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DERIVATIVES WITH TWISTS

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We study derivatives on an interval of length ℓ (or the associated circle of the same length), and certain pseudo-differential operators that arise as their fractional powers. We compare different translations across the interval (around the circle) that are characterized by a twisting angle. These results have application in the study of twist quantum field theory.

Consider the Hilbert space $\mathbb{K} = L^2([0,\ell]; dx)$ over the interval of length ℓ. It is well known that the skew-symmetric operator of differentiation $D = \frac{d}{dx}$ with the domain of smooth, compactly-supported functions yields a one-parameter family of skew-adjoint extensions, parameterized by an angle χ. Each extension has an orthonormal basis of eigenfunctions for D given by,

$$f_k(x) = \frac{1}{\sqrt{\ell}} e^{ikx}, \quad \text{for } k \in K = \frac{2\pi}{\ell}\mathbb{Z} - \frac{\chi}{\ell}. \tag{1}$$

The angle χ specifies a twist, and (1) extends each f_k to a smooth function on $\mathbb{R}$ satisfying

$$f_k(x+\ell) = e^{-i\chi} f_k(x). \tag{2}$$

1. Motivation

We described the twisted interval above in terms of pure mathematics; yet twisting plays several roles in physics. First, one often encounters parallel transport about a closed trajectory. The physical role of twisting includes the fact that the condition (2) ensures that angular momentum zero is not allowed, $0 \notin K$. Hence twisting provides an infra-red regularization, which can be useful in the study of massless fields. In fact, this author has taken advantage of these properties in recent works, see [1, 2, 3] and other works cited there.

These investigations led to the genesis of the current paper, for in the detailed estimates one must compare different twists. This comparison can be carried out using the bounds that we establish here.

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In a different direction, one is fascinated by the possibility to measure twisting in the laboratory. A team of British physicists has accomplished this recently, according to a report in *Science* [4]. One can complement photon helicity (which takes two values) with a measurement of angular momentum (which takes values in correspondence with the magnitude of the twist). In this way, one has the potential to revolutionize communications through the increase in the density of data transmission, for the possibility to measure twisting allows an individual photon to carry more information than the single bit associated with helicity.

2. Fractional Derivatives

Let $\mathcal{T} = \{\ell, \chi\}$ denote the interval and the twist, and let $\mathcal{D}_{\mathcal{T}}$ denote the finite linear span of the set of basis vectors $\{f_k\}$. Define $D_{\mathcal{T}}$ as the closure of the derivative operator $\frac{d}{dx}$ defined on the domain $\mathcal{D}_{\mathcal{T}}$. The vectors f_k form an eigenbasis for $D_{\mathcal{T}}$, so $-iD_{\mathcal{T}}$ is essentially self adjoint, and the spectrum of its closure is K. Define the *energy* operator for a unit mass as

$$\mu_{\mathcal{T}} = (I - D_{\mathcal{T}}^2)^{1/2}\,, \tag{3}$$

which for large frequency Fourier modes is asymptotically equal to $|k|$. The large-frequency behavior of $\mu_{\mathcal{T}}^{\delta}$ for $\delta > 0$ is similar to the absolute value of a fractional derivative of order δ. We also use the function $\mu(k) = (1+k^2)^{1/2}$ defined in Fourier space.

3. Comparison of Derivatives

Consider a pair of twists $\mathcal{T}$ and $\mathcal{T}' = \{\ell, \chi'\}$ on a fixed interval, and the corresponding orthonormal bases $\{f_k = \frac{1}{\sqrt{\ell}} e^{ikx}\}$ and $\{g_{k'} = \frac{1}{\sqrt{\ell}} e^{ik'x}\}$, for $k \in K$ and for $k' \in K'$ respectively. Assume that

$$\chi \neq \chi' (\mathrm{mod}\ 2\pi)\,, \tag{4}$$

so the momentum sets are disjoint, $K \cap K' = \phi$.

Proposition 3.1 (Domains). *The self-adjoint operator* $\mu_{\mathcal{T}}^{\delta}$ *has the following properties*:

(i) *If* $0 \le \delta < \frac{1}{2}$, *then the domain of the operator* $\mu_{\mathcal{T}}^{\delta}$ *contains* $\mathcal{D}^{\{\mathcal{T}'\}}$.
(ii) *If* $0 \le \delta < 1$, *then the domain of the sesqui-linear form* $\mu_{\mathcal{T}}^{\delta}$ *contains* $\mathcal{D}^{\{\mathcal{T}'\}} \times \mathcal{D}^{\{\mathcal{T}'\}}$.
(iii) *For* $0 \le \delta < 1$, *the matrix elements of* $\mu_{\mathcal{T}}^{\delta}$ *in the basis* $\{g_{k'}\}$ *are*

$$\langle g_{k_1'}, \mu_{\mathcal{T}}^{\delta} g_{k_2'} \rangle = \frac{4}{\ell^2} \sin^2\left(\frac{\chi - \chi'}{2}\right) \sum_{k \in K} \frac{\mu(k)^{\delta}}{(k_1' - k)(k_2' - k)}\,, \quad \textit{where } k_1', k_2' \in K'\,. \tag{5}$$

Proof. (i) It is sufficient to show that the domain of $\mu_{\mathcal{T}}^{\delta}$ contains each $g_{k'}$, which we now demonstrate. There exists a unitary operator U that relates the bases f_k and $g_{k'}$. The matrix elements $U_{kk'} = \langle f_k, g_{k'} \rangle$ of U satisfy

$$g_{k'} = \sum_{k \in K} U_{kk'} f_k \tag{6}$$

and direct computation yields

$$U_{kk'} = \frac{2e^{i(\chi-\chi')/2}}{\ell(k'-k)} \sin\left(\frac{\chi-\chi'}{2}\right) = \langle f_k, g_{k'} \rangle \,. \tag{7}$$

By definition each vector f_k lies in the domain of $\mu_{\mathcal{T}}^{\delta}$, and hence so does any finite linear combination of these basis vectors. Let $\Lambda < \infty$ denote a parameter and define an approximating sequence $g_{k',\Lambda}$ to $g_{k'}$ by

$$g_{k',\Lambda} = \sum_{\substack{k \in K \\ |k| \le \Lambda}} U_{kk'} f_k \in \mathcal{D}^{\{\mathcal{T}\}} \,. \tag{8}$$

Clearly $\|g_{k'} - g_{k',\Lambda}\| \to 0$ as $\Lambda \to \infty$. If in addition it is the case that $\mu_{\mathcal{T}}^{\delta} g_{k',\Lambda}$ converges as $\Lambda \to \infty$, then $g_{k'}$ lies in the domain of the (self-adjoint) closure of $\mu_{\mathcal{T}}^{\delta}$.

Since f_k is an eigenvector of $\mu_{\mathcal{T}}^{\delta}$, we infer

$$\mu_{\mathcal{T}}^{\delta} g_{k',\Lambda} = \mu_{\mathcal{T}}^{\delta} \sum_{\substack{k \in K \\ |k| \le \Lambda}} U_{kk'} f_k = \sum_{\substack{k \in K \\ |k| \le \Lambda}} U_{kk'} \mu(k)^{\delta} f_k \,, \tag{9}$$

so that

$$\|\mu_{\mathcal{T}}^{\delta} g_{k',\Lambda}\|^2 = \sum_{\substack{k \in K \\ |k| \le \Lambda}} |\mu(k)^{\delta} U_{kk'}|^2 = \sum_{\substack{k \in K \\ |k| \le \Lambda}} \frac{4\mu(k)^{2\delta}}{\ell^2(k-k')^2} \sin^2\left(\frac{\chi-\chi'}{2}\right) \,. \tag{10}$$

This sum over k is finite, and for $\delta < \frac{1}{2}$ the bound on the sum is uniform in Λ. Furthermore, for $\Lambda < \Lambda'$,

$$\mu_{\mathcal{T}}^{\delta}(g_{k',\Lambda} - g_{k',\Lambda'}) = \sum_{\substack{k \in K \\ |\Lambda| < |k| \le \Lambda'}} U_{kk'} \mu(k)^{\delta} f_k \,, \tag{11}$$

from which one infers

$$\begin{aligned} \|\mu_{\mathcal{T}}^{\delta}(g_{k',\Lambda} - g_{k',\Lambda'})\|^2 &= \sum_{\substack{k \in K \\ |\Lambda| < |k| \le \Lambda'}} |\mu(k)^{\delta} U_{kk'}|^2 \\ &= \sum_{\substack{k \in K \\ |\Lambda| < |k| \le \Lambda'}} \frac{4\mu(k)^{2\delta}}{\ell^2(k-k')^2} \sin^2\left(\frac{\chi-\chi'}{2}\right) \,. \end{aligned} \tag{12}$$

Since $2\delta < 1$, the sum on the right of (12) converges and

$$\|\mu_{\mathcal{T}}^{\delta}(g_{k',\Lambda} - g_{k',\Lambda'})\| \le o(1) \,, \tag{13}$$

as $\Lambda \to \infty$. Thus $\mu_{\mathcal{T}}^{\delta} g_{k',\Lambda}$ converges to a limit as Λ, $\Lambda' \to \infty$, completing the proof of (i).

(ii) This follows immediately from (i).

(iii) Take the inner product of $g_{k'_1}$ with the representation (6) and use (7) to obtain

$$\langle g_{k'_1}, \mu_{\mathcal{T}}^{\delta} g_{k'_2}\rangle = \sum_{k\in K} U_{kk'}\mu(k)^{\delta}\langle g_{k'_1}, f_k\rangle = \sum_{k\in K}\langle g_{k'_1}, f_k\rangle \mu(k)^{\delta}\langle f_k, g_{k'_2}\rangle \,. \tag{14}$$

Substituting the values in (7) for the matrix elements of U yields (5) and completes the proof. □

Remark. We give here a second derivation of the identity (5) in the case that $\delta = 0$. Let a, b be non-integers and consider the convergent sum, which for $a \neq b$ equals

$$F(a,b) = \sum_{n\in\mathbb{Z}} \frac{1}{(n+a)(n+b)} = \frac{\pi \sin(\pi(b-a))}{(b-a)\sin(\pi a)\sin(\pi b)} \,. \tag{15}$$

One can obtain the value above by considering the contour integral of a meromorphic function,

$$\int_{C_n} \pi\cot(\pi z)\frac{1}{(z+a)(z+b)}\, dz \,, \tag{16}$$

taken on a sequence of circular contours C_n, centered at the origin and of radius $n+\frac{1}{2}$, where $n \in \mathbb{Z}_+$. For $a \neq b$ the singularities of the integrand are simple poles. Assuming also that $n > |a|, |b|$, the contour C_n encloses $2n+3$ poles of the integrand, as follows. The function $\pi\cot(\pi z)$ has a pole at each integer, with residue 1, and C_n encloses $2n+1$ of these poles. The other two poles occur at $z = -a$, $-b$. On the contour C_n of length $O(n)$ the function $\pi\cot(\pi z)$ is bounded uniformly in n, and the function $1/|(z+a)(z+b)|$ tends to zero as $O(1/n^2)$ as $n \to \infty$. Therefore the magnitude of the integrals (16) converge to zero as $O(1/n)$.

Using the Cauchy integral theorem we infer that the sum of the residues of the integrand vanish. Using the addition law for sines, this yields in the $n \to \infty$ limit,

$$F(a,b) = \frac{\pi\cot(\pi a)}{(b-a)} - \frac{\pi\cot(\pi b)}{(b-a)} = \frac{\pi\sin(\pi(b-a))}{(b-a)\sin(\pi a)\sin(\pi b)} \,. \tag{17}$$

Letting $b \to a$ gives $F(a,a) = \sum_{n\in\mathbb{Z}}(n+a)^{-2} = \pi^2/\sin^2(\pi a)$. Thus in case $b-a$ is integer,

$$\pi^{-2}\sin^2(\pi a)F(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } 0 \neq b-a \in \mathbb{Z} \end{cases} \,. \tag{18}$$

Parameterize the sum (5) in the case $\delta = 0$ as follows: take $\ell k'_j = 2\pi n'_j - \chi'$ and $\ell k = 2\pi n - \chi$ with n, $n'_j \in \mathbb{Z}$; take $a = \frac{(\chi'-\chi)}{2\pi}$ and $b = \frac{(\chi'-\chi)}{2\pi} + n'_1 - n'_2$. Then a, b are non-integer, while $b-a$ is integer. In terms of these variables, (5) has the form

$$\langle g_{k'_1}, g_{k'_2}\rangle = \pi^{-2}\sin^2(\pi a)F(a,b) \,, \tag{19}$$

which by (18) equals $\delta_{k'_1 k'_2}$.

Proposition 3.2 (Relative Bound). *Let $0 \leq \delta < \frac{1}{2}$ and $\delta < \frac{1}{2}\delta'$. Then $\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}$ is a bounded transformation with norm M,*

$$\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\| \leq M\,, \tag{20}$$

where $M = M(\delta,\delta',\ell)$ can be chosen independently of χ, χ'.

Definition 3.3. ($\boldsymbol{\ell_{1,\infty}}$ Norm). Consider an orthonormal basis $\mathcal{B} = \{e_i\}$ for the Hilbert space $\mathcal{H}$ and a closed linear transformation X with domain containing $\mathcal{B}$ as a core. Let $X_{ij} = \langle e_i, Xe_j\rangle$ denote the matrix elements of X in the basis. Define the $\ell_{1,\infty}$ norm of X with respect to the basis $\mathcal{B}$ as

$$\|X\|_{\mathcal{B}_{1,\infty}} = \left(\sup_i \left(\sum_j |X_{ij}|\right) \sup_{j'} \left(\sum_{i'} |X_{i'j'}|\right)\right)^{1/2}. \tag{21}$$

In case X is self-adjoint or skew-adjoint, the $\ell_{1,\infty}$ norm reduces to

$$\|X\|_{\mathcal{B}_{1,\infty}} = \sup_i \left(\sum_j |X_{ij}|\right). \tag{22}$$

Lemma 3.4. ($\boldsymbol{\ell_{1,\infty}}$ ***Estimate***). *The $\ell_{1,\infty}$ norm given in Definition* 3.3 *dominates the operator norm $\|X\|$ of X,*

$$\|X\| \leq \|X\|_{\mathcal{B}_{1,\infty}}\,. \tag{23}$$

Proof. Let $f = \sum_i f_i e_i$ and $g = \sum_i g_i e_i$ be unit vectors. Consider $\langle f, Xg\rangle = \sum_{ij} \bar{f}_i X_{ij} g_j$. Thus the Schwarz inequality yields

$$\begin{aligned} |\langle f, Xg\rangle| &\leq \sum_{ij} |f_i X_{ij} g_j| \\ &\leq \left(\sum_{ij} |f_i^2 X_{ij}|\right)^{1/2} \left(\sum_{ij} |X_{ij} g_j^2|\right)^{1/2} \\ &\leq \left(\sum_i |f_i|^2\right)^{1/2} \left(\sup_i \sum_j |X_{ij}|\right)^{1/2} \left(\sup_j \sum_i |X_{ij}|\right)^{1/2} \left(\sum_j |g_j^2|\right)^{1/2} \\ &= \|X\|_{\mathcal{B}_{1,\infty}}\,, \end{aligned} \tag{24}$$

from which the claim follows. □

Lemma 3.5. *Let χ, $\chi' \in (0,\pi)$. Then there exists a constant $J = J(\ell) < \infty$ such that*

$$\sup_{\chi,\chi'} \sup_{\substack{k\in K \\ k'\in K'}} \frac{\mu(k'-k)}{\ell|k'-k|} \sin\left(\frac{|\chi-\chi'|}{2}\right) \leq J\,. \tag{25}$$

Proof. The momentum difference in the denominator has the value $\ell(k'-k) = 2\pi n+\chi-\chi'$ for $n\in\mathbb{Z}$. If $n\neq 0$, then $\ell|(k'-k)|\geq\pi$. In this case, $\mu(k'-k)/\ell|k'-k|$ is uniformly bounded, as long as ℓ is bounded away from zero. On the other hand, if $n=0$, then

$$\frac{\mu(k'-k)}{\ell|k'-k|}\sin\left(\frac{|\chi-\chi'|}{2}\right) = \frac{\mu((\chi-\chi')/\ell)}{|\chi'-\chi|}\sin\left(\frac{|\chi-\chi'|}{2}\right) . \tag{26}$$

Since $|\mathrm{sin}x/x|$ is bounded, it follows that (26) is bounded uniformly in χ, χ' for fixed ℓ. □

Lemma 3.6. *Let α, $\beta\in\mathbb{R}$ with $\alpha+\beta>1$, and define $\gamma=\min\{\alpha,\beta,\alpha+\beta-1\}$. Then there are constants $0<M_\pm=M_\pm(\alpha,\beta)<\infty$ such that*

$$F(k)=\frac{1}{\ell}\sum_{p\in K}\mu(k-p)^{-\alpha}\mu(p)^{-\beta} \tag{27}$$

satisfies the upper bound

$$F(k)\leq\begin{cases} M_+\mu(k)^{-\gamma}\,, & \text{if } \alpha,\beta\neq 1\\ M_+\mu(k)^{-\gamma}\ln(1+\mu(k))\,, & \text{if } \alpha=1 \text{ or } \beta=1\end{cases}\,, \tag{28}$$

and the lower bound

$$F(k)\geq\begin{cases} M_-\mu(k)^{-\gamma}\ln(1+\mu(k))\,, & \text{if } \alpha=1,\beta\leq 1, \text{ or if } \beta=1,\alpha\leq 1\\ M_-\mu(k)^{-\gamma}\,, & \text{otherwise}\end{cases}\,. \tag{29}$$

Furthermore the same bounds hold if the sum ranges over $p\in K'$ in place of $p\in K$.

Proof. The proofs for the lattices K and K' are the same. Furthermore $\mu(p)$ is even, so it is no loss of generality to assume that $\beta\geq 0$. Divide the p sum into three disjoint regions:

$$I=\left\{p:|p|\leq\frac{1}{2}|k|\right\}\,,\quad II=\left\{p:\frac{1}{2}|k|<|p|<2|k|\right\}\,,\quad\text{and}\quad III=\{p:2|k|\leq|p|\}\,, \tag{30}$$

and denote the corresponding sums F_I, etc.

First we prove the upper bounds. In region I, it is the case that $\mu(k-p)\leq\mu(3k/2)\leq\text{const.}\,\mu(k)$, so $\mu(k-p)^{|\alpha|}\leq\text{const.}\,\mu(k)^{|\alpha|}$. Also $\mu(k-p)\geq\mu(k/2)\geq\text{const.}\,\mu(k)$, so $\mu(k-p)^{-|\alpha|}\leq\text{const.}\,\mu(k)^{-|\alpha|}$. Therefore for either sign of α, there is a constant $\tilde{M}_1$ such that $\mu(k-p)^{-\alpha}\leq\tilde{M}_1\mu(k)^{-\alpha}$, and

$$F_I(k)\leq\tilde{M}_1\mu(k)^{-\alpha}\frac{1}{\ell}\sum_{p\in I}\mu(p)^{-\beta}\leq M_1\mu(k)^{-\alpha}\begin{cases}1\,, & \text{if } \beta>1\\ \ln(1+\mu(k))\,, & \text{if } \beta=1\\ \mu(k)^{-\beta+1}\,, & \text{if } \beta<1\end{cases}$$

$$\leq M_1\mu(k)^{-\gamma}\begin{cases}1\,, & \text{if } \beta\neq 1\\ \ln(1+\mu(k))\,, & \text{if } \beta=1\end{cases}\,. \tag{31}$$

Similarly, in region II use the bound $\mu(p) \geq \text{const.}\,\mu(k)$, as well as the bound $\mu(p) \leq \text{const.}\,\mu(k)$ to obtain $\mu(p)^{-\beta} \leq \tilde{M}_2\mu(k)^{-\beta}$. Therefore

$$F_{II}(k) \leq \tilde{M}_2\mu(k)^{-\beta}\frac{1}{\ell}\sum_{p\in II}\mu(k-p)^{-\alpha} \leq M_2\mu(k)^{-\beta}\begin{cases} 1\,, & \text{if } \alpha > 1 \\ \ln(1+\mu(k))\,, & \text{if } \alpha = 1 \\ \mu(k)^{-\alpha+1}\,, & \text{if } \alpha < 1 \end{cases}$$

$$\leq M_2\mu(k)^{-\gamma}\begin{cases} 1\,, & \text{if } \alpha \neq 1 \\ \ln(1+\mu(k))\,, & \text{if } \alpha = 1 \end{cases}\,. \tag{32}$$

Finally, in region III, it is the case that $\mu(k-p) \geq \text{const.}\,\mu(p)$ and also $\mu(k-p) \leq \mu(3p/2) \leq \text{const.}\,\mu(p)$. Thus

$$F_{III}(k) \leq \tilde{M}_3\frac{1}{\ell}\sum_{p\in III}\mu(p)^{-\alpha-\beta} \leq M_3\mu(k)^{-\alpha-\beta+1} \leq M_3\mu(k)^{-\gamma}\,. \tag{33}$$

In all three cases, and hence for the union of the regions, there is a constant M_+ such that $F(k)$ satisfies the upper bound

$$F(k) \leq M_+\mu(k)^{-\gamma}\begin{cases} 1\,, & \text{if } \alpha,\beta \neq 1 \\ \ln(1+\mu(k))\,, & \text{if } \alpha = 1 \text{ or } \beta = 1 \end{cases}\,. \tag{34}$$

In order to obtain a lower bound, use the above inequalities in the opposite direction. It is convenient to assume that $|k|$ is sufficiently large so that the sets I and II are both non-empty. On the set of $|k|$ too small to achieve this, direct inspection shows that any contribution to $F(k)$ from region III is bounded below by a strictly positive constant. Then observe that in region I, it is the case that $\mu(k-p)^{-|\alpha|} \geq \text{const.}\,\mu(k)^{-|\alpha|}$. Also in region I one has $\mu(k-p)^{|\alpha|} \geq \text{const.}\,\mu(k)^{|\alpha|}$. Thus regardless of the sign of α there is a new constant $\tilde{M}_1$ such that $\mu(k-p)^{-\alpha} \geq \tilde{M}_1\mu(k)^{-\alpha}$. We infer that

$$F_I(k) \geq \tilde{M}_1\mu(k)^{-\alpha}\frac{1}{\ell}\sum_{p\in I}\mu(p)^{-\beta} \geq M_1\mu(k)^{-\alpha}\begin{cases} 1\,, & \text{if } \beta > 1 \\ \ln(1+\mu(k))\,, & \text{if } \beta = 1 \\ \mu(k)^{-\beta+1}\,, & \text{if } \beta < 1 \end{cases}\,. \tag{35}$$

Similarly in region II we use the bound $\mu(p) \leq \text{const.}\,\mu(k)$ and the assumption $\beta \geq 0$ to obtain $\mu(p)^{-\beta} \geq \tilde{M}_2\mu(k)^{-\beta}$. Therefore

$$F_{II}(k) \geq \tilde{M}_2\mu(k)^{-\beta}\frac{1}{\ell}\sum_{p\in II}\mu(k-p)^{-\alpha}$$

$$\geq M_2\mu(k)^{-\beta}\begin{cases} 1\,, & \text{if } \alpha > 1 \\ \ln(1+\mu(k))\,, & \text{if } \alpha = 1 \\ \mu(k)^{-\alpha+1}\,, & \text{if } \alpha < 1 \end{cases}\,. \tag{36}$$

Taking the greater lower bound from (35) with (36) yields the lower bound on $F(k)$ in (29) and completes the proof. □

Proof of Proposition 3.2. Let $T = \mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}$. The matrix elements for T^*T in the $\{g_{k'}\}$ basis are

$$\begin{aligned}\langle g_{k_1'}, T^*T g_{k_2'}\rangle &= \langle g_{k_1'}, \mu_{\mathcal{T}}^{2\delta} g_{k_2'}\rangle \mu(k_1')^{-\delta'}\mu(k_2')^{-\delta'} \\ &= \frac{4}{\ell^2}\sin^2\left(\frac{\chi-\chi'}{2}\right)\sum_{k\in K}\frac{\mu(k_1')^{-\delta'}\mu(k)^{2\delta}\mu(k_2')^{-\delta'}}{(k_1'-k)(k_2'-k)} \,. \qquad (37)\end{aligned}$$

The $\ell_{1,\infty}$ estimate of the operator norm of this self-adjoint matrix yields

$$\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\|^2 = \|T\|^2 = \|T^*T\| \le \|T^*T\|_{\mathcal{B}_{1,\infty}} \,. \qquad (38)$$

Therefore

$$\begin{aligned}\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\|^2 &\le \sup_{k_1'\in K'}\left(\sum_{k_2'\in K'}|\langle g_{k_1'}, T^*T g_{k_2'}\rangle|\right) \\ &\le \frac{4}{\ell^2}\sin^2\left(\frac{\chi-\chi'}{2}\right)\sup_{k_1'\in K'}\left(\sum_{\substack{k\in K\\ k_2'\in K'}}\frac{\mu(k_1')^{-\delta'}\mu(k)^{2\delta}\mu(k_2')^{-\delta'}}{|(k_1'-k)(k-k_2')|}\right) \,. \qquad (39)\end{aligned}$$

Apply Lemma 3.5 to obtain the twist-independent bound

$$\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\|^2 \le 4J^2 \sup_{k_1'\in K'}\left(\mu(k_1')^{-\delta'}\sum_{\substack{k\in K\\ k_2'\in K'}}\frac{\mu(k)^{2\delta}\mu(k_2')^{-\delta'}}{\mu(k_1'-k)\mu(k-k_2')}\right) \,. \qquad (40)$$

This bound is a sum of positive terms, so if it is convergent, it must be summable in any order. Apply Lemma 3.6 to the k_2' sum to obtain

$$\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\|^2 \le 4J^2\ell M(1,\delta') \sup_{k_1'\in K'}\left(\mu(k_1')^{-\delta'}\sum_{k\in K}\frac{\mu(k)^{2\delta-\delta'}\ln(1+\mu(k))}{\mu(k_1'-k)}\right) \,. \qquad (41)$$

Since $2\delta-\delta'<0$, we infer that $\mu(k)^{2\delta-\delta'}\ln(1+\mu(k)) \le M_1\mu(k)^{-\epsilon}$ for a constant $M_1<\infty$ and any $\epsilon\in(0,\delta'-2\delta)$. Thus apply Lemma 3.6 to the remaining k sum to obtain

$$\begin{aligned}\|\mu_{\mathcal{T}}^{\delta}\mu_{\mathcal{T}'}^{-\delta'}\|^2 &\le 4J^2\ell M(1,\delta') \sup_{k_1'\in K'}\left(\mu(k_1')^{-\delta'}\sum_{k\in K}\frac{\mu(k)^{-\epsilon}}{\mu(k_1'-k)}\right) \\ &\le 4J^2\ell^2 M(1,\delta')M(1,\epsilon)\sup_{k_1'\in K'}\mu(k_1')^{-\delta'-\epsilon}\ln(1+\mu(k_1')) < \infty \,. \qquad (42)\end{aligned}$$

This bound does not involve the angles χ, χ', so the estimate is uniform in these parameters. Renaming the final bound to be the constant $M^2 = M(\delta,\delta',\ell)^2$ completes the proof. □

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