Constructive Quantum Field Theory

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Background: Isolate; Explain

- Newton: Gravity Calculus
- Maxwell: Electromagnetism Symmetry
- Gibbs/Boltzmann: Statistical Physics Probability

Cornerstones of 20th Century Physics:

- Bohr/Schrödinger/Heisenberg: Quantum
  Hilbert Space, Functional Analysis, PDE
- Einstein: Relativity Geometry/Symmetry

Hilbert's 6th Problem: Axiomatize Physics

- Formulated by Hilbert in 1900 at Paris ICM
- Einstein: "… the creative principle resides in mathematics…, therefore I hold it true that pure thought can grasp reality, as the ancients dreamed."
- Wightman: "A great physical theory is not mature until it has been put in a precise mathematical form."

So: Can One Isolate and Explain Relativity + Quantum Theory?

- Field theory predicts phenomena, including QED (Maxwell-Dirac equations), where observations agree with rules for calculation to an accuracy beyond belief.
- Gabrielse: anomalous magnetic moment of the electron: 12 digits
- Hansch: frequency measurements of quantum levels: 14 digits

QFT is the Basic Proposal

- Is quantum field theory a "theory"? Does it have a set of equations with mathematical (logically consistent) solutions that describe physics?
- Any relativistic field with non-trivial interaction appeared beyond mathematical reach.
- Constructive QFT proposed to address this question.
- Recently physics has served as a lighthouse for mathematics: not the topic of this lecture.

Problem:
Fields Singular, Mathematics Tricky

Canonical commutation:

$$\left[\varphi(\vec{x},t), \frac{\partial \varphi(\vec{y},t)}{\partial t}\right] = i\delta\left(\vec{x}-\vec{y}\right)$$

$$\{\psi(\vec{x}), \psi(\vec{y})\} = \delta\left(\vec{x}-\vec{y}\right)$$

Linear (free) theory easy.

But non-linear terms in field equations for interacting fields require “renormalization.”

Three Basic Goals

1. Construct non-linear examples that are not exactly soluble.
2. Determine certain properties.
3. Develop new mathematical methods.

Tools: Exact, Algebraic, Analytic

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Tools: Exact, Algebraic, Analytic (Infinite-Dimensional Analysis)

Short Answer

CQFT Shows that QFT:

1. Is mathematical.
2. Results non-perturbative; but guided by PT.
3. New mathematical methods and structures.

- Examples of interacting fields in $d = 2, 3$.
- Partial answers in $d = 4$. (Still challenge!)
- Some non-existence results in $d \geq 5$. ($d=4$?)
- While new mathematical structures emerged, more are needed!

Method: Functional Integral → QM

- Justify (non-Gaussian) path integral

  “Wick-rotated problem”

- Quantize the classical the field one obtains

- New expansions (Cluster, Phase-cell, RG) to establish properties

- Inequalities give all insights

Framework: 5 Axiom Schemes

- Wightman-Gårding (Vacuum Expectation Values)
- Haag-Kastler (Local Algebras)
- Symanzik-Nelson (Markov fields)
- Osterwalder-Schrader (Euclidean Green's functions or Fields)
- Haag-Ruelle (W + Isolated particle)

Wightman Axioms

- Separable Hilbert space $\mathcal{H}$ of Quantum Theory
- Unitary, positive-energy, continuous repn. of Poincaré group. $U(t) = e^{itH}$ defines H.
- Covariant, local field operators φ(f)
- Vacuum vector Ω in $\mathcal{H}$.
- Complete: $\varphi(f_1)\bullet\bullet\bullet\varphi(f_n)\Omega$ span $\mathcal{H}$.
- Clustering: Vacuum unique

Wightman Axioms+

- Fundamental Bound for hermitian field

$$\pm \varphi(f) \leq (H + I) \|f\|$$

Implies that φ(f) uniquely determines a self-adjoint operator, also denoted φ(f). For ex.

$$A = e^{i\varphi(f)} \text{ is unitary.}$$

- Sometimes also assume growth of VEVs in number n of fields, as $n!^k$

Haag-Kastler Axioms

- Local algebras of "observables"
  - Bounded $A = e^{i\varphi(f)}$
- Poincaré covariant with positive-energy representation
- Vacuum state, clustering
- Other+: Haag duality, nuclearity, split, …

Symanzik-Nelson Axioms

- Markov Fields classical,probability

Insight.

But problems:

Difficult to check

Supplanted by OS

Osterwalder–Schrader Axioms

- Classical fields Φ. Functions

$$A = \Phi(x_1) \cdots \Phi(x_n)$$

- Euclidean covariant field.

$$\begin{aligned} g : x &= (t, \vec{x}) \mapsto gx \\ \Phi_g(x) &= \Phi(gx) \end{aligned}$$

- Euclid. Inv. State: $\omega(A) = \omega(A^g)$
- Clustering

Osterwalder–Schrader Axioms

- (Time) Reflection:

$$\vartheta : x = (t, \vec{x}) \mapsto (-t, \vec{x})$$
$$\Phi_{\Theta}(x) = \Phi(\vartheta x)^*$$

- Reflection Positivity: For all A localized at positive times,

$$0 \leq \omega(A^*_{\Theta} A)$$

- Classical

Osterwalder–Schrader Axioms

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$$\vartheta : x = (t, \vec{x}) \mapsto (-t, \vec{x})$$

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- Reflection Positivity: For all A localized at positive times,

$$0 \leq \omega(A^*_{\Theta} A) = \langle \hat{A}, \hat{A} \rangle_{\mathcal{H}}$$

- Classical Quantization

Osterwalder–Schrader Axioms

Consequence: Quantization!

$$\widehat{\Phi}(t,\vec{x}) = e^{-tH}\varphi(0,\vec{x})e^{tH} = \varphi(it,\vec{x})$$

classical quantum

Haag-Ruelle Axioms

- Relativistic Wightman Field
- Isolated 1-particle hyperboloid

Yields many-particle states and corresponding S-matrix elements, Haag-Ruelle scattering theory.

Basic Yoga in CQFT

- Cutoff Theory: Preserve what you can.
- Establish Reflection Positivity: Hilbert Space
- Establish Stability: $0 \leq H$
- Expand: Get control over individual terms
- Localization: Exponential Decay (Cluster Expn.)

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leq e^{-mr}||A||\,||B||$$

r is cluster separation; m bounds mass gap

- Remove cutoff.
- Establish desired properties of limiting theory.

Scalar Bosons

Relativistic, quantum field equation in dimension d=2

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} + m^2\right)\varphi(x) + \lambda V'(\varphi(x)) = 0\ ,\ \ x = (t, x_1)$$

Existence for V any positive polynomial, and $0 \leq \lambda$.

All axioms for V a positive polynomial, $0 \leq \lambda/m^2 \ll 1$,

Or $0 < (\lambda/m^2)^{-1} \ll 1$ in pure state. (Cluster Expansion)

Unique vacuum for small λ. Phase structure for large λ.

And in dimension d=3:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla_{\vec{x}}^2 + m^2\right)\varphi(x) + \lambda\varphi(x)^3 = 0\ ,\ \ x = (t, x_1, x_2)$$

Existence for all λ.

W, HK, OS axioms and unique vacuum for $0 \leq \lambda \leq \lambda_0$.

James Glimm and A.J. Many papers and work by collaborators and students.

Scalar Bosons

Non-existence in dimension d ≥ 5:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2_{\vec{x}} + m^2 \right) \varphi(x) + \lambda \varphi(x)^3 = 0$$

Lattice approximation, $0 < \lambda$, and unbroken symmetry.

➔Connected 4-Point function (coupling) zero.

(Glimm, Jaffe)

Fröhlich, Aizenman

Brydges-Fröhlich-Sokal

A Few Highlights for Scalar Theories

- d=2 positivity result of E. Nelson
- Logarithmic Sobolev inequality of Federbush and Gross.
- d=2 Haag-Kastler theory and weak infinite volume limit Glimm-AJ
- d=2 cluster expansions approach, Glimm-AJ-Spencer.
- Fröhlich general theory, Eckmann-Magnen-Seneor Borel summability
- Phase cell localization method of Glimm-AJ for positivity in d=3 (multi-scale analysis)
- Feldman, Osterwalder d=3 cluster expansion
- Osterwalder-Schrader: Euclidean Fields. Symanzik, Nelson: Markov fields
- Correlation inequality approaches of Guerra-Rosen-Simon, Brydges-Fröhlich-Sokal, Dobrushin
- Reorganized cluster expansion of Brydges, and HT Yau.
- Elaborations of phase cell localization by Magnen, Sénéor.
- Coupling constant bounds Symanzik, Glimm-AJ, Brydges-Fröhlich-Sokal, Fröhlich, Aizenman
- Establish symmetry breaking: Glimm-AJ-Spencer, Fröhlich, Fröhlich-Simon-Spencer
- Other phase transitions: Gawedzki, Fröhlich, Simon and Spencer
- Phase diagram structure of Imbrie (similar to Piragov-Sinai in statistical mechanics).
- 1/n expansion of Kupiainen
- Multi-scale cluster expansions of Dimock-Brydges-Hurd
- Multiscale-renormalization expansions of Rivasseau and collaborators iincluding Abdesselam
- Multiscale analysis of Gallavotti and collaborators
- Multiscale analysis of Gawedski and Kupiainen
- Positive temperature methods of Gérard, Jäkel, Robl
- Applications to random walks Brydges-Spencer, Brydges-Imbrie
- Curved backgrounds: Dimock, Kay, Hollands, AJ.-Ritter, Jäkel and collaborators
- Reworking multi-scale methods by AJ-Moser-Jäkel (in progress)

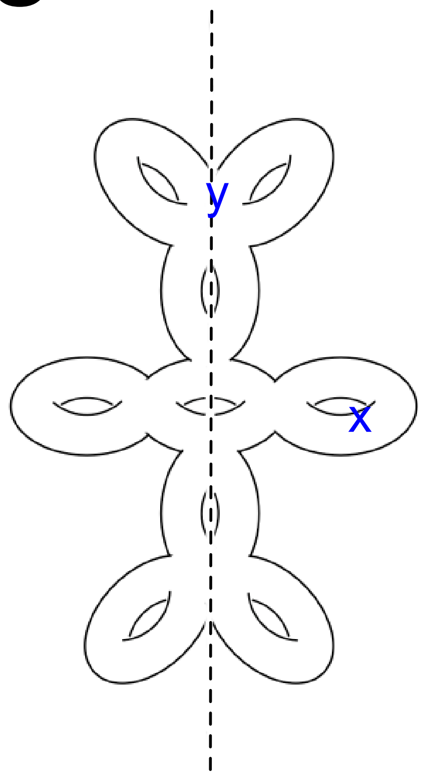
Standard Gaussian $d\mu(\Phi)$ (countably additive)

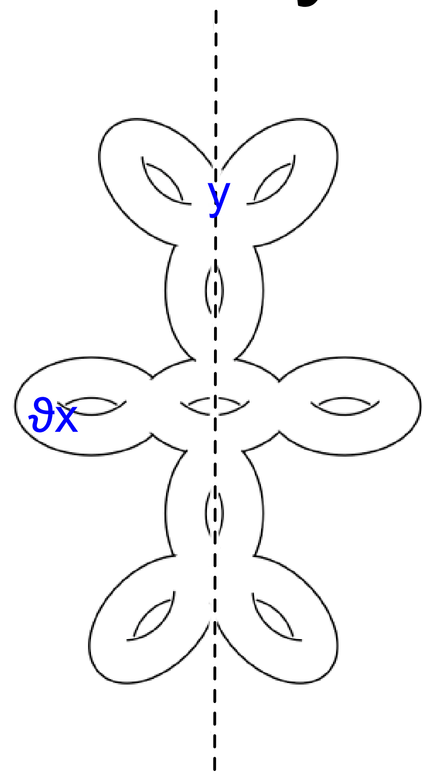
- $C=(-\Delta + m^2)^{-1}$ covariance; integral Kernel $C(x;y)$
- ϑC is RP ←→ $d\mu(\Phi)$ is RP
- Two proofs:
  - Explicit computation for ϑC
  - RP equivalent to Monotonicity of Boundary Cond.

$$C_D \leq C_N$$

Dirichlet/Neumann data on reflection hyperplane

Charges and D/N Boundary Cond.

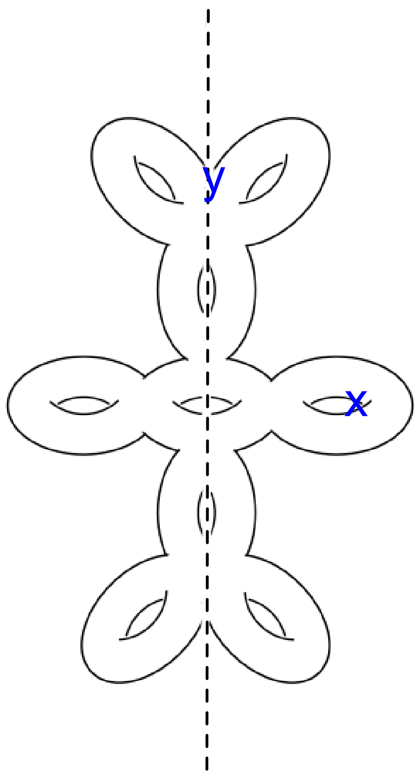


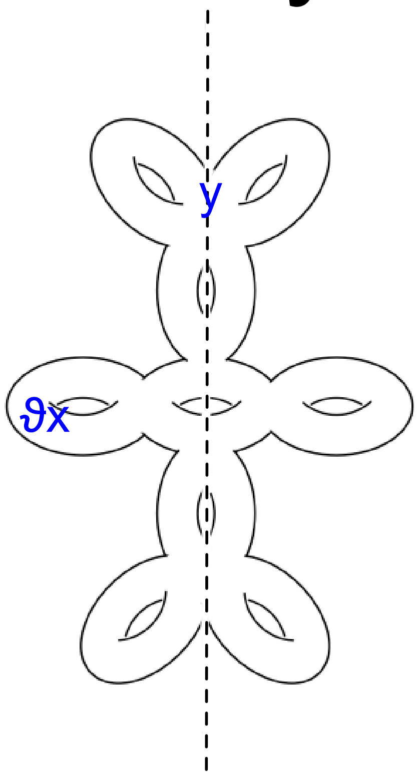


$C(x;y)$ $C(\vartheta x;y)$

$$
\begin{aligned}
C_D(x;y) &= C(x;y) - C(\vartheta x;y) \\
C_N(x;y) &= C(x;y) + C(\vartheta x;y)
\end{aligned}
$$

Charges and D/N Boundary Cond.

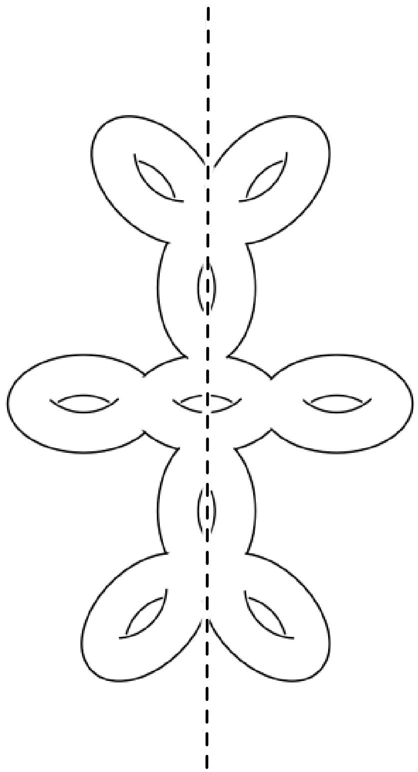




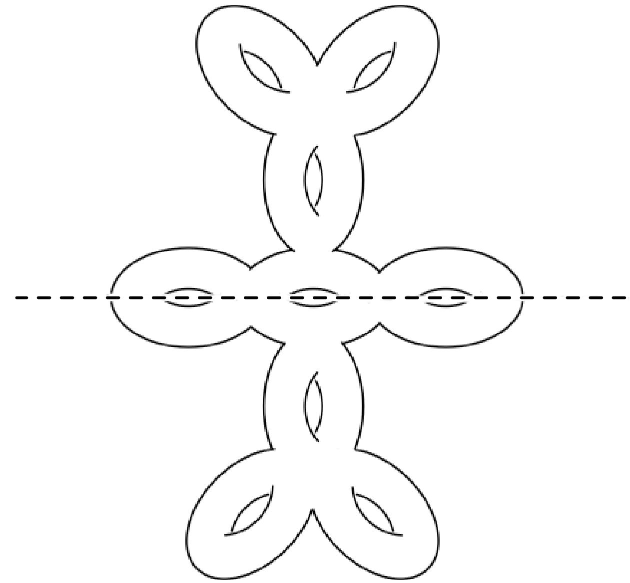
$$C_D \leq C_N$$

Triple Positivity: pointwise, operator, RP.
Classical PDE inequality for spectrum = RP.

Duality



$\mathcal{H}_1,\ \varphi_1(x),\ \Omega_1$



$\mathcal{H}_2,\ \varphi_2(\Gamma x),\ \Omega_2$

$$\langle A\varphi_1(x_1)\cdots\varphi_1(x_n)\rangle_{\Omega_1} = \langle A\varphi_2(\Gamma x_1)\cdots\varphi_2(\Gamma x_n)\rangle_{\Omega_2}$$

(Usual) Interacting Scalar Bosons

- Non-Gaussian distribution obtained as a perturbation of $d\mu_C$ by action,

$$d\mu(\Phi) = \frac{1}{Z}\, e^{-\mathcal{A}(\Phi)}\, d\mu_C(\Phi)$$

- RP if action is reflection invariant

Exact Non-Gaussian Analysis, I

- Small momentum exact bound Φ.

$$\mathcal{A}(\Phi) = \mathcal{A}(\Phi_\kappa) + \delta\mathcal{A}(\Phi_\kappa)$$

$$\mathcal{A}(\Phi_\kappa) \geq -\kappa^\epsilon \ , \ e^{-\mathcal{A}(\Phi_\kappa)} \leq e^{\kappa^\epsilon}$$

- Large momentum perturbation bound.

$$\Pr(|\delta\mathcal{A}(\Phi_\kappa)| > 1) \quad \leq \quad e^{-\kappa^\alpha}$$

- Product small. $\epsilon \quad < \quad \alpha$
- May work with effective or approx. action.

Exact Non-Gaussian Analysis, II (Phase-Cell Localization Method)

- Integrate range of momentum

$$M^n \leq |k| < M^{n+1}$$

- Estimates on one scale lead to estimates on next scale.
- Inductive scheme
- Multiscale Cluster expansion

Some Comments

- Cluster expansion methods depend heavily on mass gap
  - Uniform mass gap (in volume) enables infinite volume limit.
  - Add interaction at distance R, and using exponential decay to sum R out to infinity.

Exact n-Particle States (low energy)

Consider a polynomial $\lambda V(\varphi)$ model in d=2. Given n and $\varepsilon > 0$, one can obtain all states up to an energy

$E = (n+1)m(1-\varepsilon)$ by polynomials in fields of degree n applied to the vacuum vector Ω.

This result depends on a form of the GJS cluster expansion that should be reworked. It is the only cluster expansion giving exact low-energy one particle states, so one can isolate decay from multi-particle states, etc. Revised versions of cluster expansions have all been oriented toward n=0 (ground state) case.

Exact Non-Gaussian Analysis, III (RG Multi-scale Analysis)

- Integrate range of momentum

$$M^n \leq |k| < M^{n+1}$$

- Scale
- Identify interaction, up to error, with original one with modified coupling.
- Iterate

Example: Non-Hermitian C on R^n

$$C(t,\vec{x}) = \frac{1}{(2\pi)^d}\int e^{iEt+i\vec{k}\cdot\vec{x}}\widetilde{C}(E,\vec{k})dEd\vec{k}$$

$$\widetilde{C}(E,\vec{k}) = \frac{1}{(E+i\vec{k}\cdot\vec{v})^2+\vec{k}^2+m^2},\ \text{with } |\vec{v}|<1$$

- Time RP: One particle hamiltonian

$$h = \sqrt{\vec{k}^{\,2}+m^2} \pm \vec{k}\cdot\vec{v}$$

- Space RP:

$$\vec{x} \mapsto \vec{x} - \frac{2\vec{x}\cdot\vec{v}}{\vec{v}^2}\,\vec{v}$$

Complex Path Integral? (NO!)

$$C = K + iL\ ,\ 0 \leq K\ ,\ L = L^*$$

Yngvason criterion: Covariance C gives countably-additive complex Gaussian iff

$$K^{-1/2} L K^{-1/2} \in I_2$$

However: Gaussian Functional

$$\begin{aligned} S(f) &= \langle \Omega_0^{\mathrm{E}}, e^{i\Phi(f)} \Omega_0^{\mathrm{E}} \rangle_{\mathcal{E}} \\ &= e^{-\frac{1}{2}\langle \bar{f}, Cf \rangle_{L_2}} \end{aligned}$$

Some Open Question (scalar fields)

- Use new expansions to investigate mass gap, upper mass gap, etc., applicable in a variety of examples.
- E.g. define a useful topology on the space (landscape) of known models.
- In d=3, establish existence of isolated one-particle spectrum (upper mass gap) for weak coupling φ^4.
- In d=3, establish axioms, phase transition, mass gap and upper mass gap for strong coupling φ^4. (Develop new expansion techniques.)
- In d=4, nail down triviality of φ^4 (not asympotically free). Relevance to SM?

Model with Fermions

- Quantum (Majorana) Fermions on $\mathcal{H}$ satisfy

$$\{\psi(\vec{x},t),\psi(\vec{y},t)\}=\delta\left(\vec{x}-\vec{y}\right)$$

- Classical Fermions on $\mathcal{E}$ are Grassmann,

$$\{\Psi(x),\Psi(y)\}=0$$

- But (anti) time-ordered products of the quantum fermions are skew, so one can require

$$\langle\Psi(x)\Psi(y)\rangle_{\Omega^{\mathcal{E}}}=\langle A\psi(\vec{x},it)\psi(\vec{y},is)\rangle_{\Omega^{\mathcal{H}}}$$

Osterwalder-Schrader fields Require doubling.

SUSY, d=2, N=1, 2

- Non-Commutative Geometry: Cyclic Cocycle

$$\mathcal{C} = \{G_0, G_1, G_2, \ldots,\}\,, \qquad n^{1/2}|G_n| \to 0$$

$$\partial : \mathcal{C} \mapsto \mathcal{C} \qquad \partial^2 = 0$$

$$\partial\tau = 0\,, \quad a^2 = I\,, \quad \langle \tau, a\rangle = \langle \tau + \partial G, a\rangle$$

- JLO Pairing: Index Theory, Geometry

$$\langle \tau, a\rangle = \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Str}(a e^{-H+itda-t^2})\, dt$$

- J.-Lesniewski-Osterwalder; J.

Gauge Theory

Regularization: Lattice or Continuum

Two schools of thought.

- Wilson lattice action not only has gauge invariance, but it also satisfies RP. Requires loop observables, rather than fields.
  Erhard Seiler-Osterwalder

French school prefers continuum regularization.
- Magnen-Sénéor-Rivasseau

Simplest: Abelian Higgs, d=2

- Complex scalar φ^4 with U(1) Maxwell field
- Lattice approximation
  - Gauge appropriate for length scale
    - Lorentz gauge in ultraviolet, Unitary gauge in infrared
    - Sequence of gauges as length scale changes
  - Balaban-Imbrie-AJ,
- Uniform mass gap (Higgs effect)
  - B-Brydges-I-AJ

Yang-Mills, d=3,4

- Extensive work by T. Balaban (using lattice regularization).
- Review by Dimock, October 2011.

$$U : b \mapsto G \text{ semi-simple, compact}$$

$$\mathcal{A} = \frac{1}{g^2} \int_p \mathrm{tr}\left(I - U(\partial p)\right)$$

- Other work by Rivasseau and collaborators CMP 1993. Continuum cutoff.
- Both works have an infrared cutoff.

Dimension 4: the "Wild West"!

- Results: Bounds on effective action (finite volume) under RG flow (Balaban)
- Bounds on expectations of fields (Rivasseau)
- Need to do:
  - Need to complete bounds on effective action
  - Need to include Wilson loops (observables)
  - Need to take continuum limit (rather than uniform estimates)
  - Need to establish finite-volume gap, …..
  - Need to establish mass gap, axioms, …..

Dimension 4

Finite volume YM appears within reach.

All CQFT methods use a uniform mass gap to add interaction at a large distance. Thus one appears to need to solve the mass gap problem to take an infinite-volume limit.

But is it so? How much must one include in order to have a mathematical theory?

Does one need a theory of everything?

Modest Proposal

Renew our attempts to isolate and explain!!

Develop mathematical machine. But formulate more generally than for CQFT: not oriented toward solving one physics problem, but general applicability.

“Black box” like theory of PDO. Estimates ensure output.

Use the methods!

Modest Proposal

- Solve problems mentioned earlier in d=2,3.
- Construct quantum for SU(2) Yang-Mills in d=4.
- Construct QCD (Yang-Mills and Dirac).
- Solve YM confinement or “mass gap” problem.

Modest Proposal

- Each of these open questions are an order of magnitude more difficult than anything presently known.
- Each appears to require new mathematical structures as well as new physics insights.
- Also keep an eye on modifying our space-time, by building quantum theory into space itself.

To be Modest:

One need the attention of young, genius mathematical physicists! So send talented students our way.

I hope that this will provide a d=4 answer.