr{D}$. Furthermore, $h(\varepsilon)$ is the sum of commuting $h_j(\varepsilon)$ which act on $\mathscr{H}^{(j)}$, and since $\gamma U(\theta)$ acts as a tensor product on the $\mathscr{H} = \otimes \mathscr{H}^{(j)}$, we need only compute $\mathfrak{Z}^{(j)}(\varepsilon, 0)$ on each $\mathscr{H}^{(j)}$, and (IV.3) holds. We perform the explicit calculation on $\mathscr{H}^{(j)}$. There $h_j(\varepsilon) = h_j(\varepsilon) \otimes I$ which acts on the bosonic subspace, so $\mathfrak{Z}^{(j)}$ further factors into bosonic and fermionic parts,

$$\begin{aligned} \mathfrak{Z}^{(j)}(\varepsilon, 0; \theta) &= \mathfrak{Z}^{b(j)}(\varepsilon, 0; \theta)\, \mathfrak{Z}^{f(j)}(\varepsilon, 0; \theta) \\ &= \operatorname{Tr}_{\mathscr{H}^{b(j)}}(U^{b(j)}(\theta)\, e^{-h_j(\varepsilon)}) \operatorname{Tr}_{\mathscr{H}^{f(j)}}(\gamma U^{f(j)}(\theta)). \end{aligned} \tag{IV.6}$$

Let

$$\mathfrak{Z}^{b}(\varepsilon, 0, \theta) = \prod_{j=1}^{n} \mathfrak{Z}^{b(j)}(\varepsilon, 0, \theta) \qquad \text{and} \qquad \mathfrak{Z}^{f}(\varepsilon, 0, \theta) = \prod_{j=1}^{n} \mathfrak{Z}^{f(j)}(\varepsilon, 0, \theta).$$

Lemma IV.2. *For* $\varepsilon > 0$,

$$\mathfrak{Z}^{b}(\varepsilon, 0, \theta) = \prod_{j=1}^{n} |2 \sinh((\varepsilon + i\omega_j)\,\theta/2)|^{-2}, \tag{IV.7}$$

and

$$\mathfrak{Z}^f(\varepsilon, 0, \theta) = \prod_{j=1}^{n} (4\sin(\omega_j \theta/2)\sin((1-\omega_j)\,\theta/2)). \tag{IV.8}$$

Proof. We evaluate each $\mathfrak{Z}^{b(j)}$ and $\mathfrak{Z}^{f(j)}$ separately. For notational convenience, let us suppress the index j (or take the case $n=1$). The representation (II.7)–(II.8) shows that

$$h(1) = a_+^* a_+ + a_-^* a_- + 1 = -\partial\bar{\partial} + |z|^2. \tag{IV.9}$$

With $U_s(\varepsilon^{1/2}) = e^{i\varepsilon^{1/2} J_s}$ implementing the scale transformation (III.58), and commuting with $U^b(\theta)$ we infer that

$$\begin{aligned} \mathfrak{Z}^b(\varepsilon, 0; \theta) &= \mathrm{Tr}_{\mathscr{H}^b}(U^b(\theta)\, e^{-h(\varepsilon)}) \\ &= \mathrm{Tr}_{\mathscr{H}^b}(U_s(\varepsilon^{1/2})^*\, U^b(\theta)\, e^{-h(\varepsilon)} U_s(\varepsilon^{1/2})) \\ &= \mathrm{Tr}_{\mathscr{H}^b}(U^b(\theta)\, e^{-\varepsilon h(1)}). \end{aligned} \tag{IV.10}$$

According to the representation (IV.9), the vectors $\Omega_{n_+, n_-} \in \mathscr{D}$ are eigenvectors of $h(1)$ with eigenvalues $n_+ + n_- + 1$. Also by (II.6),

$$j^b = (z\partial - \bar{z}\bar{\partial})\,\omega = (N_+ - N_-)\,\omega, \tag{IV.11}$$

which has the same eigenfunctions with eigenvalues Ω_{N_+, n_-}. Thus

$$\begin{aligned} \mathfrak{Z}^b(\varepsilon, 0; \theta) &= \sum_{n_+, n_- = 0}^{\infty} e^{i(n_+ - n_-)\,\omega\theta} e^{-\varepsilon(n_+ + n_- + 1)} \\ &= |2\sinh((\varepsilon + i\omega\theta)/2)|^{-2}. \end{aligned} \tag{IV.12}$$

This is (IV.7) for $n=1$, as claimed.

On the fermionic space, the operator J^f is given by (III.4) as $\psi_1\psi_1^*\omega + \psi_2\psi_2^*(1-\omega) - \frac{1}{2}$. The three operators γ, $\psi_1\psi_1^*$, and $\psi_2\psi_2^*$ all commute. Furthermore the basis vectors are eigenvectors of these operators. We give a table of the corresponding eigenvalues.

	f_+	f_-	e_+	e_-
γ	-1	-1	1	1
$\psi_1\psi_1^*$	1	0	1	0
$\psi_2\psi_2^*$	1	0	0	1
J^f	$\frac{1}{2}$	$-\frac{1}{2}$	$\omega - \frac{1}{2}$	$-(\omega - \frac{1}{2})$
$U^f(\theta)$	$e^{i\theta/2}$	$e^{-i\theta/2}$	$e^{i\omega\theta - i\theta/2}$	$e^{-i\omega\theta + i\theta/2}$

As a consequence

$$\mathfrak{Z}^f(\varepsilon, 0; \theta) = \mathrm{Tr}_{\mathscr{H}^f}(\gamma U^f(\theta)) = 4 \sin((1-\omega)\,\theta/2) \sin(\omega\theta/2). \tag{IV.13}$$

This gives (IV.8) (for the case $n=1$) and completes the proof of the lemma.

We now complete the proof of the proposition. Formula (IV.4) follows by the lemma. The real analyticity of $\mathfrak{Z}(\varepsilon, 0; \theta)$ follows from the fact that $\sinh((\varepsilon + i\omega)\,\theta/2) \times \sinh((\varepsilon - i\omega)\,\theta/2)$ does not vanish if both $\varepsilon = 0$ and $\theta \notin \Upsilon_{\mathrm{sing}}$. As the excluded values of θ are a finite set of points in each period $[\theta, \theta + \theta_{\max}]$, this non-vanishing continues to hold for complex ε with $|\varepsilon|$ sufficiently small. The existence of the $\varepsilon \to 0$ limit then follows.

V. SELF-ADJOINTNESS OF THE BOSONIC HAMILTONIAN $h(\varepsilon, \lambda)$

In this section we establish the self-adjointness of certain purely bosonic Hamiltonians

$$h(\varepsilon, \lambda) = -\partial\bar{\partial} + \lambda^2\, |\partial V|^2 + \varepsilon^2\, |z|^2. \tag{V.1}$$

Throughout this section we let $\mathscr{D}$ denote $\mathscr{D}^b$.

THEOREM V.1. *Let $V(z)$ be a polynomial satisfying Assumption* E *of Subsection* II.4.

(a) *Then the operator $h(\varepsilon, \lambda)$ is essentially self-adjoint.*

(b) *Suppose in addition that $0 \leqslant \varepsilon, \lambda, \beta \leqslant 1$, and $0 < \varepsilon + \lambda, \beta$. Then the heat kernel* $\exp(-\beta h(\varepsilon, \lambda))$ *is trace class and there exists a constant $M_1 = M_1(V) < \infty$ such that*

$$\mathrm{Tr}(e^{-\beta h(\varepsilon, \lambda)}) \leqslant \frac{M_1}{\beta^{2n}(\varepsilon + \lambda)^{2n}}. \tag{V.2}$$

As a preliminary to the proof of Theorem V.1, we introduce a family of energy norms

$$\|f\|_{n, \varepsilon} = \|h(\varepsilon)^n f\|_{L^2} \tag{V.2}$$

on $L^2(\mathbb{C}^n) \cong L^2(\varepsilon^{2n})$. The following lemma is a standard result in distribution theory:

LEMMA V.2. *For fixed $\varepsilon > 0$, the family of norms $\|\cdot\|_{n, \varepsilon}$, $n \in \mathbb{Z}_+$, defines on $\mathscr{S}(\varepsilon^{2n})$ a topology equivalent to the topology given by the usual Schwartz norms.*

Related to this result is another set of inequalities.

Lemma V.3. *Given $l \in \mathbb{Z}_+$, there is a constant $M_l < \infty$ such that on $\mathscr{D} \times \mathscr{D}$,*

$$|z|^{2l} + (-\partial\bar{\partial})^l \leqslant M_l h(1)^l. \tag{V.4}$$

Remark. We mention an extremely useful tool in the inductive proof of positivity estimates, namely the *double commutator identity*. Let A, A^*, and B be defined on a common invariant domain $\mathscr{D}$. Then we have on $\mathscr{D} \times \mathscr{D}$ the identity

$$AA^*B + BA^*A = ABA^* + A^*BA + [A, [A^*, B]]. \tag{V.5}$$

The first two terms are positive, while the double commutator term may be a small correction. We have first applied this identity in an inductive fashion as a tool to derive bounds in [J].

Proof. For $l = 0, 1$, this inequality holds for $M_l = 1$. Thus we proceed by induction. Assume that (V.4) holds for $l = 0, 1, ..., k$, with $k \geqslant 1$. Then by the inductive hypothesis and the omission of two positive terms,

$$\begin{aligned} M_{k-1} h(1)^{k+1} &= (-\partial\bar{\partial} + |z|^2)\, M_{k-1} h(1)^{k-1} \,(-\partial\bar{\partial} + |z|^2) \\ &\geqslant (-\partial\bar{\partial})^{k+1} + |z|^{2(k+1)} + (-\partial\bar{\partial})\, |z|^{2k} + |z|^{2k}\, (-\partial\bar{\partial}) \\ &\quad + |z|^2\, (-\partial\bar{\partial})^k + (-\partial\partial)^k\, |z|^2. \end{aligned} \tag{V.6}$$

The first two terms on the right of (V.6) have the desired form. On the two pairs of remaining terms, we apply the double commutator identity to obtain

$$\begin{aligned} (-\partial\bar{\partial})\, |z|^{2k} + |z|^{2k}\, (-\partial\bar{\partial}) &= \partial\, |z|^{2k}\, (-\bar{\partial}) + (-\bar{\partial})\, |z|^{2k}\, \partial - k^2\, |z|^{2(k-1)} \\ &\geqslant -k^2\, |z|^{k-1}. \end{aligned} \tag{V.7}$$

Similarly, by interchanging ∂ with z,

$$|z|^2\, (-\partial\bar{\partial})^k + (-\partial\bar{\partial})^k\, |z|^2 \geqslant -k^2 (-\partial\bar{\partial})^{k-1}. \tag{V.8}$$

Insert (V.7)–(V.8) into (V.6) and use the inductive hypothesis once more to obtain

$$M_{k-1} h(1)^{k+1} \geqslant (-\partial\bar{\partial})^{k+1} + |z|^{2(k+1)} - k^2 M_{k-1} h(1)^{k-1}. \tag{V.9}$$

Since the minimum eigenvalue of $h(1)$ is 1, we infer

$$|z|^{2(k+1)} + (-\partial\bar{\partial})^{k+1} \leqslant M_{k-1} (1 + k^2)\, h(1)^{k+1}. \tag{V.10}$$

Hence taking $M_{k+1} = (1 + k^2)\, M_{k-1}$, we have completed the inductive proof of (V.4).

Proof of Theorem V.1. For $\varepsilon = \lambda = 0$, the bosonic Hamiltonian $h(\varepsilon, \lambda) = -\partial\bar{\partial}$ reduces to the Laplacian, which is essentially self-adjoint on $\mathscr{D}$. Thus we assume $0 < \varepsilon + \lambda$. The proof of (a) proceeds in two steps. Let $\mathscr{S}$ denote the Schwartz space of C^∞, complex valued functions on $\mathbb{R}^{2n}$ which decrease rapidly at infinity, and for which each derivative decreases rapidly at infinity. Since every function in the domain $\mathscr{D}$ is a polynomial times the Gaussian $\exp(-|z|^2)$, we have $\mathscr{D} \subset \mathscr{S}$. The first step in the proof is to show that the domain of the closure of $h(\varepsilon, \lambda) \restriction \mathscr{D}$ contains $\mathscr{S}$. The second step is to establish that $h(\varepsilon, \lambda) \restriction \mathscr{S}$ is essentially self adjoint.

As a consequence of Lemmas V.2 and V.3, the operator $h(\varepsilon, \lambda)$ is a continuous transformation of $\mathscr{S}$ into $\mathscr{S}$. Furthermore a sequence converging in $\mathscr{S}$ also converges in L^2. Therefore the domain of the closure of $h(\varepsilon, \lambda) \restriction \mathscr{D}$ includes all of $\mathscr{S}$. Here we use the fact that $\mathscr{D}$ is dense in $\mathscr{S}$ in the $\mathscr{S}$-topology. This follows from Lemma V.2 which ensures that elements of $\mathscr{S}$ are just limits of finite sums of eigenvectors of $h(1)$, the finite sums are elements of $\mathscr{D}$, with coefficients which decrease faster than any inverse polynomial in $n_1 + \cdots + n_n$, an eigenvalue of $h(1)$. In other words

$$(h(\varepsilon, \lambda) \restriction \mathscr{D})^{-}_{L^2} \supset (h(\varepsilon, \lambda) \restriction \mathscr{D})^{-}_{\mathscr{S}} = h(\varepsilon, \lambda) \restriction \mathscr{S}. \tag{V.11}$$

Since $h(\varepsilon, \lambda) \restriction \mathscr{S}$ is a bounded operator from $\mathscr{S}$ to $\mathscr{S}$, it has an adjoint h^+ where

$$h^+ \colon \mathscr{S}' \to \mathscr{S}'. \tag{V.12}$$

Here $\mathscr{S}'$ is the space of tempered distributions, dual to $\mathscr{S}$ in the $\mathscr{S}$ topology. But (V.11) ensures

$$(h(\varepsilon, \lambda) \restriction \mathscr{D})^* \subset (h \restriction \mathscr{S})^+. \tag{V.13}$$

So if we have an element

$$f \in \mathscr{D}(h^*) \subset L^2 \subset \mathscr{S}', \tag{V.14}$$

then

$$h^* f = h^+ f. \tag{V.15}$$

We prove that $h\mathscr{S}$ is dense in $\mathscr{H}^b = L^2(\mathbb{C})$. In other words, we show that there is no vector $f \in \mathscr{H}^b$ except zero, which is orthogonal to $h\mathscr{S}$. In other words, no vector $f \neq 0$ in domain (h^*) satisfies $h^* f = 0$. This being the case, and since $h(\varepsilon, \lambda) \geqslant (\varepsilon^2 + M^{-2}\lambda^2)^{1/2} > 0$, we infer that $f = 0$ and that $h \restriction \mathscr{S}$ is essentially self-adjoint.

In order to establish that $h\mathscr{S}$ is dense, remark that the above imbedding ensures that it is sufficient to study $f \in \mathscr{S}'$ which satisfy $h^+ f = 0$, to see whether any such f is a non-zero element of $\mathscr{H}^b$. An element of $f \in \mathscr{S}'$ which satisfies $h^+ f = 0$ is a solution to the elliptic PDE

$$\tfrac{1}{4} \Delta f = (\lambda^2 \, |\partial V|^2 + \varepsilon^2 \, |z|^2) \, f, \tag{V.16}$$

where Δ denotes the real Laplacian on $\mathbb{R}^{2n}$. By the regularity theorem for solutions to elliptic PDE's, any solution to (V.16) with polynomial V is a C^∞ function.

Furthermore, as the differential equation (V.16) is real, we can also assume without loss of generality that f is real. Multiplying by f we obtain

$$\tfrac{1}{2}\Delta(f^2) = (\nabla f)^2 + (\lambda^2 |\partial V|^2 + \varepsilon^2 |z|^2) f^2. \tag{V.17}$$

Integrating (V.17) over a ball of radius r, we have

$$\frac{1}{2}\frac{\partial}{\partial r}\int_{\mathscr{S}_r} f^2 \, d\Omega = \int_{B_r} ((\nabla f)^2 + (\lambda^2 |\partial V|^2 + \varepsilon^2 |z|^2) f^2) \, d\,\mathrm{vol}, \tag{V.18}$$

where $d\Omega$ denotes integration over the real $(2n-1)$-sphere $\mathscr{S}_r$ bounding the $2n$-ball B_r of radius r. Since the right side of (V.18) is positive, the integral of f^2 over angles at radius r is a non-decreasing function of r. However, $\int_{S_r} f^2 \, d\Omega$ multiplied by r^{2n-1} and integrated over r gives the L^2 norm of f. Thus f lies in $\mathscr{H}^b$; it can be square integrable only if $f = 0$. This completes the proof of part (a) of the theorem.

Part (b) of the theorem is a consequence of Assumption E. In particular, we use (II.37) to ensure the following useful bound.

Lemma V.4. *Let $V(z)$ be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *Then there exists a constant $M_6 = M_6(V) < \infty$ such that for $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, the inequality*

$$h(\varepsilon + \lambda) \leqslant M_6(h(\varepsilon, \lambda) + I) \tag{V.19}$$

holds on $\mathscr{D} \times \mathscr{D}$.

Proof. The bound (II.37) ensures that there exists $M = M(V) \geqslant 1$, such that

$$\lambda^2 |z|^2 \leqslant 2M^2(\lambda^2 |\partial V|^2 + 1).$$

Therefore

$$(\varepsilon + \lambda)^2 |z|^2 \leqslant 2\varepsilon^2 |z|^2 + 4M^2\lambda^2 |\partial V|^2 + 4M^2,$$

and for $M_6 \geqslant 4M^2 \geqslant 1$,

$$-\partial\bar{\partial} + (\varepsilon + \lambda)^2 |z|^2 \leqslant M_6 h(\varepsilon, \lambda).$$

This is inequality (VI.19) and the proof is complete.

We now complete the proof of Theorem V.1.6. It follows from Lemma V.4 and the essential self-adjointness of $h(\varepsilon + \lambda)$ and $h(\varepsilon, \lambda)$ on $\mathscr{D}$ that the nth eigenvalue $\mu_n(h(\varepsilon, \lambda))$ of $h(\varepsilon, \lambda)$, ordered so that $\mu_n \leqslant \mu_{n+1}$, satisfies

$$\mu_n(h(\varepsilon + \lambda)) \leqslant M_6 \mu_n(h(\varepsilon, \lambda)) + M_6 \lambda^2. \tag{V.20}$$

Hence, as we assume $\beta, \lambda \leqslant 1$,

$$\operatorname{Tr}_{\mathscr{H}^b}(e^{-\beta h(\varepsilon, \lambda)}) \leqslant e^{\beta M_6 \lambda^2} \operatorname{Tr}(e^{-\beta M_6^{-1} h(\varepsilon+\lambda)}) \leqslant e^{\beta M_6 \lambda^2}(2 \sinh(\tfrac{1}{2} \beta M_6^{-1}(\varepsilon+\lambda)))^{-2n}. \tag{V.21}$$

In the last step we use (IV.10) to evaluate $\operatorname{Tr}_{\mathscr{H}^b}(e^{-(\beta M_6^{-1} h(\varepsilon+\lambda))})$. The bound (V.2) now follows by applying the inequality $\sinh x \geqslant x$ and taking $M_1 = e^{M_6} M_6^{2n}$. This completes the proof of the theorem.

VI. THE SELF-ADJOINTNESS AND SPECTRUM OF $Q(\lambda)$ AND $H(\varepsilon, \lambda)$

In this section we establish some properties of

$$H(\varepsilon, \lambda) = h(\varepsilon, \lambda) + \lambda(W + W^*) = Q(\lambda)^2 + \varepsilon^2 |z|^2, \tag{VI.1}$$

where we studied $h(\varepsilon, \lambda)$ in the previous section, and $W = \psi_1 \psi_2^* \partial^2 V$. In Section III we have shown that the four operators $Q_1(\lambda)$, $Q_2(\lambda)$, $\tilde{Q}_1(\lambda)$, and $\tilde{Q}_2(\lambda)$ which have the same square $H(\lambda)$ are also unitarily equivalent; here $Q(\lambda)$ denotes any of the four. The first such property is essential self-adjointness, which relies on an estimate we prove in Section VII, as does an estimate on the growth of eigenvalues of $H(\varepsilon, \lambda)$ as a function of small $\varepsilon + \lambda$.

THEOREM VI.1. *Let $V(z)$ be a polynomial satisfying Assumption* E *of Subsection* II.4.

(a) *Then $Q(\lambda)$ and $H(\varepsilon, \lambda)$ are essentially self-adjoint on $\mathscr{D}$.*

(b) *Suppose that $0 \leqslant \varepsilon, \lambda \leqslant 1$. Then there exists a constant $M_2 = M_2(V) < \infty$ such that $h(\cdot) = h(\cdot, 0)$ satisfies*

$$h(\varepsilon + \lambda) \leqslant M_2(H(\varepsilon, \lambda) + I). \tag{VI.2}$$

(c) *Suppose in addition that $0 < \varepsilon + \lambda$, $\beta \leqslant 1$. Then* $\exp(-\beta H(\varepsilon, \lambda))$ *is trace class, and there exists a constant $M_3 = M_3(V) < \infty$ such that*

$$\operatorname{Tr}_{\mathscr{H}}(e^{-\beta H(\varepsilon, \lambda)}) \leqslant \frac{M_3}{\beta^{2n}(\varepsilon+\lambda)^{2n}}. \tag{VI.3}$$

We first show that essential self-adjointness of $H(0, \lambda) = Q(\lambda)^2$ on $\mathscr{D}$ ensures the essential self-adjointness of $Q(\lambda)$. This relies on the invariance of the domain $\mathscr{D}$ under $Q(\lambda)$.

LEMMA VI.2. *Let T be a symmetric operator defined on a dense, invariant domain $\mathscr{D}$. If T^2 is essentially self-adjoint on $\mathscr{D}$, then so is T.*

Proof. A symmetric operator T is essentially self-adjoint on $\mathscr{D}$, if and only if both $(T \pm iI)\,\mathscr{D}$ are dense in $\mathscr{H}$. Furthermore a positive, symmetric operator T^2 is essentially self-adjoint on $\mathscr{D}$, if and only if $(T^2+I)\,\mathscr{D}$ is dense. Let $\mathscr{D}_\pm = (T \pm iI)\,\mathscr{D}$. As $\mathscr{D}$ is an invariant domain, $\mathscr{D}_\pm \subset \mathscr{D}$. Hence if both $(T \pm iI)\,\mathscr{D}_\mp$ are dense, then both $(T \pm iI)\,\mathscr{D}$ are also dense. But $(T \pm iI)\,\mathscr{D}_\mp = (T^2+I)\,\mathscr{D}$. Therefore the essential self-adjointness of T^2 on $\mathscr{D}$ ensures the essential self-adjointness of T on $\mathscr{D}$.

LEMMA VI.3. *Let V satisfy the Assumption* E *of Subsection* II.4 *and let $0 \leqslant \varepsilon$, $\lambda \leqslant 1$. Then there exists a constant $M_4 = M_4(V)$ such that on $\mathscr{D} \times \mathscr{D}$,*

$$\lambda^2 (W+W^*)^2 \leqslant M_4(\lambda^2\, |\partial V|^2 + 1) \leqslant M_4(h(\varepsilon, \lambda) + I). \tag{VI.4}$$

Furthermore, given $\varepsilon_3 > 0$, there exists $M_4 = M_4(\varepsilon_3, V) < \infty$ such that on $\mathscr{D} \times \mathscr{D}$,

$$(1-\varepsilon_3^2)\, h(\varepsilon, \lambda) \leqslant H(\varepsilon, \lambda) + M_4^2. \tag{VI.5}$$

Proof. Since $\|\psi_\alpha^{(j)}\| = 1$, it follows that

$$\begin{aligned} \|Wf\| &= \left\| \sum_{l,m=1}^{n} \psi_1^{(l)} \psi_2^{(m)} \partial_l \partial_m V f \right\| \\ &\leqslant \sum_{l,m=1}^{n} \|\partial_l \partial_m V f\|. \end{aligned}$$

Using assumption (II.37) in the case $\partial^j = \partial_l \partial_m$, i.e., for $|j| = 2$, we infer

$$\|Wf\| \leqslant n^2 M(\|\,|\partial V|\, f\| + \|f\|). \tag{VI.6}$$

Furthermore, $W^* = \psi_2 \psi_1^* \overline{\partial^2 V}$, so $\|W^* f\|$ satisfies exactly the same bound. Note that $W^2 = W^{*2} = 0$, so $(W+W^*)^2 = W^*W + WW^*$. Hence from (VI.6) and $\lambda \leqslant 1$ we infer

$$\lambda^2 (W+W^*)^2 \leqslant \lambda^2 (W^*W + WW^*) \leqslant 4n^4 M^2 (\lambda^2\, |\partial V|^2 + I). \tag{VI.7}$$

This is the first inequality in (VI.4), with M_4 chosen appropriately. Also $h(\varepsilon, \lambda) = -\partial\bar{\partial} + \lambda^2\, |\partial V|^2 + \varepsilon^2\, |z|^2$ is a sum of three non-negative terms. Hence the second inequality in (VI.4) also follows. We also conclude that for any $\mu > 0$,

$$\begin{aligned} |\langle f, \lambda (W+W^*)\, f\rangle| &\leqslant \|f\|\; \|\lambda (W+W^*)\, f\| \\ &\leqslant \frac{1}{2\mu^2}\, \|f\|^2 + \frac{1}{2}\mu^2\, \|\lambda (W+W^*)\, f\|^2 \\ &\leqslant \frac{1}{2\mu^2}\, \langle f, f\rangle + \frac{1}{2}\mu^2 M_4 \langle f, (h(\varepsilon, \lambda) + I)\, f\rangle. \end{aligned} \tag{VI.8}$$

Thus by choosing μ so that $\varepsilon_3^2 = \frac{1}{2} M_4 \mu^2$ we infer that with a new constant M_4 depending on ε_3 and the old M_4 that on $\mathscr{D} \times \mathscr{D}$

$$\pm \lambda(W + W^*) \leqslant \varepsilon_3^2 h(\varepsilon, \lambda) + M_4^2. \tag{VI.9}$$

Since $H(\varepsilon, \lambda) = h(\varepsilon, \lambda) + \lambda(W + W^*)$, we have also established (VI.5) and the lemma.

Proof of Theorem VI.1. The second inequality (VI.4) of the lemma is the statement that $X = \lambda(W + W^*)$ is relatively bounded with respect to $h(\varepsilon, \lambda)^{1/2}$, in the sense of Rellich and of T. Kato. Therefore X is relatively bounded with respect to $h(\varepsilon, \lambda)$, with relative bound 0. We proved in Theorem V.1 that $h(\varepsilon, \lambda)$ is essentially self-adjoint on $\mathscr{D}$. We infer therefore that the perturbed operator $H(\varepsilon, \lambda) = h(\varepsilon, \lambda) + X$ is also essentially self-adjoint on $\mathscr{D}$. See Subsection V.4.1 of [K], and in particular Theorem 4.4.

Thus $H(0, \lambda)$ is essentially self-adjoint on $\mathscr{D}$. The essential self-adjointness of $Q(\lambda)$ then follows by the application of Lemma VI.2. This completes the proof of part (a) of the theorem.

To establish part (b), consider the bound (VI.5) for the case $\varepsilon_3^2 = 1/2$. Since $\beta \leqslant 1$, we have

$$-M_4^2 + \tfrac{1}{2} \beta h(\varepsilon, \lambda) \leqslant \beta H(\varepsilon, \lambda). \tag{VI.10}$$

This inequality carries over to the nth eigenvalues of the operators, and therefore to the traces,

$$\operatorname{Tr}(e^{-\beta H(\varepsilon, \lambda)}) \leqslant e^{M_4^2} \operatorname{Tr}(e^{-\beta h(\varepsilon, \lambda)/2}). \tag{VI.11}$$

The trace class property of $\exp(-\beta H(\varepsilon, \lambda))$ and the bound (VI.3) are a consequence of the bound (V.2) of Theorem V.1. We obtain $M_3 \leqslant 2^{2n} M_1 \exp(M_4^2)$, and the proof of the theorem is complete.

Define the operator $\dot{Q} = dQ(\lambda)/d\lambda$ on the domain $\mathscr{D}$, namely

$$\dot{Q} = \psi_2 \overline{\partial V} + \psi_2^* \partial V. \tag{VI.12}$$

We compute the square of $\dot{Q}$ using the commutation relations for the $\psi_\alpha^{(j)}$'s, and obtain

$$\dot{Q}^2 = |\partial V|^2. \tag{VI.13}$$

PROPOSITION VI.4. *The operator $\dot{Q}$ is essentially self-adjoint on $\mathscr{D}$. Also for any positive power k, the multiplication operator $|\partial V|^k$ is essentially self-adjoint on $\mathscr{D}$.*

Proof. Note $\dot{Q}\colon \mathscr{D} \to \mathscr{D}$, and $|\partial V|^2$ is a multiplication operator on $\mathscr{H}^b$ by the positive function $|\partial V(z)|^2$. As such, any power $|\partial V|^{2k}$ is essentially self-adjoint on $\mathscr{D}$, and hence so is $|\partial V|^k$. In fact, suppose $\mathfrak{v} = |\partial V(z)|^{2k}$, so $f \perp (\mathfrak{v} \pm i)\, \mathscr{D}$. Then

$\mathfrak{v}^* f = \mp i f$. But $\langle f, \mathfrak{v}^* f\rangle = \| |\partial V|^k f\|^2 = \mp i \|f\|^2$, which can only be the case for $f = 0$. Hence $|\partial V|^k$ is essentially self-adjoint. The essential self-adjointness of $\dot{Q}$ then follows from Lemma VI.2.

Next we give another criterion for essential self-adjointness of a symmetric operator A.

Proposition VI.5. *Let A and B be symmetric operators defined on a common, dense, invariant domain $\mathscr{D}$. Let A and B commute on $\mathscr{D}$, and suppose that on $\mathscr{D} \times \mathscr{D}$ they satisfy*

$$A^2 \leqslant B.$$

Finally suppose that every positive power B^k of B is essentially self-adjoint on $\mathscr{D}$. Then A is essentially self-adjoint on $\mathscr{D}$.

Proof. The assumed bound means that for $f \in \mathscr{D}$,

$$\begin{aligned} \|Af\| &= \langle f, A^2 f\rangle^{1/2} \leqslant \langle f, Bf\rangle^{1/2} \leqslant (\|f\| \; \|Bf\|)^{1/2} \\ &\leqslant \tfrac{1}{2}(\|Bf\| + \|f\|) \leqslant \|(B+I)\, f\|. \end{aligned}$$

The last inequality follows by the assumed positivity of B. This bound ensures that every vector in the domain of the self-adjoint closure B^- of $B \restriction \mathscr{D}$ is in the domain of A^-. Thus we need only show that $A^- \restriction \mathscr{D}(B^-)$ is essentially self-adjoint.

As A and B commute on $\mathscr{D}$, we can iterate this inequality to establish for any positive integer k,

$$\|A^k f\| \leqslant \|(B+I)\, A^{k-1} f\| = \|A^{k-1}(B+I)\, f\| \leqslant \|(B+I)^k f\|.$$

Since B^k is essentially self-adjoint on $\mathscr{D}$, this inequality extends to all vectors f in the domain of $(B^-)^k$. The operator $e^{(B^- + I)^2}$ is densely defined, so the range of its inverse Range$(e^{-(B^- + I)^2})$ is dense. Let $f = e^{-(B^- + I)^2} g$, for $g \in \mathscr{H}$. Thus $f \in C^\infty(B^-) = \bigcap_k \mathscr{D}((B^-)^k) \subset \mathscr{D}(B^-)$, and

$$\|A^k f\| \leqslant \|(B+I)^k\, e^{-(B^- + I)^2} g\| \leqslant k!^{1/2} \, \|g\|.$$

Every such vector f is an analytic vector for A, so Nelson's analytic vector theorem [N] ensures that $A \restriction \mathfrak{B}(B^-)$ is essentially self-adjoint.

We apply this result to show:

Corollary VI.6. *Let $\mu \in \mathbb{R}$ and define the operator $X = \mu \, |\partial V|^2 + W + W^*$. Then for any positive integer k, the operator X^k is essentially self-adjoint on $\mathscr{D}$.*

Proof. For all μ, the operator $X\colon \mathscr{D} \to \mathscr{D}$. The bound (VI.4) with $\lambda = 1$ yields the inequality

$$(W + W^*)^2 \leqslant M_3(|\partial V|^2 + 1).$$

Since $W+W^*$ commutes with $|\partial V|^2$ on $\mathscr{D}$, we have for any positive integer k, and new constant M_3,

$$X^{2k} \leqslant M_3^k(|\partial V|^2+1)^k.$$

In Proposition VI.4 we showed that any power of $|\partial V|$ is essentially self-adjoint on $\mathscr{D}$. Hence with $A=X^k$ and $B=M_3^k(|\partial V|^2+1)^k$, the essential self-adjointness of X^k follows from Proposition VI.5.

VII. FUNDAMENTAL A PRIORI ESTIMATES

In this section we establish fundamental *a priori* estimates on perturbations of $H(\varepsilon, \lambda)$. We are interested here in the behavior of the estimates as (ε, λ) tends to zero, so we always work in the unit square $0 \leqslant \varepsilon, \lambda \leqslant 1$, with $0<\varepsilon+\lambda$. It is no loss of generality that we assume $0 \leqslant \varepsilon, \lambda$. The Hamiltonian $H(\varepsilon, \lambda)$ is a function of ε^2, while we saw in Section III that the Hamiltonians $H(\varepsilon, \lambda)$ and $H(\varepsilon, -\lambda)$ are unitarily equivalent.

VII.1. *First Order Estimates*

We call the estimate of an operator A by $H(\varepsilon, \lambda)$ a *first order estimate*, if it is an inequality of the form

$$A \leqslant H(\varepsilon, \lambda),$$

providing an upper bound on A by $H(\varepsilon, \lambda)$. We have already established such a bound in Lemma VI.3. Let us restate this bound in the following form.

THEOREM VII.1. *Let V be a holomorphic polynomial satisfying Assumption* E *of Section* II.4. *Also let* $0 \leqslant \bar{\delta} < 1$.

(a) *Then there exists $\bar{M}=\bar{M}(\bar{\delta}, V)<\infty$ such that for all $0 \leqslant \varepsilon, \lambda \leqslant 1$, the following inequality holds on the domain $\mathscr{D} \times \mathscr{D}$,*

$$\bar{\delta} h(\varepsilon, \lambda) \leqslant H(\varepsilon, \lambda)+\bar{M}. \tag{VII.1}$$

(b) *With another constant $M=M(V)$, on the domain $\mathscr{D} \times \mathscr{D}$,*

$$\lambda^2(W+W^*)^2 \leqslant \bar{M}(H(\varepsilon, \lambda)+I). \tag{VII.2}$$

(c) *In addition, with $\bar{M}=\bar{M}(V)$, we have the operator bounds*

$$\|\partial(h(\varepsilon, \lambda)+I)^{-1/2}\|+\|\partial(H(\varepsilon, \lambda)+I)^{-1/2}\| \leqslant \bar{M}, \tag{VII.3}$$

$$\|\,|z|\,(h(\varepsilon, \lambda)+I)^{-1/2}\|+\|\,|z|\,(H(\varepsilon, \lambda)+I)^{-1/2}\| \leqslant \bar{M}(\varepsilon+\lambda)^{-1}, \tag{VII.4}$$

$$\|\,|\partial V|\,(h(\varepsilon, \lambda)+I)^{-1/2}\|+\|\,|\partial V|\,(H(\varepsilon, \lambda)+I)^{-1/2}\| \leqslant \bar{M} \lambda^{-1}, \tag{VII.5}$$

$$\|\dot{Q}(H(\varepsilon, \lambda)+I)^{-1/2}\| \leqslant \bar{M} \lambda^{-1}, \tag{VII.6}$$

$$\|W(H(\varepsilon, \lambda)+I)^{-1/2}\|+\|W^*(H(\varepsilon, \lambda)+I)^{-1/2}\| \leqslant \bar{M} \lambda^{-1}. \tag{VII.7}$$

(d) *Let* $0 \leqslant a, b$ *and* $a+b=1$. *Then*

$$\|(H(\varepsilon,\lambda)+I)^{-a/2}\,(W+W^*)(H(\varepsilon,\lambda)+I)^{b/2}\| \leqslant \bar{M}\lambda^{-1}. \tag{VII.8}$$

(e) *With the stronger Assumption* E′ *in place of Assumption* E, *each multi-derivative* ∂^j *of* V *of order* $|j| \geqslant 2$, *satisfies*

$$\|\,|\partial^j V|\,(H(\varepsilon,\lambda)+I)^{-(1-(|j|-1)\,\eta)/2}\| \leqslant \bar{M}\lambda^{-(1-(|j|-1)\,\eta)}.$$

Proof. The bound (VII.1) was established in (VI.5). Since $h(\varepsilon,\lambda) = -\partial\bar{\partial} + \lambda^2\,|\partial V|^2 + \varepsilon^2\,|z|^2$, the bounds (VII.3)–(VII.5) follow immediately from the essential self-adjointness of $H(\varepsilon,\lambda)$, Theorem VI.1.2. The bound (VII.2) is a consequence of Lemma VI.3. Using (VI.3) and the identity $W^2 = W^{*2} = 0$, we have

$$\begin{aligned}\lambda^2\,\langle f, (W+W^*)^2\, f\rangle &= \lambda^2\,\langle f, (W^*W + WW^*)\, f\rangle\\ &\leqslant 2(\lambda\,\|Wf\| + \lambda\,\|W^*f\|)^2\\ &\leqslant 4n^4M^2(\|h(\varepsilon,\lambda)^{1/2}\, f\| + \|f\|)^2\\ &\leqslant 8n^4M^2\,\langle f, h(\varepsilon,\lambda)\, f\rangle + \langle f, f\rangle.\end{aligned}$$

Thus with $\bar{M} > 8n^4M^2$, we obtain (VII.2). The bound (VII.6) follows from Proposition VI.4 and bound (VII.5). The bound (VII.7) follows from (VI.4) and the self-adjointness of $h(\varepsilon,\lambda)$. Under assumption (c), the positive numbers a and b add up to 2, so either $a \geqslant 1$ or $b \geqslant 1$. Thus the bound (VII.8) is a consequence of (VII.2) and the elementary bound $\|Xf\| \leqslant \|\,|X|\, f\|$, where $|X| = (X^*X)^{1/2}$. Similarly, part (e) follows from the bound (II.38). Thus the proof is complete.

We now give another first order inequality involving the generator J of the $U(1)$-symmetry $U(\theta)$ (and similarly $\tilde{J}$ the generator of the symmetry $\tilde{U}(\theta)$). Here $J = J^b \otimes I + I \otimes J^f$ is defined in Subsection III.1 and is essentially self-adjoint on $\mathscr{D}$. Likewise, the operator $\tilde{J}$ is defined in Subsection III.2.

Proposition VII.2. *Let* $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, *and* $0 < \varepsilon + \lambda$. *Then there is a constant* $\bar{M}_2 = \bar{M}_2(V)$ *such that on the domain* $\mathscr{D} \times \mathscr{D}$,

$$\pm J \leqslant \bar{M}(\varepsilon+\lambda)^{-1}\,(H(\varepsilon,\lambda)+I) \quad \textit{and} \quad \pm\tilde{J} \leqslant \bar{M}(\varepsilon+\lambda)^{-1}\,(H(\varepsilon,\lambda)+I). \tag{VII.9}$$

Proof. We establish the inequality for J^b. Since J^f and $\tilde{J}^f$ are bounded, and $\tilde{J}^b = J^b$, the claimed inequalities follow from the fact that $\varepsilon + \lambda \leqslant 2$. Using the representation (III.1), and $\omega_j \leqslant \frac{1}{2}$, it is sufficient to establish the bound with J^b replaced by $T_j = z_j(\partial/\partial z_j) - \bar{z}_j(\partial/\partial\bar{z}_j)$. From $AB \leqslant AA^* + B^*B$, we have

$$\pm T_j \leqslant -(\varepsilon+\lambda)^{-1}\,\bar{\partial}_j\partial_j + (\varepsilon+\lambda)\,|z_j|^2 = (\varepsilon+\lambda)^{-1}\,h_j(\varepsilon+\lambda),$$

so summing over j,

$$\pm J^b \leqslant (\varepsilon+\lambda)^{-1}\, h(\varepsilon+\lambda).$$

The desired inequality for J^b is now a consequence of (VI.2), completing the proof of (VII.9).

VII.2. *Second Order Estimates*

We call an estimate that establishes an upper bound on the square of an operator by the square of $H(\varepsilon, \lambda)$ a *second order* estimate. A typical second order estimate has the form

$$A^2 \leqslant H(\varepsilon, \lambda)^2.$$

Most of our second order estimates of the form $A^2 \leqslant B^2$ arise for operators for which the first order estimate $A \leqslant B$ is known. However, the second order inequality is an inevitable consequence of the first order inequality only for positive, *commuting* A and B. On the other hand, for self-adjoint A^2 and B^2, the inequality

$$A^2 \leqslant B^2$$

does ensure that

$$A^{2\alpha} \leqslant B^{2\alpha}, \qquad \text{for all} \quad 0 \leqslant \alpha \leqslant 1. \tag{VII.10}$$

THEOREM VII.3. *Let $V(z)$ be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4.

(a) *There exist constants $\bar{m}_2 = \bar{m}_2(V) < \infty$ and $\bar{M}_2 = \bar{M}_2(V) < \infty$ such that for all $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, the following inequalities hold on $\mathscr{D} \times \mathscr{D}$,*

$$(\partial\bar{\partial})^2 + \lambda^4\, |\partial V|^4 + \varepsilon^4\, |z|^4 \leqslant \bar{m}_2^2 (h(\varepsilon, \lambda) + I)^2 \leqslant \bar{M}_2^2 (H(\varepsilon, \lambda) + I)^2, \tag{VII.11}$$

and

$$h(\varepsilon+\lambda)^2 \leqslant \bar{m}_2^2 (h(\varepsilon, \lambda) + I)^2. \tag{VII.12}$$

(b) *Let $R = (H(\varepsilon, \lambda) + I)^{-1/2}$. For all $0 \leqslant \alpha \leqslant 2$, we have the operator inequalities,*

$$\|(-\partial\bar{\partial})^{\alpha/2}\, R^\alpha\| \leqslant \bar{M}_2^\alpha, \qquad \|\, |\partial V|^\alpha\, R^\alpha\| \leqslant \bar{M}_2^\alpha \lambda^{-\alpha}, \tag{VII.13}$$

and

$$\|\, |z|^\alpha\, R^\alpha\| \leqslant \bar{M}_2^\alpha (\varepsilon+\lambda)^{-\alpha}. \tag{VII.14}$$

(c) *For all* $0 \leqslant a, b$ *with* $a+b=2$,

$$\|R^a \partial\bar{\partial} R^b\| \leqslant \bar{M}_2^2, \qquad \|R^a\,|\partial V|^2\,R^b\| \leqslant \bar{M}_2^2 \lambda^{-2},$$
$$\|R^a\,|z|^2\,R^b\| \leqslant \bar{M}_2^2(\varepsilon+\lambda)^{-2}. \tag{VII.15}$$

(d) *Given* $a \in [0, 2]$ *and* $l = \text{degree}(V) - 1$, *then*

$$\|\,|\partial V|^{a/l}\,R^a\| \leqslant \bar{M}_2^a(\varepsilon+\lambda)^{-a}. \tag{VII.16}$$

(e) *Given* $a \in [0, 2]$, *let* $b = 2 - a/l \in [0, 2]$. *Then*

$$\|R^a\,|\partial V|^2\,R^b\| + \|R^b\,|\partial V|^2\,R^a\| \leqslant \bar{M}_2^{a+b}(\varepsilon+\lambda)^{-a}\,\lambda^{-b}. \tag{VII.17}$$

Also, for $a \in [0, 1]$ *and* $b = 1 - a/l \in [0, 1]$, *we have*

$$\|R^a(W+W^*)\,R^b\| + \|R^b(W+W^*)\,R^a\| \leqslant \bar{M}_2^{a+b}(\varepsilon+\lambda)^{-a}\,\lambda^{-b}. \tag{VII.18}$$

Remark. As a consequence of (VII.11), and the essential self-adjointness of $H(\varepsilon, \lambda)$ established in Theorem VI.1(a), we infer that for $0 < \lambda$,

$$\begin{aligned}\mathscr{D}_1 &= \text{Domain}(H(\varepsilon, \lambda)) = \text{Domain}(\partial\bar{\partial}) \cap \text{Domain}(|\partial V|^2) \cap \text{Domain}(|z|^2)\\ &= \text{Domain}(\partial\bar{\partial}) \cap \text{Domain}(|\partial V|^2).\end{aligned} \tag{VII.19}$$

Proof. Let us write $H(\varepsilon, \lambda) = h(\varepsilon, \lambda) + X$, where $X = \lambda(W+W^*)$. We begin by proving (VII.11). Note

$$\begin{aligned}H(\varepsilon, \lambda)^2 &= h(\varepsilon, \lambda)^2 + X^2 + h(\varepsilon, \lambda)\,X + Xh(\varepsilon, \lambda)\\ &\geqslant h(\varepsilon, \lambda)^2 + h(\varepsilon, \lambda)\,X + Xh(\varepsilon, \lambda).\end{aligned}$$

For any $\varepsilon_4 > 0$, $\pm(hX + Xh) \leqslant \varepsilon_4 h^2 + \varepsilon_4^{-1} X^2$, as follows from applying Schwarz and Hölder inequalities to $|\langle f, hXf\rangle| = |\langle hf, Xf\rangle|$. Thus

$$H(\varepsilon, \lambda)^2 \geqslant (1-\varepsilon_4)\,h(\varepsilon, \lambda)^2 - \varepsilon_4^{-1} X^2.$$

However, for any $\varepsilon_3 > 0$ we infer from (VII.2) that there exists $M_3 = M_3(\varepsilon_3, V)$ such that

$$\langle f, X^2 f\rangle \leqslant \varepsilon_3^2 \langle f, h(\varepsilon, \lambda)^2 f\rangle + M_3^2 \langle f, f\rangle.$$

Thus for any $\delta_5 < 1$, and with a new constant $M_5 = M_5(V, \delta_5, \varepsilon_3, M_3)$,

$$H(\varepsilon, \lambda)^2 \geqslant \delta_5 h(\varepsilon, \lambda)^2 - M_5. \tag{VII.20}$$

We now expand $h(\varepsilon, \lambda)^2$ using the double commutator identity (V.5) to obtain

$$\begin{aligned} h(\varepsilon, \lambda)^2 &= (\partial\bar{\partial})^2 + (\lambda\,|\partial V|)^4 + (\varepsilon\,|z|)^4 + \{-\partial\bar{\partial}, \lambda^2\,|\partial V|^2 + \varepsilon^2\,|z|^2\} + 2\lambda^2\,|\partial V|^2\,\varepsilon^2\,|z|^2 \\ &\geqslant (\partial\bar{\partial})^2 + (\lambda\,|\partial V|)^4 + (\varepsilon\,|z|)^4 - \sum_{j=1}^{n} [\bar{\partial}_j, [\partial_j, \lambda^2\,|\partial V|^2 + \varepsilon^2\,|z|^2]]. \qquad \text{(VII.21)} \end{aligned}$$

Here we bound some positive terms from below by zero. Let Y denote the double commutator term in (VII.21). Then

$$Y = -\varepsilon^2 n - \lambda^2 \sum_{1 \leqslant j, k \leqslant n} |\partial_j \partial_k V|^2.$$

Using bound (II.37), we have for any $\varepsilon_6 > 0$, a constant $M_6 = M_6(\varepsilon_6, V) < \infty$, such that

$$|Y| \leqslant \varepsilon_6 h(\varepsilon, \lambda)^2 + M_6.$$

Thus (VII.21) yields

$$h(\varepsilon, \lambda)^2\,(1 + \varepsilon_6) + M_6 \geqslant (\partial\bar{\partial})^2 + (\lambda\,|\partial V|)^4 + (\varepsilon\,|z|)^4,$$

which with appropriate $\bar{m}_2$ is the first inequality in (VII.11). Also (VII.20) ensures the second inequality (VII.11). The inequality (VII.12) also follows by combining with (II.37) the inequality

$$\begin{aligned} h(\varepsilon + \lambda)^2 &= (-\partial\bar{\partial} + (\varepsilon + \lambda)^2\,|z|^2)^2 \\ &\leqslant 2(\partial\bar{\partial})^2 + 2(\varepsilon + \lambda)^4\,|z|^4 \\ &\leqslant 32((\partial\bar{\partial})^2 + \varepsilon^4\,|z|^4 + \lambda^4\,|z|^4). \end{aligned}$$

The inequality (VII.11) extends to the domain of $H(\varepsilon, \lambda)$ as a consequence of the essential self-adjointness of $H(\varepsilon, \lambda)$ on $\mathscr{D}$, see Theorem VI.1. We also conclude from (II.37) combined with (VII.11)–(VII.12) that with a new constant $\bar{M}_2$

$$\bar{M}_2^2(H(\varepsilon, \lambda) + I)^2 \geqslant (\varepsilon + \lambda)^4\,|z|^4.$$

The remaining bounds of parts (b) and (c) then follow from the fact that $\partial\bar{\partial}$, $|\partial V|^2$, $|z|^2$, and $\dot{Q} = W + W^*$ are also essentially self-adjoint on $\mathscr{D}$. Thus the inequality $0 \leqslant A^2 \leqslant B^2$ ensures by (VII.10) the desired fractional inequalities.

In part (b), we bound $\lambda^2\,|\partial V|^2$ uniformly down to $\lambda = 0$. This requires a different approach. Since V is a polynomial of degree $l + 1$, there is a constant $M_7 = M_7(V)$ such that

$$|\partial V|^4 \leqslant M_7(|z|^{4l} + 1).$$

By the higher order estimates established in Lemma V.3 for the $\lambda=0$ operator $h(1)$, there is a constant M_{2l} such that

$$|z|^{4l} \leqslant M_{2l} h(1)^{2l}.$$

Applying the unitary scaling transformation $z \to \varepsilon^{1/2} z$, we obtain

$$\varepsilon^{4l}\, |z|^{4l} \leqslant M_{2l} h(\varepsilon)^{2l}.$$

We can take the constants M_{2l} and M_7 greater than 1, and we assume $\varepsilon \leqslant 1$. So with $M_8 = M_7^{2l} M_{2l}^{2l}$ we have

$$\varepsilon^{4l}\, |\partial V|^4 \leqslant M_8^{2l} (h(\varepsilon)+I)^{2l}.$$

Here $l \geqslant 1$, as the case $l=1$ gives the polynomial $V=z^2$. As $0 \leqslant a \leqslant 2$, we have $0 \leqslant a/2l \leqslant 1$, and by (VII.10),

$$\varepsilon^{2a}\, |\partial V|^{2a/l} \leqslant M_8^a (h(\varepsilon)+I)^a.$$

This remains true if we replace ε by $\varepsilon+\lambda$. Therefore the second order estimates (VII.11)–(VII.12) show that

$$(\varepsilon+\lambda)^{2a}\, |\partial V|^{2a/l} \leqslant M_8^a \bar{M}_2^a (H(\varepsilon, \lambda)+I)^a.$$

With a new choice of $\bar{M}_2$, this gives the bound

$$\|\, |\partial V|^{a/l}\, R^a \| \leqslant \bar{M}_2^a (\varepsilon+\lambda)^{-a},$$

which is (VII.16). Combined with the bound (VII.13) in the form

$$\| R^b\, |\partial V|^{2-a/l} \| \leqslant \| R^b\, |\partial V|^b \| \leqslant \bar{M}_2^b \lambda^{-b},$$

we obtain

$$\| R^b\, |\partial V|^2\, R^a \| \leqslant \bar{M}_2^{a+b} (\varepsilon+\lambda)^{-a}\, \lambda^{-b},$$

which is the desired bound on the norm of $R^b\, |\partial V|^2\, R^a$ in (VII.17). Taking adjoints completes the proof of (VII.17).

The proof of (VII.18) begins from the inequality (VI.4) for $\lambda=1$. As $W+W^*$ and $|\partial V|^2$ commute, we can square this inequality, yielding

$$(W+W^*)^4 \leqslant M_3^2 (|\partial V|^2+1)^2.$$

We showed in Corollary VI.6 that $(W+W^*)^4$ is essentially self-adjoint on $\mathscr{D}$. Thus for $0 \leqslant a \leqslant 2$, we have $0 \leqslant 2a/l \leqslant 4$ and by (VII.10),

$$|W+W^*|^{2a/l} \leqslant M_3^{a/l} (|\partial V|^2+1)^{a/l},$$

where

$$|W+W^*|=((W+W^*)^2)^{1/2}.$$

We now proceed as above to obtain with a new $\bar{M}_2$,

$$|W+W^*|^{2a/l}\leqslant \bar{M}_2^{2a}(H(\varepsilon+\lambda)+I)^a\,(\varepsilon+\lambda)^{-2a},$$

or $\|\,|W+W^*|\,R^a\|\leqslant \bar{M}_2^a(\varepsilon+\lambda)^{-a}$. We use the following:

LEMMA VII.4. *Let* $X=X^*$, *let* $0\leqslant H=H^*$, *and let* $R=(H+I)^{-1/2}$. *Suppose that* $\mathscr{D}(X)\subset\mathscr{D}(H)$. *Let* $0\leqslant a, b, \alpha, \beta$ *with* $\alpha+\beta=1$. *Then*

$$\|R^aXR^b\|\leqslant 2(\|\,|X|^\alpha\,R^a\|\;\|\,|X|^\beta\,R^b\|).$$

Proof. For any self-adjoint operator X, decompose $X=X_++X_-$ into its positive and negative parts. By the triangle inequality,

$$\|R^aXR^b\|=\|R^aX_+R^b\|+\|R^aX_-R^b\|.$$

Then as $X_+\geqslant 0$, we have

$$\|R^aX_+R^b\|=\|R^aX_+^{\alpha+\beta}R^b\|\leqslant\|X_+^\alpha\,R^a\|\;\|X_+^\beta\,R^b\|\leqslant\|\,|X|^\alpha\,R^a\|\;\|\,|X|^\beta\,R^b\|.$$

Combined with a similar inequality for $R^aX_-R^b$ we obtain the lemma.

Now let us return to the proof of Theorem VII.3. Applying the lemma with a, b as in (VII.18), we have

$$\|R^aXR^b\|\leqslant 2(\|\,|X|^\alpha\,R^a\|)(\|\,|X|^\beta\,R^b\|).$$

Thus we have the bound

$$\|R^a(W+W^*)\,R^b\|\leqslant\|\,|(W+W^*)|^\alpha\,R^a\|\;\|\,|(W+W^*)|^\beta\,R^b\|.$$

Choosing $\alpha=a/l$ and $\beta=1-a/l$, we then obtain the bound on $R^a(W+W^*)\,R^b$ in (VII.18). Interchanging a and b we derive the bound on $R^b(W+W^*)\,R^a$ in a similar fashion. This completes the proof of the theorem.

PROPOSITION VII.5. *Let V be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *Then on* $\mathscr{D}\times\mathscr{D}$,

$$Q(\lambda)^4\leqslant Q(\lambda)^4+\varepsilon^4\,|z|^4\leqslant(H(\varepsilon,\lambda)+\varepsilon I)^2. \tag{VII.22}$$

Also let $0<\lambda'$, λ *and let* $c=(\lambda',\lambda)=\max\{1,\lambda'/\lambda\}$. *Then on* $\mathscr{D}\times\mathscr{D}$,

$$Q(\lambda')^4\leqslant c^4\bar{M}_3(Q(\lambda)^2+I)^2, \tag{VII.23}$$

where $\bar{M}_3=\bar{M}_3(V)=8(1+2\bar{M}^2)\,\bar{M}_2^2$ *and* $\bar{M}$, $\bar{M}_2$ *are the constants in Propositions* VII.1 *and* VII.2, *respectively.*

Proof. Expand $H(\varepsilon, \lambda)$ using the double commutator identity (V.5). Then

$$\begin{aligned} H(\varepsilon, \lambda)^2 &= H(0, \lambda)^2 + \varepsilon^4 \, |z|^4 + \varepsilon^2 \{ |z|^2, H(0, \lambda) \} \\ &= H(0, \lambda)^2 + \varepsilon^4 \, |z|^4 + \varepsilon^2 (z H(0, \lambda) \, \bar{z} + \bar{z} H(0, \lambda) \, z) \\ &\quad + \varepsilon^2 [\bar{z}, [H(0, \lambda)]]. \end{aligned} \tag{VII.24}$$

The double commutator term is $-\varepsilon^2$, so

$$H(\varepsilon, \lambda)^2 \geqslant Q(\lambda)^4 + \varepsilon^4 \, |z|^4 - \varepsilon^2. \tag{VII.25}$$

As $0 \leqslant |\varepsilon z|^4$, and $(H^2 + \varepsilon^2 I)^2$, the inequality (VII.22) follows.

Next apply the inequality (VII.11) with $\varepsilon = 0$ to give

$$(\partial \bar{\partial})^2 + \lambda^4 \, |\partial V|^4 \leqslant \bar{M}_2^2 (Q(\lambda)^2 + I)^2. \tag{VII.26}$$

For self-adjoint A and B, we have the elementary bound $\frac{1}{2}(A+B)^2 \leqslant A^2 + B^2$. Therefore with $X = W + W^*$, we have

$$Q(\lambda')^4 = H(0, \lambda')^2 = (h(0, \lambda') + \lambda' X)^2 \leqslant 2h(0, \lambda')^2 + 2\lambda'^2 X^2. \tag{VII.27}$$

As $W^2 = 0$, we use the bound (VI.4) in the form

$$X^2 = W^* W + W W^* \leqslant M_4 (\lambda')^{-2} \, (h(0, \lambda') + I). \tag{VII.28}$$

From (VII.27)–(VII.28) we infer

$$Q(\lambda')^4 \leqslant 2(1 + M_4)(h(0, \lambda')^2 + I), \tag{VII.29}$$

and using the elementary bound once more,

$$Q(\lambda')^4 \leqslant 4(1 + M_4)((\partial \bar{\partial})^2 + \lambda'^4 \, |\partial V|^4 + I). \tag{VII.30}$$

If $\lambda' \leqslant \lambda$, then (VII.30), (VII.26) ensure

$$\begin{aligned} Q(\lambda')^4 &\leqslant 4(1 + M_4)((\partial \bar{\partial})^2 + \lambda^4 \, |\partial V|^4 + I) \\ &\leqslant 4(1 + M_4)(\bar{M}_2^2 (Q(\lambda)^2 + I)^2 + I) \\ &\leqslant 8(1 + M_4) \, \bar{M}_2^2 (Q(\lambda)^2 + I)^2, \end{aligned} \tag{VII.31}$$

where without loss of generality we assume $M_4, \bar{M}_2 \geqslant 1$.

On the other hand, if $\lambda \leqslant \lambda'$, then $(\lambda')^4 = (c\lambda)^4$, with $c \geqslant 1$. Hence (VII.30) yields

$$\begin{aligned} Q(\lambda')^4 &\leqslant 4(1 + M_4)((\partial \bar{\partial})^2 + \lambda'^4 \, |\partial V|^4 + I) \\ &\leqslant 4(1 + M_4)((\partial \bar{\partial})^2 + c^4 \lambda^4 \, |\partial V|^4 + I) \\ &\leqslant 4c^4 (1 + M_4)((\partial \bar{\partial})^2 + \lambda^2 \, |\partial V| + I) \\ &\leqslant 8c^4 (1 + M_4) \, \bar{M}_2^2 (Q(\lambda)^2 + I)^2. \end{aligned} \tag{VII.32}$$

Thus, taking $\bar{M}_3 = 8(1 + M_4) \, \bar{M}_2^2$, we have established (VII.23) for all allowed λ, λ'.

Proposition VII.6. *Let V be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *Then for $0 \leqslant \lambda, \lambda'$, there is a constant $\bar{M}_4 = \bar{M}_4(\lambda, \lambda', V)$ such that for all $\alpha \in [0, 2]$,*

$$\|f\|_{\mathscr{H}_\alpha(\lambda')} \leqslant \bar{M}_4^\alpha \, \|f\|_{\mathscr{H}_\alpha(\lambda)} \leqslant \bar{M}_4^{2\alpha} \, \|f\|_{\mathscr{H}_\alpha(\lambda')}. \tag{VII.33}$$

We may take

$$\bar{M}_4 = (2\bar{M}_3)^{1/4} \left(\frac{\max(\lambda, \lambda')}{\min(\lambda, \lambda')}\right), \tag{VII.34}$$

where $\bar{M}_3 = \bar{M}_3(V)$ is the constant in Proposition VII.5.

Proof. This is the statement that the Sobolev norms $\mathscr{H}_\alpha(\lambda)$ are equivalent for different, non-zero λ. For $0 \leqslant \alpha \leqslant 1$, this follows from the general results applied to our particular $Q(\lambda)$. We however need to use the second order estimate of Proposition VII.5. For $0 \leqslant \lambda' \leqslant \lambda$ the inequality (VII.23) ensures on $\mathscr{D} \times \mathscr{D}$ that

$$(Q(\lambda')^2 + I)^2 \leqslant 2\bar{M}_3 (Q(\lambda)^2 + I)^2 \leqslant 4\bar{M}_3^2 \left(\frac{\lambda}{\lambda'}\right)^4 (Q(\lambda')^2 + I)^2. \tag{VII.35}$$

Since $Q(\lambda)^2 = H(0, \lambda)$ is essentially self-adjoint on $\mathscr{D}$, the inequality (VII.35) ensures for $0 < \lambda' \leqslant \lambda$,

$$\|f\|_{\mathscr{H}_2(\lambda')} \leqslant (2\bar{M}_3)^{1/2} \, \|f\|_{\mathscr{H}_2(\lambda)} \leqslant 2\bar{M}_3 \left(\frac{\lambda}{\lambda'}\right)^2 \|f\|_{\mathscr{H}_2(\lambda')}. \tag{VII.36}$$

Thus for all $0 < \lambda, \lambda'$,

$$\|f\|_{\mathscr{H}_2(\lambda)} \leqslant (2\bar{M}_3)^{1/2} \left(\frac{\max(\lambda, \lambda')}{\min(\lambda, \lambda')}\right)^2 \|f\|_{\mathscr{H}_2(\lambda')},$$

which is the chain for $\alpha = 2$. The corresponding inequality for $0 \leqslant \alpha < 2$ then follows from self-adjoint, invertible A, B satisfying $A^4 \leqslant B^4$ ensuring $A^{2\alpha} \leqslant B^{2\alpha}$ for $0 \leqslant \alpha \leqslant 2$.

Similarly, we can compare Hamiltonians $H(\varepsilon, \lambda)$ for two different pairs of positive values of ε, λ. We use this comparison to establish continuity and differentiability of $\exp(-sH(\varepsilon, \lambda))$. Thus it is only necessary to choose (ε', λ') close to (ε, λ), for example, we generally take $\varepsilon'/\varepsilon, \lambda'/\lambda \in [\frac{1}{2}, 2]$. Thus it is natural to define the constant

$$c_1 = c_1(\varepsilon', \varepsilon, \lambda', \lambda) = \max\{c(\varepsilon', \varepsilon), c(\lambda', \lambda)\}, \tag{VII.37}$$

where $c(\varepsilon', \varepsilon) = \max\{1, \varepsilon'/\varepsilon\}$.

Proposition VII.7. *Let V be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4.

(a) *Let* $0 \leqslant \varepsilon, \varepsilon', \lambda, \lambda' \leqslant 1$. *Then on* $\mathscr{D} \times \mathscr{D}$,

$$(H(\varepsilon', \lambda') + 1) \leqslant \bar{M}_4^\alpha (H(\varepsilon, \lambda) + I)^\alpha, \qquad \text{for all} \quad 0 \leqslant \alpha \leqslant 2. \tag{VII.38}$$

Here $\bar{M}_4 = M_4(\lambda, \lambda', \varepsilon, \varepsilon', V)$ *has the form* $\bar{M}_4 = 32c_1^4 \bar{M}(V)$ *with* c_1 *given by* (VII.37) *and* $\bar{M}_3(V)$ *in Proposition* VII.5. *Also for all* $0 \leqslant \alpha \leqslant 1$,

$$\begin{aligned} &\|(H(\varepsilon', \lambda') + I)^\alpha \, (H(\varepsilon, \lambda) + I)^{-\alpha}\| \leqslant \bar{M}_4^\alpha, \\ &\|(H(\varepsilon, \lambda) + I)^{-\alpha} \, (H(\varepsilon', \lambda') + I)^\alpha\| \leqslant \bar{M}_4^\alpha. \end{aligned} \tag{VII.39}$$

(b) *Let* $0 \leqslant \varepsilon, \varepsilon' \leqslant \lambda$. *Then there is a constant* $\bar{M}_5 = \bar{M}_5(V)$ *which is independent of* λ *and such that for all* $0 \leqslant \alpha \leqslant 2$,

$$(H(\varepsilon', \lambda) + I)^\alpha \leqslant \bar{M}_5^\alpha (H(\varepsilon, \lambda) + I)^\alpha. \tag{VII.40}$$

Proof. We need only establish the inequality (VII.38) for $\alpha = 2$, from which it follows as above for $0 \leqslant \alpha \leqslant 2$. We have using (VII.22), (VII.23),

$$\begin{aligned} (H(\varepsilon', \lambda') + I)^2 &\leqslant 2H(\varepsilon', \lambda')^2 + 2I \\ &\leqslant 4Q(\lambda')^4 + 4\varepsilon'^4 \, |z|^4 + 2I \\ &\leqslant 8c(\lambda', \lambda)^4 \, \bar{M}_3(Q(\lambda)^4 + I) + 4c(\varepsilon', \varepsilon)^4 \, \varepsilon^4 \, |z|^4 + 2I \\ &\leqslant 8c_1^4 \bar{M}_3(Q(\lambda)^4 + \varepsilon^4 \, |z|^2) + (8c_1^4 \bar{M}_3 + 2) \, I \\ &\leqslant 8c_1^4 \bar{M}_3(H(\varepsilon, \lambda) + \varepsilon I)^2 + (8c_1^4 \bar{M}_3 + I) \, I \\ &\leqslant 32c_1^4 \bar{M}_3(H(\varepsilon, \lambda) + I)^2 = \bar{M}_4(H(\varepsilon, \lambda) + I)^2, \end{aligned}$$

which is the desired bound. The operator inequality (VII.39) is then a consequence of the essential self-adjointness of $H(\varepsilon, \lambda)$ on $\mathscr{D}$, Theorem VI.1(a).

As in the case of the bound (VII.38), the bound (VII.40) only needs to be proved for $\alpha = 2$. We use (VII.11) to establish

$$\begin{aligned} \bar{M}_2^2 (H(\varepsilon, \lambda) + I)^2 &\geqslant (\partial\bar{\partial})^2 + \lambda^4 \, |\partial V|^4 + \varepsilon^4 \, |z|^4 \\ &\geqslant (\partial\bar{\partial})^2 + \tfrac{1}{2}(\lambda^4 + \varepsilon'^4) \, |\partial V|^4. \end{aligned}$$

Hence by (II.37), with a new constant $\bar{M}_2$,

$$\bar{M}_2^2 (H(\varepsilon, \lambda) + I)^2 \geqslant (\partial\bar{\partial})^2 + \lambda^4 \, |\partial V|^2 + (\varepsilon' \, |z|)^4. \tag{VII.41}$$

For self-adjoint A and B, we have $(A + B)^2 \leqslant 2A^2 + 2B^2$. Thus by (VI.4), with ε' in place of ε,

$$\begin{aligned} H(\varepsilon', \lambda)^2 = (h(\varepsilon', \lambda) + 2\lambda(W + W^*))^2 &\leqslant 2h(\varepsilon', \lambda)^2 + \lambda^2 (W + W^*)^2 \\ &\leqslant 2h(\varepsilon', \lambda)^2 + M_3 h(\varepsilon', \lambda) + I \\ &\leqslant (2 + M_3)(h(\varepsilon', \lambda)^2 + I). \end{aligned} \tag{VII.42}$$

Likewise, for self-adjoint A, B, and C, we have $(A+B+C)^2 \leqslant 4(A^2+B^2+C^2)$. Then

$$\begin{aligned} h(\varepsilon', \lambda)^2 &= (-\partial\bar{\partial} + \lambda^2\, |\partial V|^2 + \varepsilon'^2\, |z|^2)^2 \\ &\leqslant 4((\partial\bar{\partial})^2 + \lambda^4\, |\partial V|^4 + (\varepsilon'\, |z|)^4). \end{aligned} \tag{VII.43}$$

Thus (VII.42)–(VII.43) combined with (VII.41) yield the bound (VII.40) and complete the proof.

VIII. VERTICES AND EXPECTATIONS

VIII.1. *Heat Kernel Regularization of Regular Sets of Vertices*

In this subsection we establish some basic regularizing properties of the heat kernels $e^{-sH(\varepsilon, \lambda)}$. Our presentation follows the general outline of Section V of [QHA], but we refine this in order to pay careful attention to the behavior of the heat kernel regularization at the singular endpoint $\lambda = 0$ of the interval $\lambda \in (0, 1]$.

We use the definitions of the Sobolev spaces $\mathscr{H}_\alpha(\lambda)$ introduced in Subsection I.4, rather than the case $\varepsilon = 0$ studied in [QHA]. We use $\mathscr{H}_\alpha(\varepsilon, \lambda) = \mathscr{D}((H(\varepsilon, \lambda) + I)^{\alpha/2})$, for $\alpha > 0$ and its dual space $\mathscr{H}_{-\alpha}(\varepsilon, \lambda)$ for $\alpha > 0$. We also use the spaces $\mathscr{T}_{\varepsilon, \lambda}(-\beta, \alpha)$ of bounded, linear maps from $\mathscr{H}_\alpha(\varepsilon, \lambda)$ to $\mathscr{H}_{-\beta}(\varepsilon, \lambda)$. We denote an element $X_j \in \mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)$ as a *vertex*. Let $H = H(\varepsilon, \lambda) = Q(\lambda)^2 + \varepsilon^2\, |z|^2$ and $R = R(\varepsilon, \lambda) = (H(\varepsilon, \lambda) + I)^{-1/2}$.

Definition VIII.1. We say that $X = \{X_0, ..., X_k\}$ is a regular set of $(k+1)$-vertices with respect to $H(\varepsilon, \lambda)$ if $X_j \in \mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)$, with $\eta_j = \frac{1}{2}(2 - \alpha_j - \beta_{j+1}) > 0$ for $0 \leqslant j \leqslant k$. Here $\beta_{k+1} = \beta_0$.

Definition VIII.2. Let σ_k denote the subset of $\mathbb{R}^{k+1}$ given by

$$\sigma_k = \left\{ s\colon 0 < s_j,\ j = 0, 1, ..., k, \text{ and } \sum_{j=0}^{k} s_j = 1 \right\}.$$

For a regular set of vertices X, define its *heat kernel regularization* $X(s)$ by

$$X(s) = \begin{cases} R^{\beta_0} X_0 e^{-s_0 H} X_1 e^{-s_1 H} \cdots X_k e^{-s_k H} R^{-\beta_0}, & \text{if } \ 0 < s_j, \ \text{ for } 0 \leqslant j \leqslant k, \\ 0, & \text{otherwise.} \end{cases} \tag{VIII.1}$$

Proposition VIII.3. *Let $V(z)$ be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4, *and let X be a regular set of vertices with respect to $H(\varepsilon, \lambda)$, and let $s \in \sigma_k$. Then $X(s)$ is trace class, and the trace norm $\|X(s)\|_{I_1}$ of $X(s)$ is bounded by*

$$\|X(s)\|_{I_1} \leqslant \operatorname{Tr}(e^{-H(\varepsilon, \lambda)/2}) \prod_{j=0}^{k} (4 s_j^{-(1-\eta_j)}\, \|X_j\|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)}). \tag{VIII.2}$$

Remark. The Schatten norms $\|T\|_{I_p} = (\mathrm{Tr}((T^*T)^{p/2}))^{1/p}$ are operator L_p norms on compact operators. The operators with $\|\cdot\|_{I_p} < \infty$ comprise the Schatten ideal I_p, where $1 \leqslant p \leqslant \infty$. These norms satisfy fundamental inequalities, the most important for us being $|\mathrm{Tr}(T)| \leqslant \|T\|_{I_1}$, and the Hölder inequality $\|TS\|_{I_p} \leqslant \|T\|_{I_q} \|S\|_{I_r}$, for $p^{-1} = q^{-1} + r^{-1}$.

Proof. As a consequence of Assumption E, the heat kernel e^{-s_jH} is trace class for $s_j > 0$, see Theorem VI.1(c). Then e^{-s_jH} maps every $\mathscr{H}_{\beta_j}$ into $\bigcap_{\alpha>0} \mathscr{H}_\alpha$. Let

$$T_j = R^{\beta_j} X_j R^{\alpha_j}, \qquad j = 0, 1, ..., k, \tag{VIII.3}$$

and with $\beta_{k+1} = \beta_0$, let

$$S_j = R^{-\alpha_j} e^{-s_jH} R^{-\beta_{j+1}}, \qquad j = 0, 1, 2, ..., k. \tag{VIII.4}$$

Thus for $s \in \sigma_n$,

$$X(s) = T_0 S_0 T_1 S_1 \cdots T_k S_k. \tag{VIII.5}$$

Each T_j is a bounded, sesquilinear form and the operator norm of T_j satisfies

$$\|T_j\| = \|X_j\|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)}. \tag{VIII.6}$$

Also each S_j is trace class, and in fact for $0 < s_j \leqslant 1$, the operator S_j belongs to the Schatten class $I_{s_j^{-1}}$. For such s_j, the Hölder inequality gives

$$\begin{aligned} \|S_j\|_{I_{s_j^{-1}}} &= \|R^{-\alpha_j} e^{-s_jH/4} e^{-s_jH/2} e^{-s_jH/4} R^{-\beta_{j+1}}\|_{I_{s_j^{-1}}} \\ &\leqslant \|R^{-\alpha_j} e^{-s_jH/4}\|_{I_\infty} \|e^{-s_jH/2}\|_{I_{s_j^{-1}}} \|e^{-s_jH/4} R^{-\beta_{j+1}}\|_{I_\infty}. \end{aligned} \tag{VIII.7}$$

The second factor on the right of (VIII.7) equals $(\mathrm{Tr}(e^{-H/2}))^{s_j}$. The other two factors are bounded using $\|\cdot\|_{I_\infty} = \|\cdot\|$, along with Proposition VII.7. This yields

$$\|(H+I)^{\alpha/2} e^{-sH/4}\| \leqslant 2^\alpha s^{-\alpha/2}. \tag{VIII.8}$$

Thus we obtain

$$\|S_j\|_{I_{s_j^{-1}}} \leqslant 2^{\alpha_j + \beta_{j+1}} s_j^{-(\alpha_j + \beta_{j+1})/2} (\mathrm{Tr}(e^{-H(\varepsilon, \lambda)/2}))^{s_j}. \tag{VIII.9}$$

Here $\alpha_j + \beta_{j+1} < 2$, so $2^{\alpha_j + \beta_{j+1}} < 4$. Thus for $s \in \sigma_k$, we bound (VIII.5) using Hölder's inequality by

$$\begin{aligned} \|X(s)\|_{I_1} &\leqslant \prod_{j=0}^{k} (\|T_j\|_{I_\infty} \|S_j\|_{I_{s_j^{-1}}}) \\ &\leqslant \mathrm{Tr}(e^{-H(\varepsilon, \lambda)/2}) \prod_{j=0}^{k} (4 s_j^{-(1-\eta_j)} \|X_j\|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)}). \end{aligned} \tag{VIII.10}$$

This completes the proof of (VIII.2).

Proposition VIII.4. *Let V be an holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *Let X be a regular set of $(k+1)$-vertices with respect to $H(\varepsilon, \lambda)$, and let $s \in \sigma_k$. Then $s \to X(s)$ is a continuous map from σ_k to I_1. For $s, s' \in \sigma_k$, separated by Euclidean distance $|s-s'| \leqslant \frac{1}{2}\min_j s_j$, and for $\eta' < \eta_{\min}$,*

$$\|X(s)-X(s')\|_{I_1} \leqslant \mathrm{Tr}(e^{-H(\varepsilon,\lambda)/4})\,|s-s'|^{\eta'}\left(\sum_{j=0}^{k} s_j^{-\eta'}\right)\prod_{j=0}^{k}\left(8 s_j^{-1+\eta_j}\,\|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}\right). \tag{VIII.11}$$

Proof. We use the decomposition (VIII.5) of the previous result, and let S_j' denote $R^{-\alpha_j}e^{-s_j'H}R^{-\beta_{j+1}}$. On the set of s, s' under consideration, two inequalities hold. In particular,

$$\begin{aligned} s_j' &= s_j + (s_j'-s_j) \geqslant s_j - |s_j'-s_j| \geqslant s_j - |s-s'| \\ &\geqslant s_j - \tfrac{1}{2}\min_{j'} s_{j'} \geqslant \tfrac{1}{2}s_j, \end{aligned} \tag{VIII.12}$$

and

$$|s_j - s_j'| \leqslant \tfrac{1}{2}\min_{j'} s_{j'} \leqslant \tfrac{1}{2}s_j. \tag{VIII.13}$$

Thus as in the proof of (VIII.9) we obtain

$$\begin{aligned} \|S_j'\|_{I_{s_j^{-1}}} &\leqslant 4(s_j')^{-(\alpha_j+\beta_{j+1})/2}\,(\mathrm{Tr}(e^{-s_j'/s_j\,H/2}))^{s_j} \\ &\leqslant 16(s_j)^{-(\alpha_j+\beta_{j+1})/2}\,\mathrm{Tr}(e^{-H/4})^{s_j}. \end{aligned} \tag{VIII.14}$$

In the last inequality we use (VIII.12) and $\alpha_j+\beta_{j+1}<2$.

Also $S_j - S_j'$ satisfies the Hölder continuity bound

$$\|S_j - S_j'\|_{I_{s_j^{-1}}} \leqslant s_j^{-1+(\eta_j-\eta')}\,|s_j-s_j'|^{\eta'}\,(\mathrm{Tr}(e^{-H/4}))^{s_j}. \tag{VIII.15}$$

In fact let $s_j(\alpha) = \alpha s_j + (1-\alpha)\,s_j'$ interpolate between s_j and s_j' for $0 \leqslant \alpha \leqslant 1$. Thus we have

$$S_j - S_j' = \int_0^1 \left(\frac{d}{d\alpha}\,S_j(s_j(\alpha))\right) d\alpha = (s_j - s_j')\int_0^1 R^{-\alpha_j}(He^{-s_j(\alpha)\,H})\,R^{-\beta_{j+1}}\,d\alpha. \tag{VIII.16}$$

Using (VIII.13), we have $s_j(\alpha) = s_j + (1-\alpha)(s_j'-s_j) \geqslant \frac{1}{2}s_j$.

Using (VIII.12)–(VIII.13), we have the estimate

$$\begin{aligned} \|S_j - S_j'\|_{I_{s_j^{-1}}} &\leqslant |s_j-s_j'|\;\|R^{-2-(\alpha_j+\beta_{j+1})}e^{-s_j(\alpha)\,H(\varepsilon,\lambda)}\|_{I_{s_j^{-1}}} \\ &\leqslant 16\,|s_j-s_j'|\;s_j(\alpha)^{-1-(\alpha_j+\beta_{j+1})/2}\,(\mathrm{Tr}(e^{-H(\varepsilon,\lambda)/4}))^{s_j} \\ &\leqslant 64\,|s_j-s_j'|\;s_j^{-(1-\eta')}s_j^{-(\alpha_j+\beta_{j+1})/2-\eta'}(\mathrm{Tr}(e^{-H(\varepsilon,\lambda)/4}))^{s_j} \\ &\leqslant 64\,|s_j-s_j'|^{\eta'}\;s_j^{-(\alpha_j+\beta_{j+1})/2-\eta'}(\mathrm{Tr}(e^{-H(\varepsilon,\lambda)/4}))^{s_j}. \end{aligned} \tag{VIII.17}$$

We now proceed as in the proof of Proposition VIII.3 to establish (VIII.11).

DEFINITION VIII.5. Define the subset $\sigma_k^\varepsilon \subset \sigma_k \subset \mathbb{R}^{k+1}$ by

$$\sigma_k^\varepsilon = \{ s \in \sigma_k, \varepsilon < s_j, 0 \leqslant j \leqslant k \}. \tag{VIII.18}$$

Let $X = \{X_0, X_1, ..., X_k\}$ be a regular set of vertices with respect to $H(\varepsilon, \lambda)$. The regularized Radon transform $\hat{X}_{\varepsilon'}$ of X is

$$\hat{X}_{\varepsilon'} = \int_{\sigma_k^{\varepsilon'}} X(s)\, \delta\left(1 - \sum_{j=0}^{k} s_j\right) ds_0 \cdots ds_k. \tag{VIII.19}$$

PROPOSITION VIII.6. *Let V be a holomorphic polynomial satisfying Assumption* E *of Subsection* II.4. *Let X be a regular set of $(k+1)$-vertices with respect to $H(\varepsilon, \lambda)$. Let $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, and $0 < \varepsilon + \lambda$. Then the following holds*:

(a) *The $\varepsilon' \to 0$ limit of $\hat{X}_{\varepsilon'}$ exists. In particular there exists $\hat{X} \in I_1$ such that*

$$\lim_{\varepsilon \to 0} \|\hat{X} - \hat{X}_{\varepsilon'}\|_{I_1} = 0. \tag{VIII.20}$$

We denote this limit as the Radon transform $\hat{X}$ of X, and we write

$$\hat{X} = \int_{\sigma_k} X(s)\, \delta\left(1 - \sum_{j=0}^{k} s_j\right) ds_0 \cdots ds_k. \tag{VIII.21}$$

(b) *With $\eta_{\min}$ and η_{tot} defined in* (I.35),

$$\|\hat{X}\|_{I_1} \leqslant \frac{1}{\Gamma(\eta_{\mathrm{tot}})} \operatorname{Tr}(e^{-H(\varepsilon, \lambda)/2}) \left(\prod_{j=0}^{k} 4\Gamma(\eta_j)\, \|X_j\|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)} \right). \tag{VIII.22}$$

(c) *For (ε, λ) fixed, the function $\theta \to U(\theta)\, \hat{X}$ is continuous in trace norm.*

Proof. For parts (a) and (b), we can follow the proof of Propositions VIII.3 and VIII.4. We integrate the bound of Proposition VIII.3. To show convergence, we estimate for $\varepsilon' < \varepsilon''$,

$$\|\hat{X}_{\varepsilon'} - \hat{X}_{\varepsilon''}\|_{I_1} = \int_{\sigma_k^{\varepsilon'} \setminus \sigma_k^{\varepsilon''}} \|X(s)\|_{I_1}\, \delta\left(1 - \sum_{j=0}^{k} s_j\right) ds_0 \cdots ds_k.$$

We compute the s-integrals, which converge as $\eta_{\min} > 0$, and use properties of the beta function,

$$\int_{s_j > 0} \left(\prod_{j=0}^{k} s_j^{-1+\eta_j} \right) \delta\left(1 - \sum_{j=0}^{k} s_j\right) ds_0\, ds_1 \cdots ds_k = \frac{\prod_{j=0}^{k} \Gamma(\eta_j)}{\Gamma(\sum_{j=0}^{k} \eta_j)}. \tag{VIII.23}$$

We thus obtain (VIII.22). The small tail of the integral shows convergence as $\varepsilon', \varepsilon'' \to 0$.

The proof of (c) relies on the fact that if X is a regular set of vertices with respect to $H(\varepsilon, \lambda)$ with $\eta_{\min} > 0$, then there exists $0 < \eta'$, but with η' sufficiently small, such that we may replace X_0 with $R^{-\eta'} X_0$ (namely increase β_0 to $\beta_0 + \eta'$), yet still have a regular set of vertices with respect to $H(\varepsilon, \lambda)$. Thus with this η', we have $\|R^{-\eta'} \hat{X}\|_{I_1} < \infty$, and

$$\begin{aligned}\|(U(\theta) - U(\theta'))\, \hat{X}\|_{I_1} &= \|(U(\theta) - U(\theta'))\, R^{\eta'} R^{-\eta'} \hat{X}\|_{I_1} \\ &\leqslant \|(U(\theta) - U(\theta'))\, R^{\eta'}\|\ \|R^{-\eta'} \hat{X}\|_{I_1}.\end{aligned}$$

We now show $\|(U(\theta) - U(\theta'))\, \hat{X}\|_{I_1} \to 0$ as $|\theta - \theta'| \to 0$. In particular, given $\varepsilon_1 > 0$, we choose $\theta - \theta'$ sufficiently small so that $\|(U(\theta) - U(\theta'))\, \hat{X}\|_{I_1} \leqslant \varepsilon_1$. Let P_m denote the spectral projection for $H(\varepsilon, \lambda)$ onto the subspace for which $H(\varepsilon, \lambda) \leqslant m$. We write

$$\begin{aligned}\|(U(\theta) - U(\theta'))\, R^{\eta'}\| &= \|(U(\theta - \theta') - I)\, R^{\eta'}\| \\ &\leqslant \|(U(\theta - \theta') - I)\, R^{\eta'} P_m\| + \|(U(\theta - \theta') - I)\, R^{\eta'} (I - P_m)\|.\end{aligned}$$

Choose m sufficiently large so that

$$\|R^{\eta'}(I - P_m)\| = (m+1)^{-\eta'/2} \leqslant \tfrac{1}{4}\, \varepsilon_1 (\|R^{-\eta'} \hat{X}\|_{I_1})^{-1}.$$

With this choice of m,

$$\|(U(\theta - \theta') - I)\, R^{\eta'}(I - P_m)\| \leqslant 2\, \|R^{\eta'}(I - P_m)\| \leqslant \tfrac{1}{2} \varepsilon_1 (\|R^{-\eta'} \hat{X}\|_{I_1})^{-1}.$$

However, $U(\theta)$ commutes with $H(\varepsilon, \lambda)$, so P_m also commutes with $U(\theta)$. Thus $U(\theta)$ maps the finite dimensional subspace $P_m \mathscr{H}$ into itself. On any finite dimensional subspace, $U(\theta)$ is continuous in the trace norm: for $|\theta - \theta'|$ sufficiently small,

$$\begin{aligned}\|(U(\theta - \theta') - I)\, R^{\eta'} P_m\| &= \|(U(\theta - \theta') - I)\, P_m R^{\eta'}\| \\ &\leqslant \|(U(\theta - \theta') - I)\, P_m\| \leqslant \tfrac{1}{2} \varepsilon_1 (\|R^{-\eta'} \hat{X}\|_{I_1})^{-1}.\end{aligned}$$

Thus $\|(U(\theta - \theta') - I)\, R^{\eta'}\| \leqslant \varepsilon_1 (\|R^{-\eta'} \hat{X}\|_{I_1})^{-1}$, and therefore

$$\|(U(\theta) - U(\theta'))\, \hat{X}\|_{I_1} \leqslant \|(U(\theta) - U(\theta'))\, R^{\eta'}\|\ \|R^{-\eta'} \hat{X}\|_{I_1} \leqslant \varepsilon_1,$$

to complete the proof of continuity in θ.

An immediate application of Proposition VIII.6 is the proof of Theorem I.5(a), (b), and (d). Remark that the Γ-function is convex, and $\Gamma(1) = \Gamma(2) = 1$. Therefore, $\Gamma(\eta)$ is a monotonic decreasing function for $\eta \in (0, 1]$. We may obtain an upper bound on the value of the product in (VIII.23) by replacing each $\Gamma(\eta_j)$ by $\Gamma(\eta_{\min})$. This yields the bound (I.37).

VIII.2. *Continuity and Differentiability of* $\exp(-H(\varepsilon, \lambda))$

We establish the fundamental existence of the derivatives of $e^{-sH(\varepsilon, \lambda)}$ with respect to ε and λ for $\lambda > 0$. Since we differentiate the representation of $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$ under the trace and the time integrals, we begin by establishing the existence of the derivative of $\exp(-sH(\varepsilon, \lambda))$. We consider the differentiability of the heat kernel $e^{-sH(\varepsilon, \lambda)}$ not just as an operator, but (for fixed s) as a map from the unit square in (ε, λ) to an operator in the Schatten class $I_{s^{-1}}$. We actually obtain a somewhat stronger measure of differentiability of the heat kernel, namely as a map from the unit square to a space of trace-class, smoothing operators which map $\mathscr{H}_{-\beta}$ into $\mathscr{H}_{\alpha}$. Define the unit square $\square$ (less the origin) by

$$\square = \{(\varepsilon, \lambda) \colon 0 \leqslant \varepsilon, \lambda \leqslant 1 \text{ and } 0 < \varepsilon + \lambda\}. \tag{VIII.24}$$

Define the difference quotient Δ_λ for $\lambda \neq \lambda'$ and Δ_ε for $\varepsilon \neq \varepsilon'$ by

$$\Delta_\lambda = \frac{e^{-sH(\varepsilon, \lambda)} - e^{-sH(\varepsilon, \lambda')}}{\lambda - \lambda'}, \quad \text{and} \quad \Delta_\varepsilon = \frac{e^{-sH(\varepsilon, \lambda)} - e^{-sH(\varepsilon', \lambda)}}{\varepsilon - \varepsilon'}. \tag{VIII.25}$$

If V is a holomorphic polynomial satisfying Assumption E of Subsection II.4, and if $(\varepsilon, \lambda), (\varepsilon', \lambda') \in \square$, then Δ_λ and Δ_ε are bounded, trace-class operators. The formal limits of Δ_λ as $\lambda' \to \lambda$ or of Δ_ε as $\varepsilon' \to \varepsilon$ are given by

$$Y_\lambda = Y_\lambda(\varepsilon, \lambda) = -\int_0^s e^{-tH}(2\lambda\, |\partial V|^2 + W + W^*)\, e^{-(s-t)H}\, dt \tag{VIII.26}$$

and

$$Y_\varepsilon = Y_\varepsilon(\varepsilon, \lambda) = -\int_0^s e^{-tH} 2\varepsilon\, |z|^2\, e^{-(s-t)H}\, dt. \tag{VIII.27}$$

Also define

$$\delta_\lambda = \frac{H(\varepsilon, \lambda) - H(\varepsilon, \lambda')}{\lambda - \lambda'} \quad \text{and} \quad \delta_\varepsilon = \frac{H(\varepsilon, \lambda) - H(\varepsilon', \lambda)}{\varepsilon - \varepsilon'}, \tag{VIII.28}$$

so that

$$\delta_\lambda = (\lambda + \lambda')\, |\partial V|^2 + W + W^*, \quad \text{and} \quad \delta_\varepsilon = (\varepsilon + \varepsilon')\, |z|^2. \tag{VIII.29}$$

We use

$$R = R(\varepsilon, \lambda) = (H(\varepsilon, \lambda) + I)^{-1/2}. \tag{VIII.30}$$

Let $\operatorname{Tr}(e^{-H(\varepsilon, \lambda^{\#})/2})$ denote $\max_{\lambda'' \in \{\lambda, \lambda'\}} \operatorname{Tr}(e^{-H(\varepsilon, \lambda'')/2})$. Likewise, let $\operatorname{Tr}(e^{-H(\varepsilon^{\#}, \lambda)/2})$ denote $\max_{\varepsilon'' \in \{\varepsilon, \varepsilon'\}} \operatorname{Tr}(e^{-H(\varepsilon'', \lambda)/2})$. When confusion cannot occur, we simply denote the relevant upper bound by $\operatorname{Tr}(e^{-H^{\#}/2})$.

Proposition VIII.7. *Let V be a holomorphic polynomial of degree $l+1$ that satisfies Assumption* E *of Subsection* II.4. *Also suppose that $(\varepsilon, \lambda) \in \square$ and that $(\varepsilon', \lambda') \in \square$ is sufficiently close to (ε, λ). Choose $0 \leqslant \alpha$, β with $\alpha+\beta<2$, and define $\eta = 1-(\alpha+\beta)/2$. Then*

(a) *For $0<\lambda$, s, the operators Δ_λ, Y_λ, Δ_ε, and Y_ε satisfy the bounds*

$$\begin{aligned} &\|R^{-\alpha} \Delta_\lambda R^{-\beta}\|_{I_{s-1}} + \|R^{-\alpha} Y_\lambda R^{-\beta}\|_{I_{s-1}} \\ &\qquad \leqslant O(1)(\varepsilon+\lambda)^{-\eta/2}\, \lambda^{-1+\eta/2l} s^{-1+\eta/2} \eta^{-1+\eta/2} \eta^{-1} (\operatorname{Tr}(e^{-H^{\#}/2}))^s, \end{aligned} \tag{VIII.31}$$

$$\begin{aligned} &\|R^{-\alpha} \Delta_\varepsilon R^{-\beta}\|_{I_{s-1}} \\ &\qquad \leqslant O(1)(\varepsilon+\varepsilon')(\varepsilon+\lambda)^{-2}\, s^{-1+\eta} \eta^{-1} (\operatorname{Tr}(e^{-H^{\#}/2}))^s, \end{aligned} \tag{VIII.32}$$

and

$$\|R^{-\alpha} Y_\varepsilon R^{-\beta}\|_{I_{s-1}} \leqslant O(1)(\varepsilon+\lambda)^{-1}\, s^{-1+\eta} \eta^{-1} (\operatorname{Tr}(e^{-H^{\#}/2}))^s. \tag{VIII.33}$$

(b) *For $0<\lambda$, s, the difference quotient Δ_λ converges to Y_λ, and*

$$\begin{aligned} &\|R^{-\alpha}(\Delta_\lambda - Y_\lambda)\, R^{-\beta}\|_{I_{s-1}} \\ &\qquad \leqslant O(|\lambda-\lambda'|)\, \lambda^{-2+\eta/2l} (\varepsilon+\lambda)^{-\eta/2}\, s^{-1+\eta/2} \eta^{-2} (\operatorname{Tr}(e^{-H^{\#}/2}))^s. \end{aligned} \tag{VIII.34}$$

(c) *For $0<\varepsilon$, and $0<s$, the difference quotient Δ_ε converges to Y_ε as $\varepsilon' \to \varepsilon$ in the sense that*

$$\|R^{-\alpha}(\Delta_\varepsilon - Y_\varepsilon)\, R^{-\beta}\|_{I_{s-1}} \leqslant O(|\varepsilon-\varepsilon'|)(\varepsilon+\lambda)^{-2}\, s^{-1+\eta} \eta^{-2} (\operatorname{Tr}(e^{-H^{\#}/2}))^s. \tag{VIII.35}$$

(d) *Suppose that $0<\varepsilon$, $0<s$, and $0 \leqslant \lambda$, λ'. Then the heat kernels $e^{-sH(\varepsilon, \lambda)}$ are uniformly Hölder continuous in the sense that*

$$\begin{aligned} &\|R^{-\alpha}(e^{-sH(\varepsilon, \lambda)} - e^{-sH(\varepsilon, \lambda')})\, R^{-\beta}\|_{I_{s-1}} \\ &\qquad \leqslant O(1)\, |\lambda-\lambda'|^{\eta/2l}\, \varepsilon^{-\eta} s^{-1+\eta/2} \eta^{-1} (\operatorname{Tr}(e^{-H^{\#}/2}))^s. \end{aligned} \tag{VIII.36}$$

(e) *For $0<\lambda$, $0<s$, and $0 \leqslant \varepsilon$, ε', the heat kernels $e^{-sH(\varepsilon, \lambda)}$ are uniformly Hölder continuous in the sense that*

$$\begin{aligned} &\|R^{-\alpha}(e^{-sH(\varepsilon, \lambda)} - e^{-sH(\varepsilon', \lambda)})\, R^{-\beta}\|_{I_{s-1}} \\ &\qquad \leqslant O(1)(\varepsilon^2 - \varepsilon'^2)\, \lambda^{-2} s^{-1+\eta} \eta^{-1} (\operatorname{Tr}(e^{-H^{\#}/2}))^s. \end{aligned} \tag{VIII.37}$$

Remark. We write

$$\frac{\partial}{\partial \lambda} e^{-sH(\varepsilon, \lambda)} = Y_\lambda(\varepsilon, \lambda), \qquad \frac{\partial}{\partial \varepsilon} e^{-sH(\varepsilon, \lambda)} = Y_\varepsilon(\varepsilon, \lambda),$$

where the derivatives converge in the sense of (b) and of (c), respectively. The proof of Theorem I.5(c) follows from the existence of the (ε, λ) derivatives of $e^{-sH(\theta, \lambda)}$, and the bounds (VIII.31), (VIII.32). Integrating the (ε, λ) derivatives establishes the claimed continuity, and completes the proof of Theorem 1.5.

Lemma VIII.8. *Let V be a holomorphic polynomial of degree $l+1$ satisfying Assumption* E *of Subsection* II.4.

(a) *Let $0<\lambda, \lambda', s$, let $H=H(\varepsilon, \lambda)$, and $H'=H(\varepsilon, \lambda')$. Then*

$$\Delta_\lambda = -\int_0^s e^{-tH} \delta_\lambda e^{-(s-t)H'}\, dt \tag{VIII.38}$$

and

$$\Delta_\lambda - Y_\lambda = (\lambda-\lambda') \int_0^s e^{-sH} |\partial V|^2 e^{-(s-t)H}\, dt + (\lambda-\lambda') \int_0^s dt \int_0^{s-t} du\, e^{-tH} \delta_\lambda e^{-uH} \delta_\lambda e^{-(s-t-u)H'}. \tag{VIII.39}$$

(b) *Let $0<\varepsilon+\lambda, \varepsilon'+\lambda', s$, let $H=H(\varepsilon, \lambda)$, and $H'=H(\varepsilon', \lambda)$. Then*

$$\Delta_\varepsilon = -(\varepsilon+\varepsilon') \int_0^s e^{-tH} |z|^2 e^{-(s-t)H'}\, dt \tag{VIII.40}$$

and

$$\Delta_\varepsilon - Y_\varepsilon = (\varepsilon-\varepsilon') \int_0^s e^{-tH} |z|^2 e^{-(s-t)H}\, dt + (\varepsilon-\varepsilon') \int_0^s dt \int_0^{s-t} du\, e^{-tH} \delta_\varepsilon e^{-uH} \delta_\varepsilon e^{-(s-t-u)H'}. \tag{VIII.41}$$

Proof. Expression (VIII.38) is the Duhamel formula for the derivative. This is justified for $\lambda>0$ as $\mathscr{D}(H')=\mathscr{D}(H) \supset \mathscr{D}(\delta_\lambda)$ by (VII.13) and (VII.7). To prove (VIII.39), we also use the representation (VIII.26). Then $\delta_\lambda - (2\lambda\, |\partial V|^2 + W + W^*) = (\lambda-\lambda)\, |\partial V|^2$. Hence

$$\Delta_\lambda - Y_\lambda = (\lambda-\lambda') \int_0^s e^{-tH} |\partial V|^2 e^{-(s-t)H}\, dt + \int_0^s e^{-tH} \delta_\lambda (e^{-(s-t)H'} - e^{-(s-t)H})\, dt.$$

Expanding the last term by the Duhamel formula yields (VIII.39). The proof of (VIII.40), (VIII.41) is similar, using $\mathscr{D}(H) \cap \mathscr{D}(H') \supset \mathscr{D}(|z|^2)$, see (VII.14).

Recall $\eta = 1-((\alpha+\beta)/2)>0$. Define a, b as

$$a = 2-\alpha-\eta/2 \qquad \text{and} \qquad b = 2-\beta-\eta/2. \tag{VIII.42}$$

Then

$$\frac{3\eta}{2} \leqslant a, \quad b < 2 \text{ and } a + b = 2 + \eta, \quad \text{and} \quad 2 - \left(\frac{a + b + \alpha + \beta}{2}\right) = \frac{\eta}{2}. \tag{VIII.43}$$

In case $\alpha = \beta$, then $a = b = 1 + \eta/2$.

Lemma 9. *Assume the above choice of a and b, and let $\lambda/2 \leqslant \lambda' \leqslant 2\lambda$. Then for all $\eta' \in [0, \eta]$,*

$$\| R^a \, |\partial V|^2 \, R^b \| \leqslant O(1)(\varepsilon + \lambda)^{-\eta'/2} \, \lambda^{-2 + \eta^1/2l}, \tag{VIII.44}$$

and

$$\| R^a \delta_\lambda R^b \| \leqslant O(1)(\varepsilon + \lambda)^{-\eta'/2} \, \lambda^{-1 + \eta'/2l}. \tag{VIII.45}$$

Proof. Since both $R^a \delta_\lambda R^b$ and its adjoint $R^b \delta_\lambda R^a$ have the same norm, we may assume $a \geqslant b$. Thus $b \geqslant \eta$, $a \geqslant 1 + \eta/2$. Let us first bound the $W + W^*$ contribution to δ_λ. Let $\alpha_1 + \beta_1 = 1$. By Lemma VII.4,

$$\| R^a (W + W^*) \, R^b \| \leqslant \| \, |W + W^*|^{\alpha_1} \, R^a \| \; \| \, |W + W^*|^{\beta_1} \, R^b \|.$$

Take $\beta_1 = \eta/2b$. Then by (VI.4), we have for $0 \leqslant \alpha \leqslant 1$,

$$|W + W^*|^\alpha \leqslant M_3^{\alpha/2} (|\partial V|^2 + 1)^{\alpha/2} \leqslant (2M_3)^{\alpha/2} \, (|\partial V|^\alpha + 1).$$

Hence by (VII.16),

$$\| \, |W + W^*|^{\beta_1} \, R^b \| \leqslant O((\varepsilon + \lambda)^{-\eta/2}).$$

Also $\alpha_1 = (1 - \eta/2l) < 1 < a$. Therefore by (VII.13),

$$\| \, |W + W^*|^{\alpha_1} \, R^a \| \leqslant O(\lambda^{-1 + \eta/2l}).$$

Combining these estimates,

$$\| R^a (W + W^*) \, R^b \| \leqslant O(1)(\varepsilon + \lambda)^{-\beta/2} \, \lambda^{-1 + \eta/2l},$$

as desired.

The bound on $(\lambda + \lambda') \, |\partial V|^2$ is somewhat more involved. We use the representation for the fractional power of a positive number, $s^{-a} = (\sin(\pi a)/\pi) \int_0^\infty t^{-a} (t + s)^{-1} \, dt$, valid for $0 < a < 1$. Substituting $a/2$ for a, we obtain

$$R^a = \frac{\sin(\pi a/2)}{\pi} \int_0^\infty t^{-a/2} R(t)^2 \, dt, \qquad \text{for} \quad 0 < a < 2. \tag{VIII.46}$$

Here $R(t) = (H(\varepsilon, \lambda) + (t+1)\, I)^{-1/2}$. Note

$$\| R(t)^{\eta} \| \leqslant (1+t)^{-\eta/2} \qquad \text{for} \quad \eta > 0. \tag{VIII.47}$$

Write

$$\begin{aligned} R^a\, |\partial V|^2\, R^b &= \sum_{j=1}^{n} R^{\eta/2} \partial_j V R^{a-\eta/2}\, \overline{\partial_j V}\, R^b + \sum_{j=1}^{n} R^{\eta/2} [R^{-a-\eta/2}, \partial_j V]\, \overline{\partial_j V}\, R^b \\ &= \sum_{j=1}^{n} R^{\eta/2} \partial_j V R^{a-\eta/2}\, \overline{\partial_j V}\, R^b + \sum_{j=1}^{n} \frac{\sin(\pi(a-\eta/2)/2)}{\pi} \\ &\quad \times \int_0^\infty t^{-(a-\eta/2)/2} R(t)^2\, R^{\eta/2} X_j R(t)^2\, \overline{\partial_j V}\, R^b\, dt, \end{aligned} \tag{VIII.48}$$

where

$$X_j = [\partial_j V, H(\varepsilon, \lambda)] = -[\partial_j V, \partial \bar{\partial}] = \sum_{k=1}^{n} (\partial_j \partial_k V)\, \bar{\partial}_k.$$

Bound the first term on the right of (VIII.48) using (VII.13), (VII.16) and the fact that $a - \eta/2 \geqslant 1$. Therefore using Lemma VII.4 and (VII.16),

$$\begin{aligned} \| R^{\eta/2} \partial_j V R^{a-\eta/2}\, \overline{\partial_j V}\, R^b \| &\leqslant \| R^{\eta/2}\, |\partial_j V|^{\eta/2l} \| \; \| \, |\partial_j V|^{(1-\eta/2l)}\, R^{(1-\eta/2l)} \| \\ &\qquad \times \| R^{a-\eta/2-1+\eta/2l}\, |\partial_j V|\, R^b \| \\ &\leqslant O(\varepsilon+\lambda)^{-\eta/2}\, \lambda^{-1+\eta/2l}\, \| R^{a-\eta/2-1+\eta/2l}\, |\partial_j V|\, R^b \|. \end{aligned}$$

Since $a + b - 1 - \eta/2 + \eta/2l = 1 + \eta/2 + \eta/2l > 1$, we have by (VII.13),

$$\| R^{\eta/2} \partial_j V R^{a-\eta/2}\, \overline{\partial_j V}\, R^b \| \leqslant O((\varepsilon+\lambda)^{-\eta/2}\, \lambda^{-2+\eta/2l}). \tag{VIII.49}$$

Next we bound the second term on the right of (VIII.48). Note $b > \eta/2$, so

$$\begin{aligned} &\| R(t)^2\, R^{\eta/2} (\partial_j \partial_k V)\, \bar{\partial}_k R(t)^2\, \overline{\partial_j V}\, R^b \| \\ &\quad \leqslant \| R(t)^{1+\eta/2} \| \; \| R(t)^{1-\eta/2}\, R^{\eta/2}\, \partial_j \partial_k V \| \; \| \bar{\partial}_k R(t) \| \\ &\qquad \times \| R(t)\, |\overline{\partial_j V}|^{1-\eta/2l} \| \; \| \, |\overline{\partial_j V}|^{\eta/2l}\, R^b \| \\ &\quad \leqslant O(1)(1+t)^{-1/2-\eta/4}\, \lambda^{-2+\eta/2l} (\varepsilon+\lambda)^{-\eta/2}. \end{aligned} \tag{VIII.50}$$

Here we use the estimate (VIII.47) along with (II.37), Lemma VII.4, and (VII.13). From (VIII.50) we infer

$$\begin{aligned} &\left\| \int_0^\infty t^{-a/2+\eta/4} R(t)^2\, R^{\eta/2} X_j R(t)^2\, \overline{\partial_j V}\, R^b\, dt \right\| \\ &\qquad \leqslant O(1)(\varepsilon+\lambda)^{-2\eta/2}\, \lambda^{-2+\eta/2l} \int_0^\infty t^{-a/2+\eta/4} (1+t)^{-1/2-\eta/4}\, dt. \end{aligned} \tag{VIII.51}$$

As $a \geqslant 1 + \eta/2$ and $a/2 \geqslant 1/2 + \eta/4$, we bound (VIII.49) from above by

$$O(1)(\varepsilon + \lambda)^{-\eta/2} \lambda^{-2+\eta/2l} \int_0^\infty t^{-1/2}(1+t)^{-1/2-\eta/4}\, dt$$
$$\leqslant O(1)(\varepsilon + \lambda)^{-\eta/2} \lambda^{-2+\eta/2l}. \tag{VIII.52}$$

Combined with (VIII.47), and $\delta_\lambda = (\lambda + \lambda')\, |\partial V|^2 + W + W^*$, the proof of the lemma is complete.

Proof of Proposition VIII.7. (a) Let us bound $R^{-\alpha} \Delta_\lambda R^{-\beta}$. Define $\Delta_\lambda(t) = -e^{-tH} \delta_\lambda e^{-(s-t) H'}$, the density of Δ_λ in (VIII.38). For $0 < t < s$ we estimate $\Delta_\lambda(t)$ with Hölder's inequality obtaining

$$\begin{aligned} &\|R^{-\alpha} \Delta_\lambda(t)\, R^{-\beta}\|_{I_s - 1} \\ &\quad\leqslant \|R^{-\alpha} e^{-tH/4}\|\, \|e^{-tH/2}\|_{I_t - 1}\, \|e^{-tH/4}\, \delta_\lambda e^{-(s-t)\, H'/4}\| \\ &\qquad\times \|e^{-(s-t)\, H'/2}\|_{I_{(s-t)} - 1}\, \|e^{-(s-t)\, H'/4} (R')^{-\beta}\|\, \|(R')^\beta\, R^{-\beta}\|. \end{aligned} \tag{VIII.55}$$

As λ' is sufficiently close to λ, we may assume $\lambda/2 \leqslant \lambda' \leqslant 2\lambda$. Thus Proposition (VII.7) ensures that

$$\|R^\beta (R')^{-\beta}\| + \|(R')^\beta\, R^{-\beta}\| \leqslant O(1). \tag{VIII.56}$$

Choose $0 \leqslant a, b$ as in (VIII.42). Then

$$\begin{aligned} \|R^{-\alpha} \Delta_\lambda(t)\, R^{-\beta}\| &\leqslant O(1)\, t^{-\alpha/2} (s-t)^{-\beta/2}\, (\mathrm{Tr}(e^{-H^\#/2}))^s\, \|R^\alpha \delta_\lambda R^b\| \\ &\leqslant O(1)\, t^{-(\alpha+a)/2} (s-t)^{-(\beta+b)/2}\, (\mathrm{Tr}(e^{-H^\#/2}))^s\, (\varepsilon+\lambda)^{-\eta/2}\, \lambda^{-1+\eta/2l}, \end{aligned} \tag{VIII.57}$$

where in the last step we use Lemma VIII.9. We integrate this bound over t. Using the definition of the beta function, we obtain

$$\begin{aligned} \|R^{-\alpha} \Delta_\lambda R^{-\beta}\|_{I_s - 1} &\leqslant \int_0^s \|R^{-\alpha} \Delta_\lambda(t)\, R^{-\beta}\|_{I_s - 1}\, dt \\ &\leqslant O(1)(\varepsilon + \lambda)^{-\eta/2}\, \lambda^{-1+\eta/2l} s^{-1+\eta/2} (\mathrm{Tr}(e^{-H^\#/2}))^s \\ &\qquad \times \frac{\Gamma((2-\alpha-a)/2)\, \Gamma((2-\beta-b)/2)}{\Gamma((4-\alpha-\beta-a-b)/2)}. \end{aligned} \tag{VIII.58}$$

The relations (VIII.42), (VIII.43) show that the ratio of Γ functions equals

$$\frac{\Gamma(\eta/4)^2}{\Gamma(\eta/2)} \leqslant O(\eta^{-1}). \tag{VIII.59}$$

Inserting (VIII.59) in (VIII.58) gives the bound on Δ_λ in (VIII.31). The bound on Y_λ is similar, with 2λ in Y_λ substituting for $\lambda + \lambda'$ in Δ_λ.

In order to bound $\varDelta_\varepsilon$, we use the representation (VIII.40). We proceed as above defining $\varDelta_\varepsilon(t)$, and using Hölder's inequality to establish

$$\|R^{-\alpha}\varDelta_\varepsilon(t)\, R^{-\beta}\|_{I_s-1} \leqslant O(1)\, t^{-\alpha/2}(s-t)^{-\beta/2}\, (\mathrm{Tr}(e^{-H^{\#}/2}))^s\, \|e^{-tH/4}\delta_\varepsilon e^{-(s-t)\, H'/4}\|. \tag{VIII.60}$$

Here we use the estimate (VII.40) to establish equivalence of the $\mathscr{H}(\varepsilon, \lambda)$ norms for varying ε but fixed λ.

Now define

$$a = 1 - \left(\frac{\alpha - \beta}{2}\right), \qquad b = 1 + \left(\frac{\alpha - \beta}{2}\right). \tag{VIII.61}$$

Thus the constants a and b satisfy

$$a, b \in (0, 2), \qquad \text{and} \qquad a + b = 2, \tag{VIII.62}$$

and

$$1 - \left(\frac{\alpha + a}{2}\right) = \frac{\eta}{2} = 1 - \left(\frac{\beta + b}{2}\right). \tag{VIII.63}$$

We infer that $\Gamma(1-((\alpha+a)/2))\, \Gamma(1-((\beta+b)/2))\, \Gamma(2-((\alpha+\beta+a+b)/2))^{-1} = \Gamma(\eta/2)^2/\Gamma(\eta)$. Proceeding as above, but with (VII.14), (VII.15) providing the bound on $\|R^a\delta_\varepsilon R^b\|$, we obtain

$$\|R^{-\alpha}\varDelta_\varepsilon R^{-\beta}\|_{I_s-1} \leqslant O(1)(\varepsilon+\varepsilon')(\varepsilon+\lambda)^{-2}\, s^{-1+\eta}\eta^{-1}(\mathrm{Tr}(e^{-H^{\#}/2}))^s. \tag{VIII.64}$$

Similarly we obtain the bound on Y_λ.

(b) In order to establish convergence of $\varDelta_\lambda$ to Y_λ, we use the representation (VIII.39). We bound each of the terms in (VIII.39) separately. For the first term, the density can be bounded by using the method above and Lemma VIII.9, yielding

$$\begin{aligned} &|\lambda - \lambda'| \int_0^s \|R^{-\alpha}e^{-tH}\, |\partial V|^2\, e^{-(s-t)\, H}R^{-\beta}\|_{I_s-1}\, dt \\ &\qquad \leqslant O(1)\, |\lambda-\lambda'|\, (\varepsilon+\lambda)^{-\eta/2}\, \lambda^{-2+\eta/2l} s^{-1+\eta/2}\eta^{-1}(\mathrm{Tr}(e^{-H/2}))^s. \end{aligned} \tag{VIII.65}$$

This converges to zero as $\lambda' \to \lambda$ with λ, and as $\eta \leqslant 1$, it is bounded by (VIII.34).

For the second term, let

$$F(t, u) = (\lambda - \lambda')\, R^{-\alpha}e^{-tH}\delta_\lambda e^{-uH}\delta_\lambda e^{-(s-t-u)\, H'}R^{-\beta}.$$

Choose a, b as in (VIII.42), and define

$$a_1 = \beta + \eta/4, \qquad b_1 = 2 - \beta - \eta/4. \tag{VIII.66}$$

Then

$$\alpha + a - 2 - \eta/2 < 2, \quad \text{and} \quad b + a_1 = b_1 + \beta - 2 - \eta/4 < 2. \tag{VIII.67}$$

Furthermore,

$$1 - \left(\frac{\alpha + a}{2}\right) = \eta/4, \quad \text{and} \quad 1 - \left(\frac{b + a_1}{2}\right) = \frac{\eta}{8} = 1 - \left(\frac{b_1 + \beta}{2}\right), \tag{VIII.68}$$

while

$$a + b = 2 + \eta, \qquad a_1 + b_1 = 2. \tag{VIII.69}$$

Thus it is possible to proceed as above to bound

$$\|F(t, u)\|_{I_s - 1} \leqslant O(|\lambda - \lambda'|)\, t^{-(\alpha + a)/2} u^{-(b + a_1)/2} (s - t - u)^{-(b_1 + \beta)/2} \\ \times \|R^a \delta_\lambda R^b\| \; \|R^{a_1} \delta_\lambda R^{b_1}\| \; (\operatorname{Tr}(e^{-H^\#/2}))^s.$$

Use Lemma VIII.9 to bound $\|R^a \delta_\lambda R^b\|$ and use (VII.2), (VII.15) to bound $\|R^{a_1} \delta_\lambda R^{b_1}\|$. Hence

$$\|F(t, u)\|_{I_s - 1} \leqslant O(1)\, t^{-(\alpha + a)/2} u^{-(b + a_1)/2} (s - t - u)^{-(b_1 + \beta)/2} \\ \times |\lambda - \lambda'|\, (\varepsilon + \lambda)^{-\eta/2}\, \lambda^{-2 + \eta/2l} (\operatorname{Tr}(e^{-H^\#/2}))^s. \tag{VIII.70}$$

Integrating this bound, we obtain

$$\left\| \int_0^s dt \int_0^{s-t} du\, F(t, u) \right\|_{I_s - 1} \\ \leqslant O(1)\, s^{-1 + \eta/2} \frac{\Gamma(\eta/8)^2\, \Gamma(\eta/4)}{\Gamma(\eta/2)} \\ \times |\lambda - \lambda'|\, (\varepsilon + \lambda)^{-\eta/2}\, \lambda^{-2 + \eta/2l} (\operatorname{Tr}(e^{-H^\#/2}))^s. \tag{VIII.71}$$

Using $\Gamma(\eta/8)^2\, \Gamma(\eta/4)/\Gamma(\eta/2) \leqslant O(\eta^{-2})$, we therefore have a bound of the form (VIII.36). The two inequalities (VIII.63), (VIII.6a) then prove (VIII.36).

The proof of (c) is similar, and we omit the details. As in the proof of (VIII.33)–(VIII.34), we use the equivalence of $\mathscr{H}(\varepsilon, \lambda)$ norms for different values of ε. As $\varepsilon > 0$, we can assume $\varepsilon/2 < \varepsilon' < 2\varepsilon$ and use the bound (VII.38) with $\lambda' = \lambda$.

In order to estimate the Hölder continuity of $e^{-sH(\varepsilon, \lambda)}$, we can use the bounds on Δ_λ, Δ_ε. For example, if $|\lambda'| \leqslant O(\lambda)$, then by (VIII.33),

$$\|R^{-\alpha}(e^{-sH(\varepsilon, \lambda)} - e^{-sH(\varepsilon, \lambda')})\, R^{-\beta}\| \\ = |\lambda - \lambda'|\; \|R^{-\alpha} \Delta_\lambda R^{-\beta}\|_{I_s - 1} \\ \leqslant O(1)\, |\lambda - \lambda'|^{\eta/2l}\, \lambda^{1 - \eta/2l} (\varepsilon + \lambda)^{-\eta/2}\, \lambda^{-1 + \eta/2l} s^{-1 + \eta/2} \eta^{-1} (\operatorname{Tr}(e^{-H^\#/2}))^s \\ \leqslant O(1)\, |\lambda - \lambda'|^{\eta/2l}\, (\varepsilon + \lambda)^{-\eta/2}\, s^{-1 + \eta/2} \eta^{-1} (\operatorname{Tr}(e^{-H^\#/2}))^s. \tag{VIII.72}$$

This estimate is uniform in λ as $\lambda \to 0$ with ε fixed. Thus $e^{-sH(\varepsilon,\lambda)}$ also converges as $\lambda \to 0$ with ε fixed. The proof of the remaining bounds proceeds in a similar fashion.

IX. THE HOLONOMY EXPANSION FOR $\mathfrak{Z}(\varepsilon, \lambda; I; \theta)$ AND THE PROOF OF THEOREM I.1

We prove Theorem I.1 in this section. We derive formulas for the gradient of $\mathfrak{Z}$, with respect to the variables (ε, λ), and we define and estimate a holonomy expansion for each component. In the first subsection we give the preliminary holonomy expansion steps, that are designed to isolate leading powers of ε and λ in the components of the gradient. Eventually we show that the coefficients of these polynomial factors determine the leading asymptotic behavior of the components of the gradient $(\varepsilon, \lambda) \to (0, 0)$. In Subsection IX.2 we introduce a special regularization that allows us to continue our expansion. We also show that the regularization can be removed under appropriate circumstances. In Subsection IX.3 we prove that a wide class of expectations, including those that occur as a result of the expansions in Subsection IX.1, remain bounded as $(\varepsilon, \lambda) \to (0, 0)$. We use and also remove the regularization introduced in Subsection IX.2 to expand the expectations in question. Combining these results, we obtain our bounds on grad $\mathfrak{Z}$.

IX.1. *Preliminary Expansion Moves*

In this section we give some further simple motivation for the holonomy moves by deriving identities for the gradient of the equivariant index (I.10), namely for

$$\mathfrak{Z}_\lambda = \frac{\partial}{\partial \lambda} \mathfrak{Z}(\varepsilon, \lambda; I; \theta) = \frac{\partial}{\partial \lambda} \langle I; \varepsilon, \lambda; \theta \rangle_0, \qquad \text{and}$$

$$\mathfrak{Z}_\varepsilon = \frac{\partial \mathfrak{Z}(\varepsilon, \lambda; I; \theta)}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} \langle I; \varepsilon, \lambda; \theta \rangle_0. \tag{IX.1}$$

In order to simplify our notation, for the remainder of this section we suppress the explicit dependence of expectations in ε, λ, and θ, writing $\langle X_0, ..., X_k \rangle_0$ in place of $\langle X_0, ..., X_k; \varepsilon, \lambda; \theta \rangle_k$. As in (VI.12), denote $\dot{Q} = \partial Q(\lambda)/\partial \lambda = \psi_2 \overline{\partial V} + \psi_2^* \partial V$. Also introduce the vertex $\mathscr{L}$ defined by

$$\varepsilon^2 \mathscr{L} = [Q(\lambda), H(\varepsilon, \lambda)] \qquad \text{so} \quad \mathscr{L} = \psi_1 \bar{z} - \psi_1^* z. \tag{IX.2}$$

We also use the notation

$$\psi_1(\partial \partial_j V) = \sum_{k=1}^{n} \psi_1^{(k)} \partial_k \partial_j V. \tag{IX.3}$$

In general, we do not indicate summation of component indices within a vertex such as $\psi_1(\partial V)$.

Proposition IX.1. *Let V satisfy both Assumption* E *of Subsection* II.4 *and Assumption* Q *of Subsection* II.5. *Let* $0<\lambda$, *and let* $\theta\notin\Upsilon_{\mathrm{sing}}$.

(a) *The λ-derivative has the representation*

$$\begin{aligned}\mathfrak{Z}_\lambda &=\varepsilon^2\langle\dot{Q},\mathscr{L};\varepsilon,\lambda;\theta\rangle_1\\ &=\varepsilon^2\langle\psi_2^*(\partial V),\psi_1\bar{z}\rangle_1-\varepsilon^2\langle\psi_2\overline{(\partial V)},\psi_1^*z\rangle_1\\ &=\lambda\varepsilon^2\sum_{j=1}^{n}\left(-\overline{c_j(\theta)}\langle(\partial_jV),\psi_1^*\overline{(\partial\partial_jV)},\psi_1\bar{z}\rangle_2+\overline{c_j(\theta)}\langle(\partial_jV),\psi_1\bar{z},\psi_1^*\overline{(\partial\partial_jV)}\rangle_2\right.\\ &\qquad\left.+c_j(\theta)\langle\overline{(\partial_jV)},\psi_1(\partial\partial_jV),\psi_1^*z\rangle_2-c_j(\theta)\langle\overline{(\partial_jV)},\psi_1^*z,\psi_1(\partial\partial_jV)\rangle_2\right),\end{aligned}\tag{IX.4}$$

where

$$c_j(\theta)=\frac{ie^{i(1-\omega_j)\,\theta/2}}{2\sin((1-\omega_j)\,\theta/2)}.\tag{IX.5}$$

(b) *The ε-derivative has the form*

$$\begin{aligned}\mathfrak{Z}_\varepsilon &=2\varepsilon\sum_{j=1}^{n}d_j(\theta)\langle\bar{z}_j,\bar{\partial}_j\rangle_1\\ &=\frac{1}{2}\,\varepsilon\sum_{j=1}^{n}\frac{1}{\sin^2(\omega_j\theta/2)}\left(\langle I\rangle_0+\langle\partial_j,\bar{\partial}_j\rangle_1\right),\end{aligned}\tag{IX.6}$$

where

$$d_j(\theta)=\frac{ie^{i\omega_j\theta/2}}{2\sin(\omega_j\theta/2)}.\tag{IX.7}$$

Remark. In (IX.4) and (IX.6) we isolate the behavior of $\mathfrak{Z}_\lambda$ and of $\mathfrak{Z}_\varepsilon$ as a function of ε and λ as $(\varepsilon,\lambda)\to(0,0)$. In Theorem IX.2.3 of the next section, we show that the expectations $|\mathfrak{Z}_\lambda/\lambda\varepsilon^2|$ and $|\mathfrak{Z}_\varepsilon/\varepsilon|$ are uniformly bounded in ε, λ for fixed $\theta\notin\Upsilon_{\mathrm{sing}}$. In fact they also do not vanish for small ε and λ, so this reflects the actual small (ε,λ)-asymptotics.

Proof. We use Proposition VIII.7 to justify differentiation of $e^{-H(\varepsilon,\lambda)}$ under the integral and the trace with value (VIII.26). The new λ-derivative vertex has the form $2\lambda\,|\partial V|^2+W+W^*$, but it conveniently also equals $Q\dot{Q}+\dot{Q}Q$. Thus we write the λ derivative as

$$\begin{aligned}\mathfrak{Z}_\lambda &=-\operatorname{Tr}\left(\gamma U(\theta)\int_0^1 e^{-tH(\varepsilon,\lambda)}(Q\dot{Q}+\dot{Q}Q)\,e^{-(1-t)\,H(\varepsilon,\lambda)}\,dt\right)\\ &=-\langle I,Q\dot{Q}+\dot{Q}Q\rangle_1.\end{aligned}\tag{IX.8}$$

We should remark that the justification of (IX.7) requires a condition on domains, which has already been taken into account in the proof of Proposition VIII.7. The vertex $Q\dot{Q}$, for example, is an element of $\mathscr{T}_{\varepsilon, \lambda}(-1, 1)$, and $\eta_0 = \eta_1 = \frac{1}{2}$.[3]

If ε were equal to zero, the derivative (IX.7) would of course be zero. For $\varepsilon \neq 0$, we can approximately cancel the $Q\dot{Q}$ term with the $\dot{Q}Q$ term by performing a holonomy on Q in the $Q\dot{Q}$-term. We do this carefully below, yielding the first equality in (IX.4). In order to derive this Q-holonomy relation we use four basic identities that we now state. The first two identities are operator relations. We have

$$e^{-tH} Q = Q e^{-tH} + \int_0^t e^{-uH} [Q, H]\, e^{-(t-u)H}\, du.$$

In our case of Q and H this equals

$$e^{-tH} Q = Q e^{-tH} + \int_0^t e^{-uH} \mathscr{L} e^{-(t-u)H}\, du.$$

Thus moving Q to the left through a heat kernel introduces a new $\mathscr{L}$-vertex into an expectation, with a coefficient ε^2, where $\mathscr{L} \in \mathscr{T}_{\varepsilon, \lambda}(0, 1) \cup \mathscr{T}_{\varepsilon, \lambda}(-1, 0)$.

The second identity expressed is the transformation law for Q under the $\mathbb{Z}_2$ and $U(1)$ groups, namely

$$\gamma U(\theta)^* \, Q U(\theta)\, \gamma = -Q. \tag{IX.9}$$

The remaining two identities are general relations for expectations. The first of these reflects cyclicity of the trace, and the action of the groups $\mathbb{Z}_2$ and $U(1)$ on the operators X_j. Define

$$X_j^{g^{-1}} = \gamma U(\theta)^* \, X_j U(\theta)\, \gamma. \tag{IX.10}$$

Then

$$\langle X_0, X_1, ..., X_k \rangle_k = \langle X_k^{g^{-1}}, X_0, ..., X_{k-1} \rangle_k. \tag{IX.11}$$

One particular case of this relation we use here is

$$\langle Q, \dot{Q} \rangle_1 = -\langle \dot{Q}, Q \rangle_1. \tag{IX.12}$$

The fourth identity is a relation between the expectation of $(k+1)$ vertices and a set of expectations of $(k+2)$ vertices, generalizing $\langle X_0 \rangle_0 = \langle I, X_0 \rangle_1$. For regular sets of vertices, we have

$$\langle X_0, X_1, ..., X_k \rangle_k = \sum_{j=1}^{k+1} \langle X_0, X_1, ..., X_{j-1}, I, X_j, ..., X_k \rangle_{k+1}. \tag{IX.13}$$

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[3] We use here a trick allowing us to apply Proposition VIII.7 without modification to the expectation of a single vertex $X_0 \in \mathscr{T}_{\varepsilon, \lambda}(-\beta, \alpha)$ in case $\alpha, \beta < 2$ instead of $\alpha + \beta < 2$. Note the identity $\langle X_0 \rangle_0 = \langle I, X_0 \rangle_1$. In the second expectation, we take $\eta_0 = 1 - \beta/2$ and $\eta_1 = 1 - \alpha/2$, with the requirement of regularity being $0 < \eta_0, \eta_1$.

The proof of this relation involves a change of variables in the definition of the Radon transform. See for instance Corollary V.4, and in particular the identity (V.68), in [QHA].

We now give a step-by-step derivation of the relation (IX.4) for $\mathfrak{Z}_\lambda$. We start from the representation (IX.8); each equality that follows arises from the application of one of the four identities cited above. We obtain

$$\begin{aligned}
\mathfrak{Z}_\lambda &= -\langle I, Q\dot{Q}\rangle_1 - \langle I, \dot{Q}Q\rangle_1 \\
&= -\langle Q, \dot{Q}\rangle_1 - \varepsilon^2\langle I, \mathscr{L}, \dot{Q}\rangle_2 - \langle I, \dot{Q}Q\rangle_1 \\
&= \langle \dot{Q}, Q\rangle_1 - \varepsilon^2\langle I, \mathscr{L}, \dot{Q}\rangle_2 - \langle I, \dot{Q}Q\rangle_1 \\
&= \langle \dot{Q}Q, I\rangle_1 + \varepsilon^2\langle \dot{Q}, \mathscr{L}, I\rangle_2 - \varepsilon^2\langle I, \mathscr{L}, \dot{Q}\rangle_2 - \langle I, \dot{Q}Q\rangle_1 \\
&= \langle I, \dot{Q}Q\rangle_1 + \varepsilon^2\langle \dot{Q}, \mathscr{L}, I\rangle_2 - \varepsilon^2\langle I, \mathscr{L}, \dot{Q}\rangle_2 - \langle I, \dot{Q}Q\rangle_1 \\
&= \varepsilon^2\langle \dot{Q}, \mathscr{L}, I\rangle_2 - \varepsilon^2\langle I, \mathscr{L}, \dot{Q}\rangle_2 \\
&= \varepsilon^2\langle \dot{Q}, \mathscr{L}, I\rangle_2 + \varepsilon^2\langle \dot{Q}, I, \mathscr{L}\rangle_2 \\
&= \varepsilon^2\langle \dot{Q}, \mathscr{L}\rangle_1,
\end{aligned} \tag{IX.14}$$

as desired.

Next we derive the second equality in (IX.4). Both $\dot{Q}$ and $\mathscr{L}$ commute with the group $U(\theta)$. We now consider the action of the second $U(1)$-group $\tilde{U}(\theta)$ defined in Subsection III.2. In particular take the two operators $\dot{Q}_+ = \psi_2^* \, \partial V$ and $\mathscr{L}_+ = -\psi_1^* z$ that are invariant under $U(\theta)$ but not under $\tilde{U}(\theta)$. Note that

$$\tilde{U}(\theta)\, Q_+\, \tilde{U}(\theta)^* = e^{i\theta} Q_+ \qquad \text{and} \qquad \tilde{U}(\theta)\, \mathscr{L}_+\, \tilde{U}(\theta)^* = e^{i\theta}\mathscr{L}_+. \tag{IX.15}$$

The related operators $Q_- = Q_+^*$ and $\mathscr{L}_- = -\mathscr{L}_+^*$ for which

$$\dot{Q} = \dot{Q}_+ + \dot{Q}_- \qquad \text{and} \qquad \mathscr{L} = \mathscr{L}_+ + \mathscr{L}_-, \tag{IX.16}$$

satisfy

$$\tilde{U}(\theta)\, Q_-\, \tilde{U}(\theta)^* = e^{-i\theta} Q_- \qquad \text{and} \qquad \tilde{U}(\theta)\, \mathscr{L}_-\, \tilde{U}(\theta)^* = e^{-i\theta}\mathscr{L}_-. \tag{IX.17}$$

Therefore, we have

$$\mathfrak{Z}_\lambda = \varepsilon^2(\langle \dot{Q}_+, \mathscr{L}_+\rangle_1 + \langle \dot{Q}_+, \mathscr{L}_-\rangle_1 + \langle \dot{Q}_-, \mathscr{L}_+\rangle_1 + \langle \dot{Q}_-, \mathscr{L}_-\rangle_1). \tag{IX.18}$$

We argue that the first and last of these four terms vanish. The middle two terms yield the second equality (IX.4).

For any vertex X_j in a regular set, define a transformed vertex

$$\tilde{X}_j = \tilde{U}(\theta')\, X_j \tilde{U}(\theta')^*. \tag{IX.19}$$

We say that X_j has a definite $\tilde{U}(1)$-charge μ_j if

$$\tilde{X}_j = e^{i\mu_j \theta'} X_j. \tag{IX.20}$$

Thus $\dot{Q}_+$ and $\mathscr{L}_+$ have $\tilde{U}(1)$-charge equal to $+1$, while $\dot{Q}_-$ and $\mathscr{L}_-$ have $\tilde{U}(1)$-charge equal to -1. Since $\tilde{U}(\theta')$ commutes both with $U(\theta)$ and with $H(\varepsilon, \lambda)$, we infer that

$$\begin{aligned}\langle X_0, X_1, ..., X_k \rangle_k &= \langle \tilde{U}(\theta')^* \, \tilde{U}(\theta') \, X_0, X_1, ..., X_k \rangle_k \\ &= \langle \tilde{U}(\theta') \, X_0, X_1, ..., X_k \tilde{U}(\theta')^* \rangle_k \\ &= \langle \tilde{U}(\theta') \, X_0 \tilde{U}(\theta')^*, \tilde{U}(\theta') \, X_1 \tilde{U}(\theta')^*, ..., \tilde{U}(\gamma') \, X_k \tilde{U}(\theta')^* \rangle_k \\ &= \langle \tilde{X}_0, \tilde{X}_1, ..., \tilde{X}_k \rangle_k. \end{aligned} \tag{IX.21}$$

If each vertex X_j has a definite $\tilde{U}(1)$-charge, then (IX.21) gives a fifth basic identity, namely

$$\langle X_0, X_1, ..., X_k \rangle_k = e^{i(\sum_{j=0}^k \mu_j) \, \alpha\theta'} \langle X_0, X_1, ..., X_k \rangle_k. \tag{IX.22}$$

This holds for arbitrary θ', so the expectation (IX.22) vanishes unless the total $\tilde{U}(1)$-charge $\sum_{j=0}^k \mu_j = 0$. The first term in (IX.18) has total $\tilde{U}(1)$-charge 2, while the last term has total charge -2, and the middle two terms each have total charge equal to zero. Thus the first and last terms vanish as claimed.

In order to establish the final identity of (IX.4), we perform a ψ_2^*-holonomy in the first term, and a ψ_2-holonomy in the second. The ψ_2-holonomy produces a holonomy factor $c_j(\theta)$, as well as a new vertex

$$[\psi_2^{(j)}, H(\varepsilon, \lambda)] = \lambda \psi_1(\partial\partial_j V).$$

The relation $\{\psi_1, \psi_2\} = \{\psi_1, \psi_2^*\} = 0$ generates the relative minus signs in the two terms produced in this way. The ψ_2^*-holonomy is treated similarly, introducing a vertex $[\psi_2^{(j)*}, H(\varepsilon, \lambda)] = -\lambda \psi_1^* \overline{(\partial\partial_j V)}$, and a complex conjugate holonomy factor. Thus we obtain (IX.4).

Next we derive the relation for $\mathfrak{Z}_\varepsilon$ in (IX.6). Again we use the existence and the representation for the ε-derivative established in Subsection VIII.2. We obtain from (VIII.27) and the above identities,

$$\mathfrak{Z}_\varepsilon = -2\varepsilon \langle I, |z|^2 \rangle_1 = -2\varepsilon \langle |z|^2, I \rangle_1 = -2\varepsilon \sum_{j=1}^n \langle z_j \bar{z}_j, I \rangle_1. \tag{IX.23}$$

Perform a z_j-holonomy. This produces the new vertex

$$-[z_j, H(\varepsilon, \lambda)] = [z_j, \partial\bar{\partial}] = -\bar{z}_j, \tag{IX.24}$$

as well as the holonomy factor $e^{i\omega_j\theta}(1-e^{i\omega_j\theta})^{-1}=d_j(\theta)$. Thus summing over j, we obtain

$$\mathfrak{Z}_\varepsilon = 2\varepsilon \sum_{j=1}^{n} d_j(\theta)(\langle \bar z_j, \bar\partial_j, I\rangle_2 + \langle \bar z_j, I, \bar\partial_j\rangle_2) = 2\varepsilon \sum_{j=1}^{n} d_j(\theta)\langle \bar z_j, \bar\partial_j\rangle_1. \qquad \text{(IX.25)}$$

Here we used (IX.13); we end with the first equality of (IX.6). We now proceed to perform a $\bar z_j$ holonomy on the first vertex. This produces another holonomy factor $e^{-i\omega_j\theta}(1-e^{-i\omega_j\theta})^{-1}=\overline{d_j(\theta)}$, and a new ∂_j-vertex. In addition there is a term with a derived vertex arising from the commutator of ∂_j with z_j. Adding the terms together, we obtain

$$\mathfrak{Z}_\varepsilon = 2\varepsilon \sum_{j=1}^{n} |d_j(\theta)|^2 \, (\langle \partial_j, \bar\partial_j, I\rangle_2 + \langle \bar\partial_j, \partial_j, I\rangle_2 + \langle I, I\rangle_1). \qquad \text{(IX.26)}$$

We simplify (IX.26) using the identity (IX.13). The terms with ∂_j and $\bar\partial_j$ vertices combine to give the term with two vertices in (IX.6). We also simplify the term $\langle I, I\rangle_1$ using the identity $\langle I, I\rangle_1 = \langle I\rangle_0$. Thus we have completed the proof of the proposition.

IX.2. *Regular Sets of Polynomial Vertices*

In this section we define one class of vertices that arise during the holonomy expansion, namely the *polynomial vertices*. We also define a *polynomial vertex regularization*, a regularization procedure that is especially useful when working with these vertices.

The allowed vertices X_j have the form

$$X_j = X_j^b \otimes X_j^f, \qquad \text{(IX.27)}$$

with X_j^f any linear operator acting on the finite dimensional space $\mathscr{H}^f$. We assume that the norms

$$\|X_j^f\|_{\mathscr{H}^f} \leqslant m_j^f \qquad \text{(IX.28)}$$

are bounded by constants $m_j^f < \infty$. Also X_j^b has the form of one of

$$\partial_l, \bar\partial_l, \qquad \text{or} \quad P_j(z, \bar z). \qquad \text{(IX.29)}$$

The case $P_j(z,\bar z)$ is any polynomial in z and $\bar z$ that satisfies the bound

$$|P_j(z,\bar z)| \leqslant m_j^b(|\partial V|^{\delta_j}+1)n \qquad \text{(IX.30)}$$

with constants $m_j^b < \infty$, and with $\delta_j = \alpha_j + \beta_j$, for some $0 \leqslant \alpha_j, \beta_j < 2$. In the case of the vertices ∂_l or $\bar\partial_l$, define $m_j^b = 1$, and take $0 \leqslant \alpha_j, \beta_j$, such that $\alpha_j + \beta_j = 1$. We call such vertices polynomial vertices.

Let $X = X_0, ..., X_k$ denote a set of vertices of the form (IX.27)–(IX.30). In analogy with our previous definition, let $\eta_j = 1 - \frac{1}{2}(\alpha_j + \beta_{j+1})$, where $\beta_{k+1} = \beta_0$. If $0 < \eta_j$ for $0 \leqslant j \leqslant k$, then we say that X is a *regular set of polynomial vertices.*

We now define a regularization which we find especially useful for polynomial vertices. If $X_j = P_j(z, \bar{z}) \otimes X_j^f$, then

$$X_j^{\text{reg}} = P_j(z, \bar{z})\, e^{-\mu |z|^2} \otimes X_j^f. \tag{IX.31}$$

Otherwise, $X_j^{\text{reg}} = X_j$.

Proposition IX.2. *Let V satisfy both Assumption* E *of Subsection* II.4 *and Assumption* Q *of Subsection* II.5. *Let $0 \leqslant \varepsilon, \lambda \leqslant 1$, and let $0 < \lambda$. Let X be a set of polynomial vertices, regular with respect to $H(\varepsilon, \lambda)$. Let $\eta_{\min} > 0$ denote the minimum value of η_j. Let X^{reg} denote the regularization above with $0 \leqslant \mu \leqslant 1$.*

(a) *X^{reg} is a regular set of vertices with respect to $H(\varepsilon, \lambda)$. There is a constant $m < \infty$, independent of μ, such that*

$$\| X_j^{\text{reg}} \|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)} \leqslant (1 + \mu^{1/2})\, \| X_j \|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)}. \tag{IX.32}$$

(b) *For $\bar{\eta}$ in the interval $0 < \bar{\eta} < \eta_{\min}$, the set X is also regular if we replace each α_j by $\alpha_j + 2\bar{\eta}$. (For convenience we also require $\bar{\eta} \leqslant 1/2$.) Then*

$$\| X_j - X_j^{\text{reg}} \|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j + 2\bar{\eta})} \leqslant m \mu^{\bar{\eta}} (\varepsilon + \lambda)^{-2\bar{\eta}}\, \| X_j \|_{\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)}. \tag{IX.33}$$

(c) *The original expectation may be recovered as the limit*

$$\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta \rangle_k = \lim_{\mu \to 0+} \langle X_0^{\text{reg}}, X_1^{\text{reg}}, ..., X_k^{\text{reg}}; \varepsilon, \lambda; \theta \rangle_k. \tag{IX.34}$$

Remark. A major advantage of regularizing a vertex of the form $P_j(z, \bar{z})$ is that we can work with each monomial in P_j as a separate vertex in its own right. Thus we are restricted to arbitrary, polynomial vertices of low degree, not only vertices that arise as derivatives of V. For example, in the regularized expression $P_j(z, \bar{z})\, e^{-\mu |z|^2}$, each vertex of the form $v_J z^J \bar{z}^{\bar{J}} e^{-\mu |z|^2}$ coming from a monomial in P_j is bounded (in the $\mathscr{H}^b$-operator norm). Thus we can manipulate these vertices individually, using holonomies, etc., at the end arriving at a new configuration of vertices of the form (IX.27)–(IX.30). In the expansion that we use in the following subsection, we end up with vertices, each of which are bounded in the appropriate space $\mathscr{T}_{\varepsilon, \lambda}(-\beta_j, \alpha_j)$, uniformly for small (ε, λ). At that point we remove the regularization, before dealing with the limit of small (ε, λ). With this method of regularization, we obtain identities for expectations of regular sets of polynomial vertices, such that the intermediate steps in deriving these identities could otherwise not be carried out (unless we are willing to work with strengthened hypotheses on V).

Proof. We establish (IX.32), (IX.33). The convergence of part (c) then follows from this bound, Theorem I.5(d), and the multi-linearity of expectations in each of its vertices. We begin from the identity

$$[H(\varepsilon,\lambda), e^{-\mu |z|^2}] = [-\partial\bar{\partial}, e^{-\mu |z|^2}] = e^{-\mu |z|^2}(z\partial + \bar{z}\bar{\partial} + I - \mu |z|^2),$$

from which we infer

$$e^{-\mu |z|^2} R^{\alpha} = R^{\alpha} e^{-\mu |z|^2} + \mu \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty t^{-\alpha/2} R(t)^2\, e^{-\mu |z|^2} \times (z\partial + \bar{z}\bar{\partial} + I - \mu |z|^2)\, R(t)^2\, dt. \tag{IX.35}$$

Thus we have the representation

$$R^{\beta_j} X_j^{\mathrm{reg}} R^{\alpha_j} = R^{\beta_j} X_j R^{\alpha_j} e^{-\mu |z|^2} + \mu \frac{\sin(\pi\alpha_j/2)}{\pi} \int_0^\infty t^{-\alpha_j/2} R^{\beta_j} X_j R(t)^2\, e^{-\mu |z|^2} \times (z\partial + \bar{z}\bar{\partial} + I - \mu |z|^2)\, R(t)^2\, dt. \tag{IX.36}$$

The operator norm of the first term on the right of (IX.36) is

$$\|R^{\beta_j} X_j R^{\alpha_j} e^{-\mu |z|^2}\| \leqslant \|R^{\beta_j} X_j R^{\alpha_j}\| = \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}. \tag{IX.37}$$

Furthermore the operator norm of the second term on the right of (IX.36) is bounded by

$$\begin{aligned}
&\mu\alpha_j/2 \int_0^\infty t^{-\alpha_j/2}\, \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}\, \|R(t)\|^{2-\alpha_j} \\
&\qquad \times (\|z e^{-\mu |z|^2}\|\, (\|\partial R(t)\| + \|\bar{\partial} R(t)\|) + \|R(t)\| + \mu\, \|z^2 e^{-\mu |z|^2}\|)\, \|R(t)\|\, dt \\
&\leqslant \mu\alpha_j/2 \int_0^\infty t^{-\alpha_j/2}\, \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}\, \|R(t)\|^{2-\alpha_j} \\
&\qquad \times (4\bar{M}\mu^{-1/2} + \|R(t)\| + 1)\, \|R(t)\|\, dt \\
&\leqslant O(\mu^{1/2}) \left(\int_0^\infty t^{-\alpha_j/2} (1+t)^{-(3-\alpha_j)/2}\, dt \right) \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)}.
\end{aligned} \tag{IX.38}$$

Here we use $\|z e^{-\mu |z|^2}\| = \sup_z |z e^{-\mu |z|^2}| \leqslant 2\,\mu^{-1/2}$. Also $\bar{M}$ is the constant in (VII.3), and $\|z^2 e^{-\mu |z|^2}\| \leqslant \mu^{-1}$. As $0 \leqslant \alpha_j < 2$, the resulting integral in (IX.38) converges, and (IX.32) holds.

To establish (IX.33), we use

$$\begin{aligned}
&R^{\beta_j}(X_j^{\mathrm{reg}} - X_j)\, R^{\alpha_j + 2\bar{\eta}} \\
&\quad = R^{\beta_j} X_j R^{\alpha_j} (e^{-\mu |z|^2} - I)\, R^{2\bar{\eta}} + \mu \frac{\sin(\pi\alpha_j/2)}{\pi} \int_0^\infty t^{-\alpha_j/2} R^{\beta_j} X_j R(t)^2 \\
&\qquad \times e^{-\mu |z|^2} (z\partial + \bar{z}\bar{\partial} + I - \mu |z|^2)\, R(t)^2\, R^{2\bar{\eta}}\, dt.
\end{aligned} \tag{IX.39}$$

Hence

$$\begin{aligned}
&\|X_j^{\text{reg}} - X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\,\alpha_j+2\bar{\eta})}\\
&\quad = \|R^{\beta_j}(X_j^{\text{reg}} - X_j)\, R^{\alpha_j+2\bar{\eta}}\|\\
&\quad \leqslant \|R^{\beta_j} X_j R^{\alpha_j}\|\ \|(e^{-\mu|z|^2} - I)\, R^{2\bar{\eta}}\| + \mu\,\frac{\sin(\pi\alpha_j/2)}{\pi}\int_0^\infty t^{-\alpha_j/2}\\
&\qquad \times \|R^{\beta_j} X_j R(t)^2\, e^{-\mu|z|^2}(z\partial + \bar{z}\bar{\partial} + I - \mu\,|z|^2)\, R(t)^2\,\|\, dt\, \|R^{2\bar{\eta}}\|.
\end{aligned} \tag{IX.40}$$

We begin with a bound on the first term on the right of (IX.40). We show that

$$\|(e^{-\mu|z|^2} - I)\, R^{2\bar{\eta}}\| \leqslant 2\bar{M}_2 \mu^{\bar{\eta}}(\varepsilon+\lambda)^{-2\bar{\eta}}, \tag{IX.41}$$

where $\bar{M}_2$ is the constant in (VII.14). Note that $|e^{-\mu|z|^2} - 1| \leqslant 2(\mu\,|z|^2)^{\bar{\eta}} = 2\mu^{\bar{\eta}}\,|z|^{2\bar{\eta}}$. Since $e^{-\mu|z|^2} - 1$ is a multiplication operator on $\mathscr{H}^b$, for any $f \in \mathscr{H}$ we have $\|(e^{-\mu|z|^2} - 1)\, f\| \leqslant 2\mu^{\bar{\eta}}\,\|\,|z|^{2\bar{\eta}} f\|$. Thus

$$\|(e^{-\mu|z|^2} - I)\, R^{2\bar{\eta}}\| \leqslant 2\mu^{\bar{\eta}}\,\|\,|z|^{2\bar{\eta}}\, R^{2\bar{\eta}}\| \leqslant 2\bar{M}_2^{2\bar{\eta}} \mu^{\bar{\eta}}(\varepsilon+\lambda)^{-2\bar{\eta}}, \tag{IX.42}$$

which is (IX.41). Then using the bound on the first term on the right side of (IX.40), we bound the first term on the right side of (IX.39) by

$$\begin{aligned}
\|R^{\beta_j} X_j R^{\alpha_j}(e^{-\mu|z|^2} - I)\, R^{2\bar{\eta}}\| &\leqslant O(\mu^{\bar{\eta}})(\varepsilon+\lambda)^{-2\bar{\eta}}\ \|R^{\beta_j} X_j R^{\alpha_j}\|\\
&= O(\mu^{\bar{\eta}})(\varepsilon+\lambda)^{-2\bar{\eta}}\ \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\,\alpha_j)}.
\end{aligned} \tag{IX.43}$$

This is a bound of the desired form.

We bound the second term on the right side of (IX.40) by the bound of the second term on the right side of (IX.36), multiplied by $\|R^{2\bar{\eta}}\| \leqslant 1$. Using (IX.38), this term is bounded by

$$O(\mu^{1/2})\ \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\,\alpha_j)}. \tag{IX.44}$$

Since $\bar{\eta} \leqslant 1/2$, also $\mu^{1/2} \leqslant \mu^{\bar{\eta}}$, so combining (IX.43) with (IX.44) completes the proof of (IX.33).

The final statement of the proposition, namely the convergence (IX.34), reduces to the bound (IX.33). In fact

$$\begin{aligned}
&\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta\rangle_k - \langle X_0^{\text{reg}}, X_1^{\text{reg}}, ..., X_k^{\text{reg}}; \varepsilon, \lambda; \theta\rangle_k\\
&\qquad = \sum_{j=0}^{k} \langle X_0^{\text{reg}}, X_1^{\text{reg}}, ..., X_j - X_j^{\text{reg}}, ..., X_k; \varepsilon, \lambda; \theta\rangle_k.
\end{aligned}$$

We can bound each term in this sum using Theorem 1.5(d) along with the estimates (IX.32) and (IX.33) on the vertices involved. Using (IX.33) on the one difference vertex $X_j - X_j^{\mathrm{reg}}$, we obtain convergence of order $\mu^{\bar{\eta}}$ as $\mu \to 0$. Thus the proof of the proposition is complete.

IX.3. *Holonomy Bounds on Polynomial Vertices*

In this section we establish the following bound. This bound, combined with the discussion in Subsection I.2 gives a proof to Theorem I.1, parts (a) and (b). Part (c) of Theorem I.1 is a consequence of the general perturbation theory established in Theorem of [QHA], applied to the case of the identity group (i.e., the case $\theta = 0$).

THEOREM IX.3. *Let V satisfy both Assumption* E *of Subsection* II.4 *and Assumption* Q *of Subsection* II.5. *Let $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, let $0 < \lambda$, and let $\theta \notin \Upsilon_{\mathrm{sing}}$. Then there is a constant $m = m(V, \theta)$ such that*

$$|\mathfrak{Z}_\lambda| \leqslant \lambda \varepsilon^2 m \qquad \text{and} \qquad |\mathfrak{Z}_\varepsilon| \leqslant \varepsilon m. \tag{IX.45}$$

We now prove a more general result than Theorem IX.3, from which (IX.45) follows. With more work, these estimates also let one verify the actual asymptotics of the gradient of $\mathfrak{Z}$, or the asymptotics of other quantities.

THEOREM IX.4. *Let V satisfy both Assumption* E *of Subsection* II.4 *and Assumption* Q *of Subsection* II.5. *Let $0 \leqslant \varepsilon$, $\lambda \leqslant 1$, let $0 < \lambda$, and let $\theta \notin \Upsilon_{\mathrm{sing}}$. Then there exists a constant $\bar{M} = \bar{M}(V, \theta, k)$ such that for all sets of vertices $X = \{X_0, X_1, ..., X_k\}$ with each X_j of the form* (*IX*.27)–(*IX*.30) *above,*

$$|\langle X_0, X_1, ..., X_k; \varepsilon, \lambda; \theta \rangle| \leqslant \bar{M} \left(\prod_{j=0}^{k} m_j^b m_j^f \right). \tag{IX.46}$$

Remark 1. Consider the expectations that occur on the right of (IX.4) and (IX.6), namely $\langle (\partial_j V), \psi_1^* \overline{(\partial \partial_j V)}, \psi_1 \bar{z} \rangle_2$, $\langle (\partial_j V), \psi_1 \bar{z}, \psi_1^* \overline{(\partial \partial_j V)} \rangle_2, ..., \langle \partial_j, \bar{\partial}_j \rangle_1$, and $\langle I \rangle_0$. Each is a sum of expectations of the form covered by the theorem. Vertices of the form P_j occur in the case of (IX.4), and the constants are $m_j^f = 1$ and $m_j^b = M$ of (II.37). Hence we infer from (IX.4) that $|\mathfrak{Z}_\lambda| \leqslant \lambda \varepsilon^2 (4n^4 \bar{M} M^3 \sup_j |c_j(\theta)|)$. Similarly, we infer from (IX.6) that $|\mathfrak{Z}_\varepsilon| \leqslant \varepsilon (6n\bar{M} \sup_j |d_j(\theta)|^2)$. Taking $m = \bar{M} \sup_{1 \leqslant j, j' \leqslant n} \{4n^4 M^3 \, |c_j(\theta)|, 6n \, |d_{j'}(\theta)|^2\}$, we see that Theorem IX.3 is a corollary of Theorem IX.4.

Remark 2. In this section we are not concerned with the θ and k-dependence, of the constant $\bar{M}$ in (IX.46). However, the k-dependence is crucial for the related bounds that we study in Subsection XI for the general expansion for the gradient of $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$.

Remark 3. We can determine the leading small (ε, λ) asymptotics of $\mathfrak{Z}_\varepsilon$ for fixed $\theta \notin \Upsilon_{\text{sing}}$. We pick out coefficients that are expectations of the form

$$\langle I, I, ..., I \rangle_{k+1} = \frac{1}{k!} \langle I \rangle_0, \tag{IX.47}$$

proportional to $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$, which is constant on the λ-axis, and as a consequence of our expansion, is continuous at $(\varepsilon, \lambda) = (0, 0)$. Thus evaluating the expectations (IX.47) on the λ-axis yields the appropriate asymptotics. In this manner, further holonomy expansion of $\mathfrak{Z}_\varepsilon$ shows

$$\mathfrak{Z}_\varepsilon = \frac{1}{2} \varepsilon \left(\sum_{j'=1}^{n} \frac{1}{\sin^2(\omega_{j'}\theta/2)} \left(\prod_{j=1}^{n} \frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j\theta/2)} \right) + O(\varepsilon^2) + O(\lambda^2) \right). \tag{IX.48}$$

Similarly, we can use further expansion of $\mathfrak{Z}_\lambda$ to determine the leading asymptotics of $\mathfrak{Z}_\lambda$ for fixed $\theta \notin \Upsilon_{\text{sing}}$. Denoting $J = \{ j_1, ..., j_n \}$, write the polynomial V as

$$V(z) = \sum_J \frac{1}{J!} v_J z^J, \tag{IX.49}$$

and let $d_J(\theta) = \prod_{l=1}^{n} d_l(\theta)^{j_l}$. Then

$$\begin{aligned} \mathfrak{Z}_\lambda = \lambda\varepsilon^2 &\left(\sum_J \binom{|J|}{2} \frac{1}{J!} \, |d_J(\theta)|^2 \, |v_J|^2 \right. \\ &\times \left. \left(\prod_{j=1}^{n} \frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j\theta/2)} \right) + O(\varepsilon) + O(\lambda) \right). \end{aligned} \tag{IX.50}$$

We leave the derivation of these two identities to the interested reader. They then yield up to fourth order,

$$\begin{aligned} \mathfrak{Z}(\varepsilon, \lambda; \theta) = &\left(\prod_{j=1}^{n} \frac{\sin((1-\omega_j)\,\theta/2)}{\sin(\omega_j\theta/2)} \right) \\ &\times \left(1 + \frac{1}{4} \varepsilon^2 \sum_{j=1}^{n} \frac{1}{\sin^2(\omega_j\theta/2)} + \frac{1}{4} \varepsilon^2\lambda^2 \sum_J \binom{|J|}{2} \frac{1}{J!} \, |d_J(\theta)|^2 \, |v_J|^2 \right. \\ &+ \left. \frac{1}{4!} \varepsilon^4 \left(\left(\sum_{j=1}^{n} \frac{1}{2\sin^2(\omega_j\theta/2)} \right)^2 + \sum_{j=1}^{n} \frac{1}{\sin^4(\omega_j\theta/2)} \right) + O(\varepsilon^2\lambda^3) \right). \end{aligned} \tag{IX.51}$$

Remark 4. Using these methods, we can show that $\mathfrak{Z}(\varepsilon, \lambda; \theta)$ is actually C^∞ in the variables (ε, λ) for $\theta \notin \Upsilon_{\text{sing}}$ fixed.

Proof of Theorem IX.4. We specify an expansion for the expectation (IX.46). The idea of this expansion is to replace any factor of z_l in a polynomial X_j^b-vertex by a new $\bar{\partial}_l$-vertex. Likewise we replace any factor of $\bar{z}_l$ in a polynomial X_j^b-vertex by a new ∂_l-vertex. The reason for doing this is that any vertex which is an allowed polynomial in z_l and $\bar{z}_{l'}$ is an element of $\mathscr{T}_{\varepsilon,\lambda}(-\beta,\alpha)$ with $\alpha+\beta\leqslant 2$, which by itself is all right. But the estimate on the $\mathscr{T}_{\varepsilon,\lambda}(-\beta,\alpha)$-norm for such a vertex diverges for small λ as $O(\lambda^{-(\alpha+\beta)/2})$. On the other hand, a ∂_l-vertex or a $\bar{\partial}_l$-vertex belongs to $\mathscr{T}_{\varepsilon,\lambda}(0,1)$, which is also all right, but it has the advantage that its $\mathscr{T}_{\varepsilon,\lambda}(0,1)$-norm is $O(1)$, uniformly for (ε,λ) in the unit square. Thus by eliminating all polynomial factors in z's or $\bar{z}$'s, we produce bosonic vertices that are bounded in the appropriate Sobolev norm $\mathscr{T}_{\varepsilon,\lambda}(0,1)$. Furthermore, the fermionic vertices X_j^f are assumed bounded in the Hilbert space operator norm. Thus all vertices produced will be bounded in the appropriate sense that there are constants m_j^b, m_j^f, such that after such an expansion

$$\prod_{j=0}^{k'} \|X_j\|_{\mathscr{T}_{\varepsilon,\lambda}(-\beta_j,\alpha_j)} \leqslant \prod_{j=0}^{k'} m_j^b m_j^f. \tag{IX.52}$$

We carry out this replacement by performing a z_l-holonomy or a $\bar{z}_l$-holonomy on each coordinate in each vertex. But before specifying the expansion, we need to make a technical detour. Until the current section of this paper, we have only required estimates on vertices with bosonic factors of the form z_j, $|z_j|^2$, $\partial_j V(z)$, or $\partial_j\partial_k V(z)$, each of which was estimated in Section VII as a consequence of Assumption E, namely with the assumed bound (II.37). In this section, however, by performing the z-holonomies or $\bar{z}$-holonomies, we obtain new sorts of vertices that are not necessarily expressed as derivatives of V, and therefore they are vertices that we cannot estimate directly. However, at the end of our expansion our expectations no longer contain these lower degree monomials. Each z_j will be replaced by a $\bar{\partial}_j$-vertex, and each $\bar{z}_j$ will be replaced by ∂_j. These new vertices *can* be estimated using Assumption E. We establish this by introducing polynomial vertex regularization in each vertex whose bosonic part has the form P_j. Thus we obtain for such vertices by a family of approximating, bounded-operator vertices that commute with each z_j and with each $\bar{z}_j$.

At the start of the expansion, every vertex $X_{j'}$ commutes with z_l and with $\bar{z}_l$. Therefore, we proceed as follows: first holonomy *all* the z-factors. The only new vertices produced in this way are $\bar{\partial}$-vertices, and these new vertices also commute with any remaining z-factors in existing vertices. Each z_l-holonomy produces a factor $d_l(\theta)$, which is singular only at points in the excluded set, $\theta\in\Upsilon_{\text{sing}}$. Furthermore, we remark that all intermediate sets of vertices produced are regular sets of vertices, as the initial vertices satisfy the bound (IX.31). As such, removing a factor z_l or $\bar{z}_l$ from a vertex leaves a monomial that still satisfies the same bound.

After completing all the z-holonomies, holonomy all the $\bar{z}$-factors. Each $\bar{z}_l$-holonomy produces a factor $\overline{d_l(\theta)}$. In addition, this produces new ∂-vertices, as well as some identity vertices. These latter occur from commuting a $\bar{z}_l$-vertex with a $\bar{\partial}$-vertex, namely a vertex of the form $[\bar{\partial}_l, \bar{z}_l] = I$. Since we are not counting the number of terms produced in the expansion, nor the k-dependence of the estimates, and the expansion involves only a finite number of holonomies, this causes no difficulty.

However, at the end of this procedure, we still do not have a uniform bound on the resulting sum of expectations. Our goal is to use bound (I.37) of Theorem I.5(d). However, even though we now have good estimates on the norm of each resulting vertex, the estimate of the expectation is still divergent; it is proportional to the trace of the heat kernel, and by Theorem VI.1(c), we have $\mathrm{Tr}(e^{-H(\varepsilon, \lambda)/2}) \leqslant O((\varepsilon+\lambda)^{-2n})$. At this point, rather than performing an estimate, we continue the expansion for exactly $n+1$ cycles of the three moves per cycle. Each cycle consists of an ε-perturbation move, a z_l-holonomy, and a $\bar{z}_l$-holonomy.

The ε-perturbation move has the form

$$\begin{aligned}&\langle X_0, ..., X_{k'}; \varepsilon, \lambda; \theta\rangle_{k'}\\ &\quad = \langle X_0, ..., X_{k'}; 1, \lambda; \theta\rangle_{k'} - \int_\varepsilon^1 \frac{d}{d\varepsilon'} \langle X_0, ..., X_{k'}; \varepsilon', \lambda; \theta\rangle_{k'}\, d\varepsilon'. \qquad \text{(IX.53)}\end{aligned}$$

The differentiation with respect to ε' produces a sum of $k'+1$ terms, each with a new $2\varepsilon'\ |z|^2$-vertex. We remove the $2\varepsilon'$-coefficient and treat it separately. The first term in (IX.53) is not expanded further. It involves the trace for $\varepsilon = 1$, and hence this term is uniformly bounded as required; by Theorem I.5(d),

$$|\langle X_0, ..., X_{k'}; 1, \lambda; \theta\rangle_{k'}| \leqslant \frac{M_2 4^{k'+n+1}}{\Gamma((k'+1)/2)(\varepsilon+\lambda)^{2n}} \prod_{j=0}^{k'} m_j^b. \qquad \text{(IX.54)}$$

We do not discuss this term further, or the analogous terms arising from another evaluation at $\varepsilon = 1$. Nor do we estimate the number of such terms (as a function of k). We now show that the remaining terms are also uniformly bounded.

The second term on the right of (IX.53) introduces a new $2\varepsilon\ |z_l|^2$-vertex, for $1 \leqslant l \leqslant n$. The two holonomies which follow have the goal of replacing z_l and $\bar{z}_l$ by a new $\bar{\partial}_l$-vertex and a new ∂_l-vertex. As above, we also obtain a finite number of additional terms from the holonomies, when z_l or $\bar{z}_l$ produce an identity vertex $[\partial_l, z_l] = I$ or $[\bar{\partial}, \bar{z}_l] = I$ arising from the holonomy of z_l or $\bar{z}_l$ past some existing ∂_l-vertex or some existing $\bar{\partial}_l$-vertex.

In addition, each ε-perturbation move produces an explicit ε'-factor, as well as an integral over the interval $\varepsilon' \in [\varepsilon, 1]$. Thus after the $3(n+1)$ additional moves, each expectation has an additional convergence coefficient

$$\begin{aligned}C_n(\varepsilon_{n+1}, \varepsilon) &= 2^n \int_{\varepsilon \leqslant \varepsilon_1 \leqslant \varepsilon_2 \leqslant \cdots \leqslant \varepsilon_{n+1}} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n\, d\varepsilon_1\, d\varepsilon_2 \cdots d\varepsilon_n\\ &= \frac{1}{n!}(\varepsilon_{n+1}^2 - \varepsilon^2)^n. \qquad \text{(IX.55)}\end{aligned}$$

Furthermore, the resulting expectation has the form

$$\int_{\varepsilon}^{1} C_n(\varepsilon_{n+1}, \varepsilon)\, 2\varepsilon_{n+1} \langle X_0, ..., X_{k'+k''}; \varepsilon_{n+1}, \lambda; \theta \rangle \, d\varepsilon_{n+1}. \tag{IX.56}$$

The vertices here are just those obtained from the ε-perturbation moves (n in number) and the additional holonomy moves ($2n$ in number). Here k'' lies in the interval $k' + 2n \leqslant k'' \leqslant k' + 3(n+1)$. The extra coefficient C_n along with the ε_{n+1}-integral produces a small factor so that

$$\int_{\varepsilon}^{1} C_n(\varepsilon_{n+1}, \varepsilon)\, 2\varepsilon_{n+1} \operatorname{Tr}(e^{-H(\varepsilon_{n+1}, \lambda)/2})\, d\varepsilon_{n+1}$$

$$\leqslant 2^{2n} M_1 \frac{1}{n!} \int_{\varepsilon}^{1} \varepsilon_{n+1}^{2n} (\varepsilon_{n+1} + \lambda)^{-2n}\, d\varepsilon_{n+1} \leqslant 2^{2n} M_1 \frac{1}{n!}. \tag{IX.57}$$

The factor $2^{2n} M_1$ arises from using the bound (VI.3) on the trace. Thus we obtain the bound (IX.46), and the proof of the theorem is complete.

X. ALGEBRAS OF OBSERVABLE $\mathfrak{A}$: TWO EXAMPLES

In this section we begin the study of the invariants $\mathfrak{Z}(\lambda; a; \theta)$ discussed in Section I. We propose some algebras $\mathfrak{A}$ of bounded operators for which $\mathfrak{Z}(\lambda; a; \theta)$ is defined for $\lambda > 0$, and for which its value can be recovered in a fashion similar to the one in Section IX for the case $a = I$. Thus we introduce the approximating functional $\mathfrak{Z}(\varepsilon; \lambda; a; \theta)$ given by (I.22), and show that for $a \in \mathfrak{A}$,

$$\mathfrak{Z}(\lambda; a; \theta) = \lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon; 0; a; \theta). \tag{X.1}$$

We give two examples of algebras $\mathfrak{A}$ and norms $|||\cdot|||$ on $\mathfrak{A}$, calling them $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and the corresponding norms $|||\cdot|||_1$ and $|||\cdot|||_2$. The algebra $\mathfrak{A}_1$ is abelian. The algebra $\mathfrak{A}_2$ is non-abelian. Both of these algebras are subsets of $\mathscr{B}(\mathscr{H})$ with the special property that they act on $\mathscr{H} = \mathscr{H}^b \otimes \mathscr{H}^f$ as subsets of $\mathscr{B}(\mathscr{H}^b) \otimes I$. Hence all elements in $\mathfrak{A}_1$ and in $\mathfrak{A}_2$ automatically commute with γ. Also

$$da = d_\lambda a = Q(\lambda)\, a - aQ(\lambda), \quad (da)^\gamma = -da, \quad \text{and} \quad d^2 a = [Q(\lambda)^2, a]. \tag{X.2}$$

We study the functional $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$ as a power series in a. We study the individual expectations,

$$\tau_k^{\mathrm{JLO}}(a_0, a_1, ..., a_k; \varepsilon, \lambda; \theta) = \langle a_0, d_\lambda a_1, ..., d_\lambda a_k; \varepsilon, \lambda; \theta \rangle_k, \tag{X.3}$$

that generalize (I.19) by including the possibility of regularization with ε, lying in the interval $0 \leqslant \varepsilon \leqslant 1$. For $\varepsilon + \lambda > 0$, we can use the methods of [QHA] to analyze (X.3) and to show that $\mathfrak{Z}(\lambda; a; \theta)$ has a convergent expansion

$$\mathfrak{Z}(\lambda; a; \theta) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \tau_{2k}^{\mathrm{JLO}}(a, a, ..., a; \varepsilon, \lambda; \theta). \tag{X.4}$$

However, these bounds are not uniform as $(\varepsilon, \lambda) \to (0, 0)$. In order to establish uniformity, we study $\mathfrak{Z}$ by means of the holonomy expansion. Generalizing the methods of Section IX, we obtain bounds for each term τ_k that are uniform in (ε, λ). But such a bound is insufficient. We also must track the k-dependence of the estimates, in order to establish that the sum over k converges uniformly as $(\varepsilon, \lambda) \to (0, 0)$. In addition, we establish such uniform $(\varepsilon, \lambda, k)$-bounds for the ε-derivative and the λ-derivative of τ_k. Thereby we gain control over the (ε, λ)-asymptotics of $\mathfrak{Z}$ for small (ε, λ). This analysis depends on making an appropriate choice of the algebra $\mathfrak{A}$; a general choice of $\mathfrak{A}$ is not amenable to a convergent holonomy expansion. We do not attempt here to make an optimal choice for $\mathfrak{A}$, but only to give two examples where the $\varepsilon \to 0$-limit and the $\lambda \to 0$-limit of $\mathfrak{Z}$ can be interchanged.

EXAMPLE $\mathfrak{A}_1$. The algebra $\mathfrak{A}_1$ denotes an algebra of bounded functions on $\mathbb{C}^n$. We take functions that also have bounded first derivatives, in the sense that it has bounded norm $|||\cdot|||_1$ defined on $\mathfrak{A}_1$ by

$$|||a|||_1 = \|a\| + \|da\|. \tag{X.5}$$

We take $\mathfrak{A}_1$ to be the algebra of functions with finite norm (X.5). The unit function $a = 1$ is an element of $\mathfrak{A}_1$ with norm 1.

The operator norm $\|a\|$ is equal to $\sup_{z, \bar{z}} |a(z, \bar{z})|$, so a is a bounded function on $\mathbb{C}^n$ if and only if a is a bounded multiplication operator. For $a \in \mathfrak{A}_1$, we also have

$$da = \psi_1(\partial a) + \psi_1^*(\bar{\partial} a), \tag{X.6}$$

so neither da nor $\|da\|$ depends on λ. As a consequence, $|||a|||_1$ is also independent of the choice of V. The requirement that $U(\theta)$ acts on $\mathfrak{A}_1$ means the following. Every $a \in \mathfrak{A}$ is a function $a = a(z_j, \bar{z}_j)$, and so

$$U(\theta)\, aU(\theta)^* = a(e^{i\omega_j\theta} z_j, e^{-i\omega_j\theta} \bar{z}_j). \tag{X.7}$$

Hence we want $|||a(e^{i\omega_j\theta} z_j, e^{-i\omega_j\theta} \bar{z}_j)|||$ to be bounded whenever $|||a(z_j, \bar{z}_j)|||$ is bounded. But this is automatically satisfied, since $U(\theta)$ is unitary and commutes with $Q(\lambda)$. Finally we check that multiplication is continuous in $\mathfrak{A}_1$ in the sense that

$$|||ab|||_1 \leqslant |||a|||_1\, |||b|||_1. \tag{X.8}$$

This Banach algebra property follows immediately from the fact that d acts as a derivation.

EXAMPLE $\mathfrak{A}_2$. In defining this algebra $\mathfrak{A}_2$, we assume that elements $a \in \mathfrak{A}_2$ are bounded. Furthermore we assume that the commutator $d_0 a = [Q(0), a] = [\psi_1 \partial, a] - [\psi_1^* \bar{\partial}, a]$, as well as the commutators $[z_j, a]$ and $[\bar{z}_j, a]$ all have the property that they can be multiplied by certain powers of z and $\bar{z}$, and still remain bounded operators.

The highest possible such power of a coordinate will depend on the degree of the polynomial V. We choose a constant $\tilde{l}$ in Section XI, but here we just suppose that a large, positive integer $\tilde{l}$ is given. Define the norm $\| \cdot \|_{\tilde{l}}$ by

$$\|T\|_{\tilde{l}} = \sup_{|l_1| + |l_2| + |\bar{l}_1| + |\bar{l}_2| \leqslant \tilde{l}} \|z^{l_1} \bar{z}^{\bar{l}_1} T z^{l_2} \bar{z}^{\bar{l}_2}\|. \tag{X.9}$$

In terms of this functional, define the norm $||| \cdot |||_2$ as

$$|||a|||_2 = \|a\| + c_{\tilde{l}} \sup \{ \|d_0 a\|_{\tilde{l}}, \|[z_{j'}, a]\|_{\tilde{l}}, \|[\bar{z}_j, a]\|_{\tilde{l}} \}. \tag{X.10}$$

We claim that for $c_{\tilde{l}} > 2\tilde{l}$, it is true that $||| \cdot |||_2$ is a Banach norm, and

$$|||ab|||_2 \leqslant |||a|||_2 \; |||b|||. \tag{X.11}$$

When we bound $|||ab|||_2$, we need to bound $\|d_0(ab)\|_{\tilde{l}}$, $\|[\bar{z}_j, ab]\|_{\tilde{l}}$, and $\|[z_j, ab]\|_{\tilde{l}}$. Let us estimate the last of these; the estimates on the other two are similar. Since

$$\|[z_j, ab]\|_{\tilde{l}} \leqslant \|[z_j, a]\, b\|_{\tilde{l}} + \|a[z_j, b]\|_{\tilde{l}}, \tag{X.12}$$

we require an estimate on (X.12). We use the identity

$$b T_1 T_2 \cdots T_k = T_1 T_2 \cdots T_k b + \sum_{j=1}^{k} T_1 \cdots T_{j-1} [b, T]\, T_{j+1} \cdots T_k. \tag{X.13}$$

Thus a term such as $\|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, b z^{l_2} \bar{z}^{\bar{l}_2}\|$ which occurs in $\|[z_j, a]\, b\|_{\bar{l}}$ can be estimated by

$$\begin{aligned} \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, b z^{l_2} \bar{z}^{\bar{l}_2}\| \leqslant{} & \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, z^{l_2} \bar{z}^{\bar{l}_2} b\| \\ & + \sum_{j=1}^{k} \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, T_1 \cdots T_{j-1} [b, T_j]\, T_{j+1} \cdots T_k\|, \end{aligned} \tag{X.14}$$

where $k = |l_2| + |\bar{l}_2| \leqslant \tilde{l}$, and each T_j is a coordinate $z_{j'}$ or $\bar{z}_{j'}$. Thus

$$\begin{aligned} \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, b z^{l_2} \bar{z}^{\bar{l}_2}\| \leqslant{} & \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\, z^{l_2} \bar{z}^{\bar{l}_2}\| \; \|b\| \\ & + \sum_{j=1}^{k} \|z^{l_1} \bar{z}^{\bar{l}_1} [z_j, a]\| \; \|T_1 \cdots T_{j-1} [b, T_j]\, T_{j+1} \cdots T_k\| \\ \leqslant{} & \|[z_j, a]\|_{\tilde{l}} \|b\| + \tilde{l}\, \|[z_j, a]\|_{\tilde{l}} \|[z_j, b]\|_{\tilde{l}}. \end{aligned} \tag{X.15}$$

Hence

$$\|[z_j, a]\, b\|_{\tilde{l}} \leqslant \|[z_j, a]\|_{\tilde{l}}\, \|b\| + \tilde{l}\, \|[z_j, a]\|_{\tilde{l}}\, \|[z_j, b]\|_{\tilde{l}}. \tag{X.16}$$

There is a similar bound on a $\|a[z_j, b]\|_{\tilde{l}}$, so for $c_{\tilde{l}} > 2l$,

$$\begin{aligned} \|ab\| + c_{\tilde{l}}\, \|[z_j, ab]\|_{\tilde{l}} \leqslant{}& \|a\|\ \|b\| + c_{\tilde{l}}\, \|[z_j, a]\|_{\tilde{l}}\, \|b\| + \tfrac{1}{2} c_{\tilde{l}}^2\ \|[z_j, a]\|_{\tilde{l}}\, \|[z_j, b]\|_{\tilde{l}} \\ & + c_{\tilde{l}}\, \|[z_j, b]\|_{\tilde{l}}\, \|a\| + \tfrac{1}{2} c_{\tilde{l}}^2\ \|[z_j, a]\|_{\tilde{l}}\, \|[z_j, b]\|_{\tilde{l}} \\ \leqslant{}& (\|a\| + c_{\tilde{l}}\, \|[z_j, a]\|_{\tilde{l}})(\|b\| + c_{\tilde{l}}\, \|[z_j, b]\|_{\tilde{l}}). \\ \leqslant{}& |||a|||_2\ |||b|||_2. \end{aligned} \tag{X.17}$$

We now can derive a bound similar to (X.17) for each of the other two various other possibilities for the supremum in (X.10), namely for $\|d_0(ab)\|_{\tilde{l}}$ or $\|[\bar{z}_j, ab]\|_{\tilde{l}}$ replacing $\|[z_j, ab]\|_{\tilde{l}}$. From these bounds we conclude that

$$\|ab\| + c_{\tilde{l}}\, \|[\bar{z}_j, ab]\|_{\tilde{l}} \leqslant |||a|||_2\ |||b|||_2, \tag{X.18}$$

and

$$\|ab\| + c_{\tilde{l}}\, \|d_0(ab)\|_{\tilde{l}} \leqslant |||a|||_2\ |||b|||. \tag{X.19}$$

Thus (X.8) does hold.

Let $\mathfrak{A}_2$ denote those elements of $\mathscr{B}(\mathscr{H}^b) \otimes I$ for which all the terms in (X.10) are defined and for which $|||a|||_2 < \infty$. The algebra $\mathfrak{A}_2$ also has the desired invariance properties. Clearly it is pointwise invariant under the action of γ. In addition, it is globally invariant under the action of $U(\theta)$, namely for $a \in \mathfrak{A}_2$,

$$U(\theta)\, aU(\theta)^* \in \mathfrak{A}_2. \tag{X.20}$$

This fact is a consequence of the unitary nature of $U(\theta)$, of the fact that $U(\theta)$ and $Q(\lambda)$ commute, and the fact that $U(\theta)$ acts in a one-dimensional fashion on the coordinates z and $\bar{z}$ that occur in the definition of the norm.

We remark that $I \in \mathfrak{A}_2$. For the identity operator, our norms have the property that

$$|||I|||_1 = |||I|||_2 = 1. \tag{X.21}$$

XI. THE HOLONOMY EXPANSION FOR $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$

In this section we study $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$, which we expand into a sum of expectations $\tau_k^{\mathrm{JLO}}(\varepsilon, \lambda, \theta)$ defined in Section I. The methods of [QHA] apply for $\varepsilon + \lambda > 0$, yielding the existence of τ_k^{JLO}, and its differentiability in ε and λ. Furthermore, suppose that a is chosen so that both $a^2 = I$, and for all θ, we have $U(\theta)\, aU(\theta)^* = a$. Then

$\mathfrak{Z}(0, \lambda; a; \theta)$ is an invariant, which means that $\mathfrak{Z}(\lambda; a; \theta)$ is a constant function of the parameter λ. We denote this invariant $\mathfrak{Z}(a; \theta)$. It represents the invariant data obtained from pairing the cochain $\tau^{\mathrm{JLO}}(\lambda; \theta)$ with a, see [QHA, C, JLO].

Here we study the $\varepsilon + \lambda \to 0$ limit, in order to establish a method to calculate $\mathfrak{Z}(0, \lambda; a; \theta)$, generalizing the case $a = I$ studied in Section IX. The bounds of [QHA] would require uniformity in θ, and this is not the case as $\varepsilon + \lambda \to 0$. Hence the methods of [QHA] do not apply. In order to work in the region where $\varepsilon + \lambda \to 0$, we need to assume that the potential $V(z)$ has the $U(1)$-symmetry of a quasi-homogeneous polynomial, in addition to obeying the elliptic bounds of Subsection II.4. For potential functions V that lie in a uniformly bounded set of potentials $\mathfrak{V}_0^\omega \subset \mathfrak{V}^\omega$ defined in Subsection I.3, the constant M in (II.37) can be chosen independent of V. Furthermore for $V \in \mathfrak{V}_0^\omega$, the order of the polynomial $V(z)$, which we denote degree(V), is uniformly bounded, and also each coefficient $|v_J|$ in series expansion (IX.49) for $V(z)$ is uniformly bounded.

We also need to restrict the observables a, and we take the algebra $\mathfrak{A}$ to be one of the two examples in Section X. In the case of the algebra $\mathfrak{A}_2$, we make the choice $\tilde{l} \geqslant 2\,(\text{degree}(V) + n)$, and this constant is uniformly bounded on each uniformly bounded set of potentials $\mathfrak{V}_0^\omega$ introduced in Section I.

Under these hypotheses, we prove that the $\varepsilon + \lambda \to 0$ limit of τ^{JLO} exists for all $\theta \notin \Upsilon_{\mathrm{sing}}$. Since this limit is not uniform in θ, and we treat it in a pointwise fashion, it is necessary to modify the norm $|||\cdot|||$ from the one we used in [QHA]—the norm in that work includes an L^∞-norm on the group. Thus we now define the norm of the functional τ_k^{JLO} to be

$$|||\tau_k^{\mathrm{JLO}}(\varepsilon, \lambda, \theta)||| = \sup_{|||a_j||| = 1} |\tau_k^{\mathrm{JLO}}(a_0, a_1, ..., a_k; \varepsilon, \lambda; \theta)|, \tag{XI.1}$$

where $|||\cdot|||$ denotes one of the norms on $\mathfrak{A}$ introduced in Section X. With these changes, we prove that the functional τ^{JLO} satisfies the "entire-cochain" estimate $|||\tau_k^{\mathrm{JLO}}|||^{1/k} \leqslant o(k^{-1/2})$, with constants that are uniform in (ε, λ), but not in θ, as $\varepsilon + \lambda \to 0$. In fact, in the case of the two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we make the θ-dependence of the bound explicit, and we also have better k-dependence than required. Let us define

$$d(\theta) = d(\mathfrak{V}_0^\omega, \theta) = \sup_{1 \leqslant j \leqslant n} \frac{1}{|\sin(\omega_j \theta/2)|}. \tag{XI.2}$$

For $\theta \notin \Upsilon_{\mathrm{sing}}$, we will establish a uniform bound of the type

$$|||\tau_k^{\mathrm{JLO}}|||^{1/k} \leqslant d(\theta)^{(2n+1)/k}\, k^{-1} \bar{M}, \tag{XI.3}$$

for a uniform constant $\bar{M}$.

Furthermore, we show that $\tau^{\mathrm{JLO}}(\varepsilon, \lambda; \theta)$ is jointly continuous as a function of (ε, λ) for θ fixed, with the same type of uniformity in k, and non-uniformity in θ.

Given these additional bounds, for $\theta \notin \Upsilon_{\mathrm{sing}}$, we conclude that our main result holds, namely

$$\begin{aligned}\mathfrak{Z}(a;\theta) &= \langle \tau^{\mathrm{JLO}}(\lambda;\theta), a\rangle \\ &= \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,\mathrm{Tr}_{\mathscr{H}}\big(\gamma U(\theta)\, a e^{\partial\bar{\partial} + it\psi_1(\partial a) - it\psi_1^*(\bar{\partial} a) - \varepsilon^2|z|^2}\big)\,dt. \qquad \text{(XI.4)}\end{aligned}$$

Here $(\partial a) = [\partial, a]$ and $(\bar{\partial} a) = [\bar{\partial}, a]$. We read off from (XI.4) that the invariant $\mathfrak{Z}(\lambda; a; \theta)$ depends on V only through its quasi-homogeneous class. In fact, the only dependence of the right side of (XI.4) on V occurs in the operator $U(\theta)$. The generator J of $U(\theta)$ introduced in Section III, depends on ω. The operator $U(\theta)$ contains no other information about V, aside from specifying the quasi-homogeneity weights.

THEOREM XI.1. *Assume that* $V \in \mathfrak{V}_0^{\omega}$ *is a holomorphic polynomial in a uniformly bounded set of potentials that satisfy both Assumption* E *and Assumption* Q *of Subsections* II.4 *and* II.5. *Let* $0 \leqslant \varepsilon, \lambda \leqslant 1$ *with* $0 < \varepsilon + \lambda$, *and let* $\theta \notin \Upsilon_{\mathrm{sing}}$. *Let* $a_j \in \mathfrak{A}$, *with* $\mathfrak{A} = \mathfrak{A}_1$ *or* $\mathfrak{A} = \mathfrak{A}_2$ *of Section* X. *In the case of* $\mathfrak{A}_2$, *choose the constant* $\tilde{l}$ *in the norm* (X.9), (X.10) *so that*

$$\tilde{l} \geqslant 2(\mathrm{degree}(V) + n). \qquad \text{(XI.5)}$$

Then,

(a) *There exists a constant* $\bar{M} = \bar{M}(\mathfrak{V}_0^{\omega}) < \infty$ *such that with* $d(\theta)$ *above,*

$$|\tau_k^{\mathrm{JLO}}(a_0, ..., a_k; \varepsilon, \lambda; \theta)| \leqslant d(\theta)^{2n+1}\, \bar{M}^{k+1}\, \frac{1}{k!}\left(\prod_{j=0}^{k} |||a_j|||\right). \qquad \text{(XI.6)}$$

(b) *Also with the same form of bound,*

$$\left|\frac{\partial \tau_k^{\mathrm{JLO}}}{\partial \varepsilon}(a_0, ..., a_k; \varepsilon, \lambda; \theta)\right| \leqslant d(\theta)^{2n+2}\, \bar{M}^{k+1}\, \frac{1}{k!}\left(\prod_{j=0}^{k} |||a_j|||\right), \qquad \text{(XI.7)}$$

and

$$\left|\frac{\partial \tau_k^{\mathrm{JLO}}}{\partial \lambda}(a_0, ..., a_k; \varepsilon, \lambda; \theta)\right| \leqslant d(\theta)^{2n+\mathrm{degree}(V)}\, \bar{M}^{k+1}\, \frac{1}{k!}\left(\prod_{j=0}^{k} |||a_j|||\right). \qquad \text{(XI.8)}$$

Remark. The proof of this theorem relies heavily on the ideas and methods of Section IX. Much of the difficulty of has been subsumed into the definitions of the algebras $\mathfrak{A}$. Based on the experience in Section IX, we introduce the algebras in Section X so that we can follow the methods in Section IX as closely as possible. We concentrate here on the new features that arise when we work with $\mathfrak{Z}(\varepsilon, \lambda; a; \theta)$ in place of $\mathfrak{Z}(\varepsilon, \lambda; I; \theta)$ that we studied in Section IX. We carry out the proof in detail for $\mathfrak{A} = \mathfrak{A}_2$. We give only rough estimates, sufficient to establish the theorem;

we do not attempt to isolate leading asymptotics as in Section IX. The results for $\mathfrak{A} = \mathfrak{A}_1$ follow analogously. Since every $a \in \mathfrak{A}_1$ satisfies $[z, a] = 0$, the expansion for $\mathfrak{A}_1$ simplifies considerably, and we omit the details.

XI.1. *Types of Vertices and Expectations*

Let $X = \{X_0, X_1, ..., X_k\}$ denote a regular set of vertices with respect to $H(\varepsilon, \lambda)$. Let $X(s) = X(s; \varepsilon, \lambda)$ denote the heat kernel regularization (VIII.1) of X, and let $\hat{X} = \hat{X}(\varepsilon, \lambda)$ denote the transform (VIII.22) of $X(s; \varepsilon, \lambda)$. Define the expectation

$$\langle X_0, ..., X_k; \varepsilon, \lambda; \theta \rangle_k = \operatorname{Tr}(\gamma U(\theta)\, \hat{X}(\varepsilon, \lambda)). \tag{XI.9}$$

This expression gives a $(k+1)$-linear functional on the vertices.

Our expansion is defined as a set of rules to replace an expectation of the form (XI.4) by a sum and/or integral of expectations of the form

$$c(\varepsilon', \lambda', \theta) \langle X'_0, X'_1, ..., X'_k; \varepsilon', \lambda', \theta \rangle_{k'}. \tag{XI.10}$$

The function $c(\varepsilon', \lambda'; \theta)$ in (XI.10) is called the coefficient of the expectation. By convention, we assign whatever numerical factors we can to the coefficient $c(\varepsilon', \lambda'; \theta)$, including holonomy factors, or powers of the parameters ε' or λ'.

Each rule that we use to define the expansion is the application of an identity between expectations. Each elementary application of a rule is called an expansion *move*. When a small number of expansion moves frequently occur together in a particular order, we combine these moves into an expansion unit that we call an expansion *step*. We define our expansion in terms of these steps, that are composed of elementary moves. The definition of the expansion proceeds by first listing the possible moves and the related steps. Then we give a set of rules telling when to apply each step.

Consider an expansion move leading to a term (XI.10). The moves will be defined so that either $k' = k$ or else $k' = k + 1$. Also $\varepsilon \leqslant \varepsilon' \leqslant 1$, as well as $\lambda \leqslant \lambda' \leqslant 1$. We classify expectations (XI.10) into 9 types, according to three possible cases for each variable $\varepsilon' \in [\varepsilon, 1]$ and for $\lambda' \in [\lambda, 1]$. The three possibilities for ε-types are

Type-$\varepsilon_{\max}$, ε' is evaluated at $\varepsilon' = 1$;

Type-$\varepsilon_{\min}$, ε' is evaluated at $\varepsilon' = \varepsilon$; and

Type-ε', ε' is integrated from ε to 1.

Similarly we have 3 possible λ-types: Type-$\lambda_{\max}$, Type-$\lambda_{\min}$, and Type-λ'.

Definition XI.2. An expectation (XI.10) is a *final term* if it is of Type $\varepsilon_{\max}$, or if it is of Type $\lambda_{\max}$. Otherwise the expectation is called an *expansion* term.

We define the expansion by rules so that the expansion terminates for any final term. But the expansion continues for any expansion term. In general, the set of vertices $X'_{j'}$ that occur in (XI.10) are the same as the vertices X_j that occur in (XI.4), up to possible renumbering. However, some of the vertices that occur in (XI.10) at the end of the expansion move may differ from those at the start of the move. We denote these as either *derived* vertices, or else as *new* vertices. We discuss these separately.

Derived Vertices. We produce a derived vertex X'_j during a holonomy move, by performing one of two possible operations on a vertex X_j present at the start of a move. Let us give two typical examples. A Y-holonomy move begins by identifying an existing vertex X_j as a product $X_j = YZ_j$. We then translate Y cyclically around the expectation, leaving the vertex Z_j in the jth-place (or the $(j+1)$st-place). The Z_j-vertex is a vertex that has been derived from the vertex X_j. Every term produced by the Y-holonomy move will have this sort of derived vertex. As remarked above, we include the phase factor $(1-\sigma)^{-1}$ associated with the holonomy move in the coefficient c.

The other example of a derived vertex produced during a Y-holonomy move comes from the (graded) commutators between Y and an existing vertex $X_{j'}$, namely

$$X'_{j'} = [\,Y, X_{j'}\,]_\gamma = YX_{j'} - X^{\gamma}_{j'}\,Y. \tag{XI.11}$$

Here $X^\gamma = \gamma X \gamma$. In this case, we say that the operator $X'_{j'} = [\,Y, X_{j'}\,]_\gamma$ is a derived $X_{j'}$-vertex. Every term produced by a Y-holonomy move will also have this type of derived vertex, or else it will have a new vertex as described below.

New Vertices. Let us now focus on how a new vertex arises in the expansion. A term (XI.10) produced by an individual move contains at most one new vertex, and in that case the number $k' = k+1$; otherwise, $k' = k$. The new vertex may arise in one of two ways. The first method may arise during a Y-holonomy move. The new vertex comes from commuting Y and a heat kernel, which we do by using the identity

$$Ye^{-sH(\varepsilon,\lambda)} = e^{-sH(\varepsilon,\lambda)}\,Y + \int_0^s e^{-tH(\varepsilon,\lambda)}[\,H(\varepsilon,\lambda),\,Y\,]\,e^{-(s-t)\,H(\varepsilon,\lambda)}\,dt. \tag{XI.12}$$

By definition, $[\,H(\varepsilon,\lambda),\,Y\,]$ is a new vertex. Any explicit factor of ε or λ in the commutator will be assigned to the coefficient $c(\varepsilon', \lambda'; \theta)$. In this case of a new vertex arising from a holonomy, $\varepsilon' = \varepsilon$ and $\lambda' = \lambda$. We call the vertex $[\,Y, H(\varepsilon,\lambda)\,]$ a “new, Y-holonomy vertex.”

The second sort of new vertex arises from a perturbation move. During such a move, we differentiate $e^{-sH(\varepsilon,\lambda)}$ with respect to λ or ε, and we generate the move using the fundamental theorem of calculus. For example, for a λ-derivative, we use the identity

$$\frac{\partial}{\partial\lambda} e^{-sH(\varepsilon,\lambda)} = \int_0^s e^{-tH(\varepsilon,\lambda)} \left(-\frac{\partial}{\partial\lambda} H(\varepsilon,\lambda) \right) e^{-(s-t)\,H(\varepsilon,\lambda)}\,dt. \tag{XI.13}$$

We generate the move with the identity

$$\begin{aligned} e^{-sH(\varepsilon,\lambda)} &= e^{-sH(\varepsilon,1)} - \int_\lambda^1 \frac{\partial}{\partial\lambda'} e^{-sH(\varepsilon,\lambda')}\,d\lambda' \\ &= e^{-sH(\varepsilon,1)} + \int_\lambda^1 d\lambda' \int_0^s dt\, e^{-tH(\varepsilon,\lambda')} \left(\frac{\partial}{\partial\lambda'} H(\varepsilon,\lambda') \right) e^{-(s-t)\,H(\varepsilon,\lambda')}. \end{aligned} \tag{XI.14}$$

Thus the perturbation expansion move generates one term with no new vertex ($k'=k$), along with one term with the new vertex of Type-λ', namely $(\partial/\partial\lambda')\,H(\varepsilon,\lambda')$. While we have shown in Proposition VIII.7 that this vertex equals

$$\frac{\partial}{\partial\lambda'} H(\varepsilon,\lambda') = 2\lambda'\,|\partial V|^2 + W + W^*, \tag{XI.15}$$

we prefer to write the new vertex in the form

$$\frac{\partial}{\partial\lambda'} H(\varepsilon,\lambda') = Q(\lambda')\,\dot{Q} + \dot{Q}Q(\lambda'), \tag{XI.16}$$

where $\dot{Q} = (\partial/\partial\lambda')\,Q(\lambda')$, as explained in Subsection IX.1. We denote such a new vertex (XI.16) as a λ-perturbation-vertex, or λ-vertex for short. We often follow the introduction of a λ-vertex by a Q-holonomy move. This leaves the vertex $\dot{Q}$ as a derived λ-vertex, as well as producing another new or derived vertex.

Likewise, in the case of an ε-perturbation move, we use the identity

$$\begin{aligned} e^{-sH(\varepsilon,\lambda)} &= e^{-sH(1,\lambda)} - \int_\varepsilon^1 \frac{\partial}{\partial\varepsilon'} e^{-sH(\varepsilon',\lambda)}\,d\varepsilon' \\ &= e^{-sH(1,\lambda)} + \int_\varepsilon^1 d\varepsilon' \int_0^s dt\, e^{-tH(\varepsilon',\lambda)} \left(\frac{\partial}{\partial\varepsilon'} H(\varepsilon',\lambda) \right) e^{-(s-t)\,H(\varepsilon',\lambda)}. \end{aligned} \tag{XI.17}$$

We thus have two terms, one with no new vertex, and another with one new vertex of Type-ε',

$$\frac{\partial}{\partial\varepsilon'} H(\varepsilon',\lambda) = 2\varepsilon'\,|z|^2. \tag{XI.18}$$

We denote this new vertex as an ε-perturbation vertex, or ε-vertex. We generally treat the coefficient $2\varepsilon'$ separately, and we say that a new ε-vertex equals $|z|^2$.

Initial Vertices. Certain vertices are present at the beginning of the expansion, before we perform the first expansion move. We name these *initial* vertices. For example, in this section our vertices a_0 and da_j are always initial vertices. For τ_k^{JLO}, these are the only initial vertices. For other expectations we perform a preliminary expansion and include the vertices produced in this way as initial vertices. Other vertices are introduced during the expansion, namely the *new* vertices. A vertex retains its designation in the sense that a vertex such as an ε-perturbation vertex may change through holonomy or through derivation, but it retains its identity as an ε-vertex. For example, if $X_{j'}$ in (XI.15) is a new, ε-perturbation vertex, then $[Y, X_{j'}]$ is a new, derived, ε-perturbation vertex. Note that the identity operator I is a possible vertex. Certain derivations, as $[\partial, z]$, could produce an identity vertex, but such a vertex retains its identity as a derived vertex of its original type, in this case presumably of type ∂.

For a term (XI.9) at some point in the expansion, we let k_i denote the number of initial vertices, and let k_n denote the number of new vertices. In this case, then

$$k_i + k_n = k + 1. \tag{XI.19}$$

Each of the initial vertices may be a derived, initial vertex; likewise, each of the new vertices may be a derived, new vertex. We sometimes wish to count the vertices in a different way. For example, we let k_∂ denote the total number of either ∂-vertices or $\bar{\partial}$-vertices (produced by $\bar{z}$ or z-holonomies).

XI.2. *The Expansion for $\mathfrak{Z}$ and $\mathfrak{Z}_\varepsilon$*

In this subsection we give the holonomy expansion that establishes (XI.6), (XI.7). Consider first the case of $\mathfrak{Z}$. We study the expectation $\langle a_0, da_1, da_2, ..., da_k\rangle$, for $a_j \in \mathfrak{A}$. The first expansion move is an ε-perturbation (XI.17). We do not expand further the final term which is evaluated at $\varepsilon = 1$, or any final term occurring later in the expansion. The ε'-derivative term produces a sum of $(k+1)$-expectations; each of these expectations contains one new vertex $|z|^2$. The number of terms reflects the fact that this vertex occurs in one of $(k+1)$-possible locations. We then follow this ε-perturbation move by a z_j-holonomy, for each $1 \leqslant j \leqslant n$. This has the effect of replacing the vertex $|z|^2$ by a $\bar{z}_j$-vertex. In addition it either introduces a new, $\bar{\partial}_j$-vertex, or else it produces a z_j-derivation of an existing vertex. Furthermore, we assign the overall $-2\varepsilon'$-factor and the integral over $\varepsilon' \in [\varepsilon, 1]$ to the coefficient of the term.

We now continue by performing $2n$ additional cycles of these two moves on each expansion term. Since we only holonomy z_j-coordinates, these will commute with all the $\bar{\partial}_{j'}$ vertices that the expansion produces. Further the holonomies always commute with other vertices that are functions of z, ψ_j, and or their adjoints. Thus the only derived vertices produced by this particular expansion are derived, initial vertices. At the end of the expansion, each expectation is either a final term, or an expansion term. The expansion terms have exactly $(2n+1)$ perturbation vertices equal to z_j. They will also have exactly k_∂-vertices $\bar{\partial}_j$, and $(2n+1-k_\partial)$-derivations of initial vertices. (A particular vertex may be derived more than once.) Each z_j-holonomy produces one multiplicative $d_j(\theta)$-holonomy factor.

In order to estimate the result of the expansion, we require an estimate on each individual expectation that the expansion produces. We perform this estimate by estimating the trace norm of the Radon transform of the heat kernel regularization of the regular set of vertices. Thus we call this an *estimate on expectations*. Complementing this, we require an estimate on the number of terms produced and we call the latter an *entropy estimate*.

XI.2.1. *Entropy Estimates*

According to the discussion in the previous section, the expansion proceeds by a succession of cycles, composed of an ε-perturbation move followed by a z_j-holonomy move of the coordinate produced in each $|z_j|^2$ vertex of ε-type. We start from one $(k+1)$-vertex expectation. After the first introduction of a $|z|^2$-vertex we have $(k+1)$-expectations with $(k+2)$-vertices. After the initial holonomy by each of the n-coordinates, we have $n(k+1)(k+2)$-expectations each of which has $(k+3)$-vertices, plus $n(k+1)^2$ $(k+2)$-expectations with $(k+2)$-vertices and an additional derivation on one initial vertex. Here we give an estimate for this type of counting, so we can use the estimate systematically. We repeat the perturbation/holonomy cycle a total of $2n+1$ times.

A useful technique to simplify the counting is to diagram the expansion, as we do in Fig. 1. Each label in the figure indicates a set of expectations with the number

$$\begin{tikzcd}
k_{\mathrm{i}} \arrow[d, "\epsilon"'] & & & & & \\
k_{\mathrm{i}}+1 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+1 \arrow[d, "\epsilon"'] & & & & \\
k_{\mathrm{i}}+2 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+2 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+2 \arrow[d, "\epsilon"'] & & & \\
k_{\mathrm{i}}+3 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+3 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+3 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+3 \arrow[d, "\epsilon"'] & & \\
k_{\mathrm{i}}+4 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+4 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+4 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+4 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+4 \arrow[d, "\epsilon"'] & \\
k_{\mathrm{i}}+5 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+5 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+5 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+5 \arrow[d, "\epsilon"'] & k_{\mathrm{i}}+5 \arrow[r, "z"] \arrow[d, "z"'] & k_{\mathrm{i}}+5 \arrow[d, "\epsilon"'] \\
{} & {} & {} & {} & {} & {}
\end{tikzcd}$$

FIG. 1. Start of the expansion for τ_k, with $k_{\mathrm{i}}=k+1$.

of vertices in each of these expectations equal, and equal to the value of the label. For example, "$k_{\mathrm{i}}+4$" stands for a set of expectations with $k_{\mathrm{i}}+4$-vertices. The arrows in the diagram indicate how an expectation evolves according to the expansion rules given above, as we progress from through the expansion, starting from the top which represents a single expectation with k-vertices. The rows in the diagram all contain expectations with a fixed number of vertices, this number increasing by one for each lower row. Thus the horizontal arrows indicate terms produced by a z-holonomy, that have one additional derivation of an initial vertex, but that have no new $\bar{\partial}$-vertex produced by that holonomy. On the other hand, vertical arrows in the diagram indicate some part of an expansion move that produces a new vertex. Thus a vertical arrow may either indicate an ε-perturbation move (that always produces a new $|z|^2$ vertex) or it may indicate those terms in a z-holonomy move that produce a $\bar{\partial}$-vertex. We label the vertical arrows in the corresponding way, with ε indicating an ε-perturbation and z indicating the relevant set of terms produced by a z-holonomy. The terms that occur after exactly N-moves all lie on the diagonal with slope 1 that intersects the vertical axis at the row labelled by $k_{\mathrm{i}}+N$.

Let us now describe in more detail the expansion that we diagram in Fig. 1. We require that all paths begin at the top, namely at the vertex labelled k_{i}. Each expansion move (i.e., each perturbation move or certain terms produced in each holonomy move) represents one unit of length on the path p. Let $|p|$ denote the length of a path in the figure of length $|p|$. Each path p is therefore composed of $|p|$ arrows, and each path p of length m represents a set of terms produced after m expansion moves. The expansion described above for τ_k is comprised of exactly $4n+4$ moves, so each path has that length. The crucial fact is that every term present at the end of the expansion is created by following some path p in Fig. 1.

Let $\mathscr{P}$ denote the set of all possible paths p starting at k_{i}. Let $\mathscr{P}(m)\subset\mathscr{P}$ denote those paths of length m. Let $|\mathscr{P}(m)|$ denote the number of paths in $\mathscr{P}(m)$,

$$|\mathscr{P}(m)|=\sum_{p\in\mathscr{P}(m)}1. \tag{XI.20}$$

Related to this, let $\mathscr{N}(p)$ denote the total number of expansion terms produced by following a particular path p, and let $\mathscr{N}_{\mathrm{exp}}(m)$ denote the total number of expansion terms produced after m expansion moves, and let $\mathscr{N}_{\mathrm{Final}}(m)$ denote the number of final terms produced in the expansion move m. Finally, let $\mathscr{N}_{\mathrm{Tot}}(m)$ denote the total number of expansion terms and final terms produced after m moves. We give the following combinatoric estimate:

Proposition XI.3. *After an even number $|p|=m$ of the expansion moves,*

$$|\mathscr{P}(m)|=2^{m/2}, \qquad \text{and} \qquad |\mathscr{N}(p)|\leqslant n^{m/2}2^{k+m}m!. \tag{XI.21}$$

Also $\mathscr{N}_{\mathrm{Final}}(m)=0,$

$$\mathscr{N}_{\mathrm{Final}}(m+1)=\mathscr{N}_{\mathrm{Exp}}(m), \qquad \text{and} \qquad \mathscr{N}_{\mathrm{Exp}}(m)\leqslant 2^k4^mn^{m/2}m!. \tag{XI.22}$$

The total number of terms is bounded by

$$\mathcal{N}_{\mathrm{Tot}}(m) = \mathcal{N}_{\mathrm{Exp}}(m) + \sum_{m'=1}^{m/2} \mathcal{N}_{\mathrm{Final}}(2m'-1) \leqslant 2^{k+1}4^m n^{m/2} m! \tag{XI.23}$$

Proof. We begin with some elementary observations. Clearly

$$\mathcal{N}_{\mathrm{Exp}}(m) \leqslant \sum_{p \in \mathcal{P}(m)} \mathcal{N}(p).$$

Therefore

$$\mathcal{N}_{\mathrm{Exp}}(m) \leqslant |\mathcal{P}(m)| \sup_{p \in \mathcal{P}(m)} |\mathcal{N}(p)|. \tag{XI.24}$$

Also, one final term is produced by each ε-perturbation move, which alternates with z-holonomy moves. Since we start in the first step with an ε-perturbation move, $\mathcal{N}_{\mathrm{Final}}(m)$ vanishes for even values of m, and $\mathcal{N}_{\mathrm{Final}}(m+1) = \mathcal{N}_{\mathrm{Exp}}(m)$, as claimed in (XI.22).

Figure 1 clearly includes the possible moves in the expansion outlined above. Since one can make an ε-perturbation move only in the vertical direction and one can make the following z-holonomy move a horizontal or a vertical direction, after m moves there are $2^{m/2}$ possible paths. Thus the first bound in (XI.21) is the exact number of elements in $\mathcal{P}(m)$.

Let us evaluate $\mathcal{N}(p)$ on two extreme paths: the vertical path which we denote p_{vertical}, and the path going as far to the right as possible, which we denote $p_{\mathrm{horizontal}}$. First consider p_{vertical}. Each arrow on this path increases the number of vertices by one. As explained in the definition of the ε-perturbation move and of the z-holonomy, this can be done at a point with ℓ-vertices in exactly ℓ-ways. However, for an ε-move, we regard $|z|^2$ as a sum of n individual vertices $|z_j|^2$, for we will holonomy each of these individually. Thus as $k_{\mathrm{i}} = k+1$,

$$\mathcal{N}(p_{\mathrm{vertical}}) = n^{m/2}\frac{(k+m)!}{k!} = n^{m/2}\binom{k+m}{l} m! \leqslant n^{m/2}2^{k+m}m!, \tag{XI.25}$$

as we state in (XI.21). At the other extreme, the path $p_{\mathrm{horizontal}}$ yields an additional vertex only at every second expansion step. Again, as explained above, each ε-perturbation step produces a number of terms equal to n times the number of existing vertices. On the other hand, each z-holonomy move produces $k_{\mathrm{i}} = k+1$ terms, each of which contains a new derivation of one of the $(k+1)$-initial vertices. Thus this path ends up in the row with $(k+m/2)$-vertices and

$$\mathcal{N}(p_{\mathrm{horizontal}}) = n^{m/2}(k+1)^{m/2}\frac{(k+m/2)!}{k!}. \tag{XI.26}$$

Since $(k+1)^{m/2} \leqslant (k+m)!/(k+m/2)!$, we infer that

$$\mathcal{N}(p_{\text{horizontal}}) \leqslant n^{m/2} \frac{(k+m)!}{k!} = \mathcal{N}(p_{\text{vertical}}). \tag{XI.27}$$

The paths p with $|p| = m$ that are intermediate between p_{vertical} and $p_{\text{horizontal}}$ in the extent of their excursion to the right, all end up on the diagonal connecting the endpoints of p_{vertical} and $p_{\text{horizontal}}$, and they produce an intermediate number of terms. In every case, $\mathcal{N}(p) \leqslant \mathcal{N}(p_{\text{vertical}})$, and (XI.21) holds in all cases. The bound (XI.22) then is a consequence of (XI.21), (XI.24).

The bound (XI.23) now follows, as

$$\begin{aligned} \mathcal{N}_{\text{Exp}}(m) + \sum_{m'=1}^{m/2} \mathcal{N}_{\text{Final}}(2mq'-1) &\leqslant \mathcal{N}_{\text{Exp}}(m) + \sum_{m'=1}^{m/2} \mathcal{N}_{\text{Exp}}(2m'-2) \\ &\leqslant 2^k 4^m n^{m/2} m! + \sum_{m'=1}^{m/2} 2^k 4^{2m'-2} n^{m'-1} \\ &\leqslant 2^{k+1} 4^m n^{m/2} m!, \end{aligned} \tag{XI.28}$$

so Proposition XI.2 is proved.

XI.2.2. *Estimates on Expectations*

We consider here the expansion of

$$\tau_k^{\text{JLO}}(a_0, ..., a_k; \varepsilon, \lambda; \theta) = \langle a_0, da_1, ..., da_k; \varepsilon, \lambda; \theta \rangle_k. \tag{XI.29}$$

with $k_{\text{i}} = k+1$ initial vertices. We wish to estimate the magnitude of all terms (both expansion terms and final terms) produced after m expansion moves. For expansion terms, we are interested in the case $m = 4n+2$, as the expansion is defined to terminate at that point, after $2n+1$ cycles each with 2 moves.

It is clear that each individual expectation $\mathfrak{I}$ present at the end of the expansion has a total number of vertices equal to $k_{\text{tot}} = k_{\text{i}} + k_\varepsilon + k_\partial$, where k_ε denotes the number of ε-vertices, and k_∂ denotes the number of ∂ and $\bar{\partial}$-vertices. For all terms, $k_{\text{i}} = k+1$. Also $k_\varepsilon \leqslant 2n+1$, with equality for expansion terms, and $k_\partial + k_{\text{der}} = k_\varepsilon$, with k_{der} equal to the number of derivations of initial vertices. Thus $k+1+k_\varepsilon \leqslant k_{\text{tot}} \leqslant k+4n+3$.

Given this data, we will reason that at the end of the expansion, any initial vertex (derived or not) has an operator norm that can be estimated in the $\mathfrak{A}_2$ topology. In order to simplify our notation, let us count as one of the derivations arising in the expansion, the derivation da_j, for $1 \leqslant j \leqslant k$, that is present in every such initial vertex at the beginning of the expansion. Thus we denote by $(a_j)^{\text{derived}}$ the form of the vertex a_j or da_j as it appears in the expectation $\mathfrak{I}$.

Lemma XI.4. *Let for* $0 \leqslant \lambda \leqslant 1$. *There is a constant* $\bar{M}_1 = \bar{M}_1(\mathfrak{B}_0^{\omega}) < \infty$ *such that every expectation* $\mathfrak{I}$ *at the end of the expansion of* $\mathfrak{Z}$, *each derived, initial vertex* $(a_j)^{\text{derived}}$ *satisfies the bound*

$$\|(a_j)^{\text{derived}}\| \leqslant \bar{M}_1 \, |||a_j|||_2. \tag{XI.30}$$

Proof. As argued above, the only derivation of a_0 or da_j, for $1 \leqslant j \leqslant k$ introduced during the expansion arises from the explicit da_j present at the start of the commutators with individual coordinates (or their complex conjugates) that are introduced as derivations to vertices during various z_j-holonomies (or $\bar{z}_j$-holonomies) carried out during the expansion. Within each expectation, the holonomies are limited by the structure of the expansion to at most $2n+1$ in number. Thus $(a_j)^{\text{derived}}$ will have a very specific form that we now describe in more detail. First consider a term with an initial derivation. We split this derivation into the sum,

$$\begin{aligned} da &= [\,Q(0), a\,] + [\,\lambda \dot{Q}, a\,] \\ &= \psi_1(\partial a) - \psi_1^*(\bar{\partial} a) + \lambda \psi_2[\overline{\partial V}, a] + \lambda \psi_2^*[\partial V, a]. \end{aligned} \tag{XI.31}$$

Here (∂a) denotes $[\,\partial, a\,]$, and $(\bar{\partial} a)$ denotes $[\,\bar{\partial}, a\,]$. Hence each of these terms contains at least one commutator, and the terms with a commutator of ∂V are a finite sum of terms of the form $P_a(z)[z_\ell, a]\, P_b(z)$. Here $P_a(z)$ and $P_b(z)$ are two monomials in z of total degree at most degree$(V) - 2$, multiplied by a coefficient v_J occurring in the expansion (IX.49) of the potential V.

Recall that v_J is uniformly bounded for $V \in \mathfrak{Z}_0$. In this case, we make an exception and retain λv_J as part of the vertex, rather than putting it into the overall coefficient of the expansion. We make this exception in order to deal immediately with the magnitude of the constant depending on these coefficients. The exponents of each monomial z^J in V must satisfy the constraint $\sum_{j=1}^{n} J_j \omega_j = 1$. Therefore each exponent must lie in the interval $0 \leqslant J_j \leqslant \omega_j^{-1}$, and can take at most $\omega_j^{-1} + 1$ different values. This yields a rough upper bound on the number of monomials in a polynomial potential V of class ω, namely

$$\prod_{j=1}^{n} (\omega_j^{-1} + 1). \tag{XI.32}$$

Likewise, we have a bound on the degree of V, namely degree$(V) \leqslant \sum_{j=1}^{n} \omega_j^{-1}$. Combining these two estimates, we infer that the maximum number of possible commutators arising from each monomial in the potential is

$$\left(\sum_{j=1}^{n} \omega_j^{-1} \right) \left(\prod_{j=1}^{n} (\omega_j^{-1} + 1) \right). \tag{XI.33}$$

Each of these terms has a coefficient that is bounded by $\sup_J(|v_J|(1/J!))$. Hence we include in the constant $\bar{M}_1$ a contribution

$$\left(\sum_{j=1}^{n} \omega_j^{-1}\right)\left(\prod_{j=1}^{n} (\omega_j^{-1}+1)\right) \sup_J \left(|v_J| \frac{1}{J!}\right), \tag{XI.34}$$

in order to give a rough upper bound on the number of terms and the magnitude of their coefficients.

The terms with a commutator of $\overline{\partial V}$ have a similar form but are functions of $\bar{z}$ rather than of z. These derivations and multiplications by monomials are followed by possibly none, but at most $2n+1$ additional commutators with linear coordinates. In summary, the initial vertex $(a_0)^{\text{derived}}$ may appear at the end of the expansion in its original form with no derivations, and thus satisfies the bound $\|(a_0)^{\text{derived}}\| = \|a_0\| \leqslant |||a_0|||_2$. Alternatively, the derived vertex $(a_j)^{\text{derived}}$, for $0 \leqslant j \leqslant k$ contains at least one commutator with one of the operators ∂, $\bar{\partial}$, z, or $\bar{z}$. In more detail, if the derived vertex has an ∂ or an $\bar{\partial}$ commutator, then in addition it may have up to $2n+1$ additional commutators with coordinates. On the other hand, if the derived vertex does not contain a commutator with ∂ or with $\bar{\partial}$, then it may also be multiplied by monomials in the coordinates of total degree degree$(V)-2$, as well as having up to $2n+1$ additional commutators with coordinates. In every case, the total degree of monomials in coordinates that may occur in a derived vertex is less than or equal to degree$(V)+2n$. Thus by choosing $\tilde{l} \geqslant 2$ degree(V) $+2n$, we are assured that every derived, initial vertex $(a_j)^{\text{derived}}$ satisfies (XI.30). (In fact, we have an extra degree(V) factors of the coordinates in reserve; we will use these in the section when we bound $\mathfrak{Z}_\lambda$.) We choose $\bar{M}_1$ sufficiently large to account for the size of the coefficients $|v_J|$ in the potential and the possible number of factors. Since we estimate the additional commutators after the first using the factors of the coordinates in the $|||\cdot|||_2$-norm, and this leads to a numerical factor at most 2^{2n+1}, we choose overall the constant taking (XI.31), (XI.34) into account. Thus we take

$$\bar{M}_1(\mathfrak{B}_0^\omega) \geqslant 2^{2n+1}(2n)\left(1+\left(\sup_J |v_J| \frac{1}{J!}\right)\left(\sum_{j=1}^{n} \omega_j^{-1}\right)\left(\prod_{j=1}^{n} (\omega_j^{-1}+1)\right)\right). \tag{XI.35}$$

This completes the proof of the lemma.

We now proceed to bound each of the expectations present at the end of the expansion.

Proposition XI.5. *Assume that $V \in \mathfrak{B}_0^\omega$ is a holomorph polynomial in a uniformly bounded set of potentials that satisfy both Assumptions* E *and Assumption* Q *of Subsections* II.4 *and* II.5. *Let $0 \leqslant \varepsilon$, $\lambda \leqslant 1$ with $0 < \varepsilon + \lambda$, and let $\theta \notin \Upsilon_{\text{sing}}$. Let $a_j \in \mathfrak{A}_2$ of Section* X, *and consider the expansion of a $(k+1)$-vertex expectation* (XI.29). *Then there is a constant $\bar{M}_1 = \bar{M}_1(\mathfrak{B}_0^\omega)$ such that after termination of the expansion,* *every resulting term* $\mathfrak{I}$ (*either final term or expansion term*), *with* k_{tot} *vertices, satisfies the bound*

$$|\mathfrak{I}| \leqslant d(\theta)^{2n+1}\, \bar{M}_1^{k+1} \frac{1}{\Gamma((1/2)(k+1+k_{\mathrm{tot}}))} \prod_{j=0}^{k} |||a_j|||_2. \qquad \text{(XI.36)}$$

Proof. We use the basic bounds of Proposition VIII.6(b) on regular sets of vertices, coupled with the bounds of Section VII on the individual vertices involved. We also use the method of Section IX to analyze the coefficient $c(\varepsilon', \lambda', \theta)$ of each expectation, which gets contributions from the factors of ε'' generated during the ε-perturbation moves, and factors of $d_j(\theta)$, that arise during the holonomies.

We apply the bound (VIII.22) to each expectation present at the end of the expansion of (XI.29). As in Section IX, expansion terms will be expectations depending on a regularization parameter ε' that is integrated over the interval $[\varepsilon, 1]$. We must ensure three things: first, we must identify all possible vertices X that are present, and bound them in an appropriate norm $\|X\|_{\mathscr{T}_{\varepsilon',\lambda}(-\beta,\alpha)}$, paying careful regard to the dependence of this norm on the parameters ε' and λ.

Second, the various η_j must all be positive, bounded uniformly away from zero, and sum to $\eta_{\mathrm{tot}} \geqslant \frac{1}{2}(k+1+k_{\mathrm{tot}})$. This lower bound on η_{tot} will produce the overall factor $\Gamma(\frac{1}{2}(k+1+k_{\mathrm{tot}}))^{-1}$ in our bound.

Third, the coefficient of each expectation, multiplied by $\mathrm{Tr}(e^{-H^{\#}/2})$ and by any parameter-dependent norms from the vertex estimates and integrated over ε', must be bounded uniformly by $O(1)^{k+1}\, d(\theta)^{2n+1}$ as (ε, λ) varies over the set $0 \leqslant \varepsilon, \lambda \leqslant 1$. Here the constant $O(1)$ should only depend on the uniformly bounded class of potentials $\mathfrak{V}_0^{\omega}$ to which V belongs. The constant $d(\theta)$ is given in (XI.2).

These steps are straightforward. In Lemma XI.3 we have already bounded the derived, initial vertices; each such vertex lies in $\mathscr{T}_{\varepsilon',\lambda}(-\beta, \alpha)$ with exponents $\alpha = \beta = 0$, and has norm bounded by (XI.30). The only other possible vertices that occur in the expansion are ε-vertices of the form z_j or $\bar{\partial}_j$ vertices. Each vertex $z_j \in \mathscr{T}_{\varepsilon',\lambda}(0,1) \cup \mathscr{T}_{\varepsilon',\lambda}(-1,0)$, so it can be estimated either with $\beta = 1$ and $\alpha = 0$ or else with $\beta = 0$ and $\alpha = 1$. The norm is given by (VII.4) as $\|z_j\|_{\mathscr{T}_{\varepsilon',\lambda}(0,1)} = \|z_j\|_{\mathscr{T}_{\varepsilon',\lambda}(-1,0)} \leqslant \bar{M}(\varepsilon'+\lambda)^{-1}$. Similarly, the $\bar{\partial}$ vertices also belong to $\mathscr{T}_{\varepsilon',\lambda}(0,1) \cup \mathscr{T}_{\varepsilon',\lambda}(-1,0)$, but by (VII.3) they have norm $\leqslant \bar{M}$ that is uniform in the parameters. Thus the product of the norms of the vertices X_j is bounded by

$$\left(\prod_{j=0}^{k_{\mathrm{tot}}} \|X_j\|_{\mathscr{T}_{\varepsilon',\lambda}(-\beta_j,\alpha_j)}\right) \leqslant (\bar{M}^{k_\varepsilon + k_{\bar{\partial}}}(\varepsilon'+\lambda)^{-k_\varepsilon}) \left(\prod_{j=0}^{k} \bar{M}_1\, |||a_j|||_2\right). \qquad \text{(XI.37)}$$

Second, we argue that the set of vertices present at the end of the expansion is a regular set. In fact since we have the freedom, we may choose $\beta = 0$ for all vertices, and then α_j equals 0 for the initial vertices and 1 for the remaining vertices. Thus every η_j equals either $\frac{1}{2}$ or 1, and

$$\eta_{\mathrm{tot}} = \sum_{j=0}^{k} \eta_j = \sum_{j=0}^{k} \tfrac{1}{2}(2-\alpha_j) = (k+1) + \tfrac{1}{2}(k_{\mathrm{tot}} - k - 1) = \tfrac{1}{2}(k+1+k_{\mathrm{tot}}). \qquad \text{(XI.38)}$$

Third, we analyze the overall coefficient of the expectation similarly to the analysis (IX.55)–(IX.57). We include the holonomy factors $|d_j(\theta)| \leqslant d(\theta)$, of which at most $(2n+1)$ occur. In place of (IX.57) we obtain for expansion terms the bound on the coefficient, up to a term $O(1)$ per vertex, as

$$d(\theta)^{2n+1} \int_{\varepsilon}^{1} C_{k_\varepsilon - 1}(\varepsilon', \varepsilon)(2\varepsilon')(\varepsilon' + \lambda)^{-k_\varepsilon} \operatorname{Tr}(e^{-H^{\#}/2})\, d\varepsilon'. \tag{XI.39}$$

Here we include in this bound the factor $(\varepsilon' + \lambda)^{-k_\varepsilon}$ arising from the contribution of ε-vertices to (XI.37). For expansion terms, we use the bound (VI.3) to ensure $\operatorname{Tr}(e^{-H^{\#}/2}) \leqslant 2^{2n} M_1 (\varepsilon' + \lambda)^{-2n}$. Thus we obtain

$$\begin{aligned} d(\theta)^{2n+1} &\int_{\varepsilon}^{1} C_{k_\varepsilon - 1}(\varepsilon', \varepsilon)(2\varepsilon')(\varepsilon' + \lambda)^{-k_\varepsilon} \operatorname{Tr}(e^{-H^{\#}/2})\, d\varepsilon' \\ &\leqslant d(\theta)^{2n+1}\, 2^{2n} M_1 \int_{\varepsilon}^{1} C_{k_\varepsilon - 1}(\varepsilon', \varepsilon)(2\varepsilon')(\varepsilon' + \lambda)^{-k_\varepsilon} (\varepsilon' + \lambda)^{-2n}\, d\varepsilon' \\ &\leqslant d(\theta)^{2n+1}\, 2^{2n} M_1 \frac{1}{(k_\varepsilon - 1)!} \\ &\leqslant d(\theta)^{2n+1}\, 2^{2n} M_1 \frac{1}{(2n)!}. \end{aligned} \tag{XI.40}$$

In the last step, we use $k_\varepsilon - 1 = 2n$ at the end of the expansion. For final vertices, the integral over ε' is replaced by the evaluation of the integrand at the endpoint $\varepsilon' = 1$, and the constant is uniformly bounded. Combining these three estimates, and choosing a new constant $\bar{M}_1$, completes the proof of Proposition XI.4.

Proof of Theorem XI.1.a. We bound the expectation (XI.29) using (XI.23) and (XI.36) by

$$\begin{aligned} &|\langle a_0, da_1, ..., da_k; \varepsilon, \lambda; \theta \rangle| \\ &\quad\leqslant (\mathcal{N}_{\mathrm{tot}}(m)) \, (\sup_{\mathfrak{I}} |\mathfrak{I}|) \\ &\quad\leqslant 4^m n^{m/2} m!\, d(\theta)^{2n+1} \frac{1}{\Gamma((1/2)(k+1+k_{\mathrm{tot}}))} \prod_{j=0}^{k} (2\bar{M}_1 \, |||a_j|||_2). \end{aligned} \tag{XI.41}$$

At the end of the expansion $m = 2(2n+1)$, so in general $m \leqslant 4n+2$. Likewise, the constant $4^m n^{m/2} m! \leqslant 4^{4n+2} n^{2n+1} (4n+2)!$. The constant n is a fixed dimension (determined by the class $\mathfrak{B}^{\omega}$ of potentials). Also, $k_{\mathrm{tot}} \geqslant k + 1 + k_\varepsilon$. Thus with a new constant $\bar{M}_1$ giving the constant $\bar{M}$, the inequality (XI.6) holds.

We remark that a more careful tracking of the constants could be done, leading to a modified entropy estimate and to the growth of the n-dependence of the overall constants in (XI.6) to be $O(1)^n$, rather $O(n!)$ to some power.

We also remark that the bound (XI.7) of Theorem XI.1 on $\partial\tau_k^{\mathrm{JLO}}/\partial\varepsilon$ can be established in exactly the same manner. Before starting the above expansion, we make two preliminary moves. First differentiate τ_k^{JLO} with respect to ε, producing one ε-vertex in each of $k+1$ terms. Next perform an z_j-holonomy on this vertex. We can now follow the exact same steps as above, except that we have one additional holonomy, leading to one additional factor of $d(\theta)$ as well as modified constants. We omit further details, but now concentrate on bounding the λ-derivative of τ_k^{JLO}.

XI.3. *The Expansion for* $\mathfrak{Z}_\lambda$

We now study the λ-derivative of $\mathfrak{Z}$. As with $\mathfrak{Z}$ and with $\mathfrak{Z}_\varepsilon$, we expand into the individual components τ_k^{JLO}. As above, we consider $a_j \in \mathfrak{A}_2$, for the analogous bounds with $a \in \mathfrak{A}_1$ are more straightforward. Our strategy is to perform a preliminary expansion on the derivative, after which we can proceed as in Subsection XI.2, with some extra initial vertices. These extra initial vertices will always be bounded in some appropriate $\mathscr{T}_{\varepsilon,\lambda}(-\beta, \alpha)$.

Begin from the identity

$$\begin{aligned}(\partial\tau_k/\partial\lambda)&(a_0, ..., a_k; \varepsilon, \lambda, \theta)\\ &= -\sum_{j=0}^{k} \langle a_0, da_1, ..., da_j, (Q\dot{Q} + \dot{Q}Q), da_{j+1}, ..., da_k\rangle_{k+1}\\ &\quad + \sum_{j=1}^{k} \langle a_0, da_1, ..., da_{j-1}, [\dot{Q}, a_j], da_{j+1}, ..., da_k\rangle_k. \end{aligned} \tag{XI.42}$$

Perform a Q-holonomy in the parts of the first term that have a $Q\dot{Q}$-vertex. This holonomy produces a term that cancels the $\dot{Q}Q$ contribution to the vertex. Due to the symmetry of the expectation under conjugation of each operator by γ, and the fact that each a is invariant under such conjugation, the expectations vanish if the integer k is odd. Thus using $daQ = -Qda + d^2a$, we obtain

$$\begin{aligned}(\partial\tau_k/\partial\lambda)&(a_0, ..., a_k; \varepsilon, \lambda, \theta)\\ &= \sum_{j=0}^{k} (-1)^j \Bigg(\langle da_0, da_1, ..., da_j, \dot{Q}, da_{j+1}, ..., da_k\rangle_{k+1}\\ &\quad + \sum_{l=1}^{j} (-1)^l \langle a_0, da_1, ..., d^2a_l, ..., da_j, \dot{Q}, da_{j+1}, ..., da_k\rangle_{k+1}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=j+1}^{k}(-1)^{l+1}\langle a_0, da_1, ..., da_j, \dot{Q}, ..., d^2a_l, ..., da_k\rangle_{k+1}\\
&+\varepsilon^2\sum_{l=0}^{j}\langle a_0, da_1, ..., da_l, \mathscr{L}, da_j, \dot{Q}, da_{j+1}, ..., da_k\rangle_{k+2}\\
&+(-1)^j\varepsilon^2\langle a_0, da_1, ..., da_j, \dot{Q}, \mathscr{L}, da_{j+1}, ..., da_k\rangle_{k+2}\\
&+\varepsilon^2\sum_{l=j+1}^{k}(-1)^l\langle a_0, da_1, ..., da_j, \dot{Q}, ..., da_l, \mathscr{L}, ..., da_k\rangle_{k+2}\\
&+\sum_{j=1}^{k}\langle a_0, da_1, ..., da_{j-1}, [\dot{Q}, a_j], da_{j+1}, ..., da_k\rangle_k\Bigg). \qquad \text{(XI.43)}
\end{aligned}
$$

The final sum in (XI.43) involves a commutator with $\dot{Q}$ and an a_j. This set of terms is easier than the others, as the commutator vertex has a bounded operator norm that can be estimated as in the proof of Lemma XI.3. For the terms in this sum, we start right out to perform $2n+1$ alterations of ε-perturbation moves with z-holonomy moves, as in the preceding subsection. Then we bound the results as in the proof of Proposition XI.4.

Now we discuss the remaining terms in (XI.43). At this point we split each of these terms into a sum of two terms that we analyze separately. The idea is that $\dot{Q}=\psi_2\,\overline{\partial V}+\psi_2^*\,\partial V$, and that the two terms will be the start of different expansions. For the $\psi_2^*\,\partial V$ parts, perform z-holonomies on each z-coordinate in the $\psi_2^*\,\partial V$ vertex. Likewise, for the $\psi_2\,\overline{\partial V}$ parts, perform $\bar{z}$-holonomies on each $\bar{z}$-coordinate in the $\psi_2\,\overline{\partial V}$ vertex.

The z-holonomies produce $\bar{\partial}$-vertices, as well as derived, initial vertices; the latter have additional z-commutators. We further expand any term with derivations and d^2a_j-vertices. We use the identity (for a fermionic vertex X)

$$[z, dX]=d([z, X])+\{[z, Q], X\}. \qquad \text{(XI.44)}$$

Symbolically, we write this as

$$[z, dX]=d((X)^{\text{derived}})-\{\psi_1, X\}. \qquad \text{(XI.45)}$$

Concretely, we have, for example,

$$[z_\ell, d^2(a_j)]=d([z_\ell, da_j])-\{\psi_1^{(\ell)}, da_j\}. \qquad \text{(XI.46)}$$

After each holonomy move, we use (XI.44) to expand any subsequent $z_{\ell'}$-derivation of vertex dX that contains a second d. Subsequent $z_{\ell'}$-derivations of the second term in (XI.46) with one d need not be expanded. The second term is a vertex that can be estimated in operator norm by $\bar{M}\,|||da_j|||_2$. However, we rewrite the first term using (XI.44). The point of this is to assure that one of the d-derivations can be estimated using an operator Sobolev-norm, rather than the $|||\cdot|||_2$-norm.

For example, at the end of the expansion when we estimate a vertex $X = d((da)^{\text{derived}})$, we split it into two parts,

$$X = d((da)^{\text{derived}}) = Q(da)^{\text{derived}} + (da)^{\text{derived}}\, Q. \tag{XI.47}$$

We then write

$$\| Q(da)^{\text{derived}} \|_{\mathscr{T}_{\varepsilon,\lambda}(-1,0)} = \|(da)^{\text{derived}}\| \leqslant \bar{M}\, |||a|||_2, \tag{XI.48}$$

and

$$\|(da)^{\text{derived}}\, Q\|_{\mathscr{T}_{\varepsilon,\lambda}(0,1)} = \|(da)^{\text{derived}}\| \leqslant \bar{M}\, |||a|||_2. \tag{XI.49}$$

This choice of β and α is always possible, because in any expectation there is at most one $d^2(a_j)$-vertex. All the remaining vertices either have $\alpha + \beta = 0$, namely $(da_j)^{\text{derived}}$-vertices, or else they have $\alpha + \beta = 1$, namely εz_j-vertices, $\varepsilon \bar{z}_j$-vertices, ∂-vertices, $\bar{\partial}$-vertices, or $\mathscr{L}$-vertices. Even if these vertices are adjacent, it does not cause a problem.

The vertices with the $\psi_2\, \overline{\partial V}$-vertices will be treated similarly, using however the corresponding relations for $[\bar{z}_\ell, d^2(a_j)]$, etc. These holonomies will produce a large number of factors $d_\ell(\theta)$ or their complex conjugates. But after all these holonomies are complete we will be able to estimate all the resulting vertices as operators with uniformly bounded norms in the appropriate $\mathscr{T}_{\varepsilon,\lambda}(-\beta, \alpha)$-spaces, as (ε, λ) varies over the positive, unit square.

Vertices of the form $\mathscr{L}$ are elements of $\mathscr{T}_{\varepsilon',\lambda}(-1,0) \cap \mathscr{T}_{\varepsilon',\lambda}(0,1)$, with norm $\|\mathscr{L}\|_{\mathscr{T}_{\varepsilon',\lambda}(0,1)} = \|\mathscr{L}\|_{\mathscr{T}_{\varepsilon',\lambda}(-1,0)} \leqslant \bar{M}(\varepsilon' + \lambda)^{-1}$. However, these vertices also contribute an ε^2-convergence factor to the overall coefficient of the expectation in which they appear. The treatment of the α and β for these vertices then parallels the treatment for the ε-vertices or the ∂-vertices. Thus no difficulty arises from these vertices.

At this point we now continue as in the expansion of the previous section, with two minor modifications. In any case we perform $(2n+2)$ cycles of moves, alternating ε-perturbation moves with holonomy moves. The terms that already have undergone z-holonomies (and therefore contain $\bar{\partial}$-vertices) will be expanded with z-holonomy moves as before. On the other hand, the first modification of the expansion is to specify that terms that already have undergone $\bar{z}$-holonomy moves (and contain ∂-vertices) will be expanded with further $\bar{z}$-holonomy moves, rather than z-holonomies. These single holonomy moves replace a $|z_j|^2$-perturbation vertex by a z_j-vertex and introduce a possible ∂_j-vertex. This prevents the occurrence of nonzero commutators between subsequent holonomies and the ∂-vertices already present. In other respects, there is no difference in practical terms between this procedure and the one used in Subsection XI.2. The second modification to the previous expansion will be that every holonomy of a vertex $d((da_j)^{\text{derived}})$ will be expanded using (XI.44), or the corresponding relation with $\bar{z}$.

With these modifications, we obtain combinatoric factors that can always be estimated by $\bar{M}_2^k$ times a power of $n!$. Thus by following the proof of Theorem XI.1(a), we establish Theorem XI.1(c), and complete the proof of Theorem XI.1.

XI.4. *The* $(\varepsilon, \lambda) \to (0, 0)$ *Limits of* $\tau -$ JLO *and of* $\mathfrak{Z}$

In this section we remark that the results of the paper yield the proof of Theorem 1.7. This result illustrates an important point about quantum field theory. It shows by example that the invariant information contained in the function $\mathfrak{Z}(\lambda; a; \theta)$ can be better behaved at a singular point such as $\lambda = 0$, than the expectations τ_k^{JLO} out of which we build the quantity $\mathfrak{Z}(\lambda; a; \theta)$. The former are regular at least for $\theta \notin \Upsilon_{\mathrm{sing}}$. The latter, however, are singular.

We combine our earlier results to investigate the continuity of the *JLO*-cocycle $\tau = \{\tau_k\}$ as a function of (ε, λ) at $(0, 0)$. The following is an immediate corollary of Theorem XI.1. It shows that $\tau_k(a_0, ..., a_k; \varepsilon, \theta)$ is continuous at $(\varepsilon, \lambda) = (0, 0)$ in a restricted sense.

Proposition XI.6. *Let* $V \in \mathfrak{B}_0^\omega$ *be a holomorphic polynomial satisfying Assumptions* E *and* Q *of Section* II *and lying in a uniformly bounded set of potentials as defined in Subsection* I.3. *Let* $a_j \in \mathfrak{A}_a$ *where* $\mathfrak{A} = \mathfrak{A}_1$ *or* $\mathfrak{A}_2$ *of Section* X. *Then for each* $\theta \notin \Upsilon_{\mathrm{sing}}$, *the limit*

$$\tau(\alpha\theta) = \lim_{\varepsilon \to 0} \lim_{\lambda \to 0} \tau(\varepsilon, \lambda, \theta) = \lim_{\lambda \to 0} \lim_{\varepsilon \to 0} \tau(\varepsilon, \lambda, \theta) \tag{XI.50}$$

exists and defines functional on $\mathfrak{A}$ *satisfying the estimate*

$$|\tau_k(a_0, ..., a_k; \theta)| \leqslant d(\theta)^{2n+1} \bar{M}^{k+1} \frac{1}{k!} \prod_{j=0}^{k} |||a_j|||. \tag{XI.51}$$

This limiting functional is not bounded uniformly as a function of θ. The invariant constructed out of this limit is. In particular, for $a \in \mathfrak{A}_2$, such that $a^2 = I$ and $aU(\theta) = U(\theta)\, a$ for all θ, the function $\lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon, 0; a; \theta)$ is equal to the invariant $\mathfrak{Z}(0, \lambda; a; \theta)$, and for $\lambda > 0$ we have established that this function is λ-independent. As a consequence of Proposition VIII.6(c), this function for $\lambda > 0$ is also continuous in θ. Hence as the limit

$$\mathfrak{Z}(0, 0; a; \theta) = \lim_{\lambda \to 0} \mathfrak{Z}(0, \lambda; a; \theta)$$

is the limit of a constant function of λ, the limit is continuous as a function of θ. But by the continuity of $\mathfrak{Z}$ in (ε, λ),

$$\mathfrak{Z}(0, 0; a; \theta) = \lim_{\varepsilon \to 0} \mathfrak{Z}(\varepsilon, 0; a; \theta).$$

Hence we have proved Theorem I.7 in the case $\mathfrak{A} = \mathfrak{A}_2$. Analogously, a similar result holds for $a \in \mathfrak{A}_1$, rather than $a \in \mathfrak{A}_2$.

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