QUANTUM FIELD THEORY ON CURVED BACKGROUNDS VIA THE EUCLIDEAN FUNCTIONAL INTEGRAL

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Abstract. We give a mathematical construction of free Euclidean quantum fields on certain curved backgrounds. We focus on generalizing Osterwalder-Schrader quantization, as these methods have proved useful to establish estimates for interacting fields on flat space-times. In this picture, the role of physical positivity is played by positivity under reflection of imaginary time, so it is necessary to assume a certain Killing symmetry. It follows that Killing fields on spatial sections give rise to self-adjoint generators that are densely defined on the quantum field Hilbert space. We construct a Fock representation and Schrödinger interpretation using a method which involves localizing certain integrals over the full manifold to integrals over a codimension one submanifold. Further, we prove properties of the model under well-behaved perturbations of the space-time metric. As this class of space-times includes Euclidean anti-de Sitter space (AdS), we make contact with the AdS/CFT correspondence.

Contents

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Introduction

The present article presents a construction of a Euclidean quantum field theory on time-independent, curved backgrounds. Earlier work on field theories on curved space-time (Kay [36], Dimock [15], Bros et al. [8]) uses real-time/Lorentzian signature and algebraic techniques reminiscent of $\mathcal{P}(\varphi)_2$ theory from the Hamiltonian point of view [24]. In contrast, the present treatment uses the Euclidean functional integral [25] and Osterwalder-Schrader quantization [40, 41]. Experience with constructive field theory on $\mathbb{R}^d$ shows that the Euclidean functional integral provides a powerful tool, so it is interesting also to develop Euclidean functional integral methods for manifolds.

Euclidean methods are known to be useful in the study of black holes, and a standard strategy for studying black hole (BH) thermodynamics is to analytically continue time in the BH metric [11]. The present paper implies a mathematical construction of scalar fields on any static, Euclidean black hole background. The applicability of the Osterwalder-Schrader quantization procedure to curved space depends on unitarity of the time translation group and the time reflection map which we prove (theorem 1.7). The Osterwalder-Schrader construction has universal applicability; it contains the Euclidean functional integral associated with scalar boson fields, a generalization of the Berezin integral for fermions, and a further generalization for gauge fields [3]. It also appears valid for fields on Riemann surfaces [31], conformal field theory [20], and may be applicable to string theory.

Our paper has many relations with other work. Wald [46] studied metrics with Euclidean signature, although he treated the functional integral from a physical rather than a mathematical point of view. Brunetti et al [9] developed the algebraic approach (Haag-Kastler theory) for curved space-times and generalized the work of Dimock [14]. They describe covariant functors between the category of globally hyperbolic spacetimes with isometric embeddings, and the category of $*$-algebras with unital injective $*$-monomorphisms.

The examples studied in this paper—scalar quantum field theories on static space-times—have physical relevance. A first approximation to a full quantum theory (involving the gravitational field as well as scalar fields) arises from treating the sources of the gravitational field classically and independently of the dynamics of the quantized scalar fields [7]. The weakness of gravitational interactions, compared with elementary particle interactions of the standard model, leads one to believe that this approximation is reasonable. It exhibits nontrivial physical effects which are not present for the scalar field on a flat spacetime, such as the Hawking effect [28] or the Fulling-Unruh effect [45]. Density perturbations in the cosmic microwave background (CMB) are calculated using scalar field theory on certain curved backgrounds [37].

Witten [49] used quantum field theory on Euclidean anti-de Sitter space in the context of the AdS/CFT correspondence [27, 38]. Field theory on a compact d-dimensional manifold M is conjectured to be equivalent to super-gravity or string theory on a D-dimensional manifold Y where $D = 10$ or 11, corresponding to type IIB strings or M-theory. Here Y should have the form $X \times W$ (at least asymptotically), where W is a closed manifold and X has boundary M, and where the metric on X has a double pole at the boundary. Then relationships should hold which are analogous to those known between CFT on $M = S^d$ and super-gravity on a Euclidean version of

$$X \times W = \mathrm{AdS}_{d+1} \times S^{D-d-1}.$$

The latter super-gravity theory gives rise to effective scalars on Euclidean AdS_{d+1}, which the present paper places on a mathematical footing and relates to the well-established framework of constructive field theory.

In the final section we summarize our results and some open problems.

Notation and conventions. We use notation, wherever possible, compatible with standard references on relativity [48] and quantum field theory [25]. We use Latin indices $a, b = 0 \ldots d-1$ for spacetime indices, reserving Greek indices $\mu, \nu = 1 \ldots d-1$ for spatial directions. We include in our definition of 'Riemannian manifold' that the underlying topological space must be paracompact (every open cover has a locally finite open refinement) and connected. The notation $L^2(M)$ with no explicit mention of the measure is used when M is a C^∞ Riemannian manifold, and implicitly refers to the Riemannian volume measure on M, which we sometimes denote by $d$vol. The latter is associated to the standard inner product for differential forms on M, namely $\langle \omega, \eta \rangle_M = \int_M \omega \wedge *\eta$. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators $\mathcal{H} \to \mathcal{H}$. We also let $\mathcal{H}_M$ denote $L^2(M)$. Let $G = I(M) = \mathrm{Iso}(M)$ denote the isometry group, while $\mathfrak{K}$ is its Lie algebra, the Killing fields. For ψ a smooth map between manifolds, we use ψ^* to denote the pullback operator $(\psi^* f)(p) = f(\psi(p))$.

1. Reflection Positivity

1.1. **Static, hyperelliptic spacetimes.** Let M be a d-dimensional connected C^∞ Riemannian manifold, with a codimension one submanifold $B \subset M$, such that M is a disjoint union

$$M = \Omega_- \cup B \cup \Omega_+ \tag{1.1}$$

where $\Omega_\pm$ are open and $\partial\Omega_\pm = B$. Assume there is a smooth isometric involution θ on M so that

$$\theta\Omega_\pm = \Omega_\mp \quad \text{and} \quad \theta B = B \text{ pointwise}. \tag{1.2}$$

When conditions (1.1) and (1.2) are satisfied, we say that M is a *double*. One can always construct doubles by taking a Riemannian manifold M with $\partial M \neq \emptyset$ and gluing M to its mirror image $\overline{M}$, assuming the metric is sufficiently flat near the boundary.

A spacetime M is *stationary* if there is a nontrivial one-parameter group of isometries $\phi_t, t \in \mathbb{R}$. It follows that M is a foliation. One-parameter subgroups of the isometry group $\mathrm{Iso}(M)$ are in bijective correspondence with Killing vector fields. The map is given by associating a flow to its infinitesimal generator. For a Killing field ξ on a d-dimensional manifold, an *adapted coordinate system* consists of a chart $(t, x^\alpha), \alpha = 1, \ldots, d-1$ such that $\xi = \partial/\partial t$ and the metric has the form

$$ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + (dt + A_\alpha dx^\alpha)^2 F, \qquad F \equiv \xi_a \xi^a \tag{1.3}$$

In the neighborhood of any point, there always exists an adapted coordinate system.

A spacetime is said to be *static* if it is stationary and, in addition, there exists a hypersurface Σ which is orthogonal to all orbits $\{\phi_t(p) : t \in \mathbb{R}\}$. Assuming ξ is everywhere nonzero on Σ, then in a neighborhood of Σ, every point p will lie on a unique orbit. Choose coordinates (x^μ) on Σ, and label each point p in this neighborhood with the parameter t of the orbit which starts on Σ and ends at p, and the coordinates (x^μ) of the starting point. The metric components in this coordinate system are t-independent. The surface Σ_t, defined as the set of all points with coordinate t in the orthogonal direction, is the image of Σ under the isometry ϕ_t. Each Σ_t is also orthogonal to ξ.

The coordinate system (1.3) adapted to a *static* Killing field further simplifies to

$$ds^2 = F\, dt^2 + h_{\mu\nu}dx^\mu dx^\nu, \quad 1 \leq \mu, \nu \leq d-1 \tag{1.4}$$

where F and $h_{\mu\nu}$ are only functions of $x_1, \ldots, x_{d-1}$. It is clear from (1.4) that the natural time-translation and time-reflection maps are isometries for all points in the neighborhood where these coordinates are defined.

1.2. **Analytic continuation.** The Euclidean approach to quantum field theory on a curved background has advantages since elliptic operators are easier to deal with than hyperbolic operators. To obtain physically meaningful results one must presumably perform the analytic continuation back to real time. In general, Lorentzian spacetimes of interest may not be sections of 4-dimensional complex manifolds which also have Riemannian sections, and even if they do, the Riemannian section need not be unique. Thus, the general picture of extracting physics from the Euclidean approach is a difficult one where further investigation is needed.

Fortunately, for the class of spacetimes treated in the present paper (static spacetimes), the embedding within a complex 4-manifold with a Euclidean section is guaranteed, and in such a way that Einstein's equation is preserved.

Lemma 1.1 (Analytic continuation)**.** *Let g be a static solution to the vacuum Einstein equations, expressed in the form $g = -u^2 dt^2 + h_{ij} dx^i dx^j$. Then the Riemannian counterpart*

$$u^2 d\tau^2 + h_{ij} dx^i dx^j$$

also satisfies those equations.

For a proof we refer the reader to [12]. An identical argument applies to the family of complex tensor fields

$$g(\alpha) = -u^2(\alpha dt + \theta_j dx^j)^2 + h_{jk} dx^j dx^k \ , \tag{1.5}$$

so that if $\mathrm{Ric}(g) = \lambda g$ for some constant λ, then the complex tensor field $g(\alpha)$ satisfies the same equation for all $\alpha \in \mathbb{C}^*$.

1.3. **Time reflection.**

Definition 1.1 (Time reflection)**.** A **time reflection map** $\theta : M \to M$ is an isometric involution which fixes pointwise a smooth codimension-one hypersurface. This means that $\theta \in \mathrm{Iso}(M)$, $\theta^2 = 1$ and $\theta(b) = b$ for all $b \in B$.

We now discuss two important examples of time reflection.

Example 1.1 (Static manifolds)**.** Suppose there exists a globally defined, static Killing field ξ. Fix some hypersurface $B \subset M$ to which ξ is orthogonal. Define a global function $t : M \to \mathbb{R}$ by setting $t(b) = 0 \ \forall \ b \in B$, and otherwise define $t(p)$ to be the unique number t such that $\phi_t(b) = p$ for some $b \in B$, where $\{\phi_t\}$ is the one-parameter group of isometries determined by ξ. Finally, define θ to map a point $p \in M$ to the corresponding point on the same ξ-trajectory but with $t(\theta(p)) = -t(p)$.

Example 1.2 (Schottky doubles)**.** Suppose that Ω_+ is an oriented Riemannian manifold with connected boundary. Then Ω_+ may be glued to its classical mirror image Ω_- along the common boundary to yield an oriented, boundaryless manifold called the *Schottky double* [2]. Under suitable assumptions on the metric, the double also has a smooth Riemannian structure. In any such situation, θ may simply defined as the mirror reflection map from $\Omega_+ \to \Omega_-$.

The time-reflection map given by a hypersurface-orthogonal Killing field is not unique, but depends on a choice of the *initial hypersurface*. This choice is completely arbitrary and is equivalent to a global unitary transformation. In the present paper, we assume that the initial hypersurface has been fixed, and denote it by Σ. We also choose the coordinate t mentioned above so that $\Sigma = \{t = 0\}$. The initial hypersurface will be used to define time-zero fields. The coordinate t plays the role of *imaginary time* in quantum field theory, so "time reflection" $t \to -t$ is related to Hermitian conjugation of the evolution operator e^{-itH}.

1.4. **Fundamental assumptions.** Since the Schwartz space admits no natural generalization to the setting of Riemannian manifolds, we work with test function in $H^{-1}(M)$. This is a convenient choice for several reasons: the norm on $H^{-1}(M)$ is related in a simple way to the free covariance, and further, Dimock [16] has given an appealing proof of reflection positivity in Sobolev space.

The real Sobolev spaces $H^{\pm 1}(M)$, consisting of those distributions on M whose local coordinate expressions are in $H^{\pm 1}(\mathbb{R}^d)$, are equivalently described as completions of $C_c^\infty(M)$ with respect to

$$\|u\|_{\pm 1}^2 = \langle u, (-\Delta + m^2)^{\pm 1} u \rangle_M \tag{1.6}$$

for $m > 0$. These are real Hilbert spaces and they satisfy setwise inclusions

$$H^1(M) \subset L^2(M) \subset H^{-1}(M).$$

Also, $|\langle u, v\rangle| \leq \|u\|_1 \|v\|_{-1}$ so the inner product extends to a bilinear pairing of H^1 with H^{-1}. The spaces $H^{\pm 1}$ are dual with respect to this pairing, and $-\Delta + m^2$ is unitary from H^1 to H^{-1}.

Let $(Q, \mathcal{O}, \mu)$ be a probability measure space, let $\mathcal{M}$ be the space of random variables, which are simply measurable functions on Q, and let

$$\mathcal{E} = \mathcal{L}^2(Q, \mathcal{O}, d\mu).$$

Note that $\mathcal{E}$ is a Hilbert space distinct from $L^2(M)$. We denote its inner product by $\langle\ ,\ \rangle_{\mathcal{E}}$ to emphasize this distinction.

For any set $\mathcal{T}$, a *family of random variables indexed by* $\mathcal{T}$ is a map

$$\Phi : \mathcal{T} \to \mathcal{M}.$$

In quantum field theory, $\mathcal{T}$ is called the space of *test functions*, Q is the topological dual of $\mathcal{T}$ and is called the space of classical paths or simply *path space*, and integrals with respect to the measure $d\mu$ are called *path integrals* or *(Euclidean) functional integrals*.

Definition 1.2 (Fundamental assumptions)**.**

(1) The algebra generated by monomials of the form $\Phi(f_1)\dots\Phi(f_n)$ is dense in $\mathcal{E}$.
(2) $\mathcal{T} = H^{-1}(M)$, and
(3) The measure μ is a Gaussian probability measure with covariance $C = (-\Delta + m^2)^{-1}$.

Note that C is the integral operator whose kernel is the Green's function for the Laplacian. Since our Laplacian is defined with Euclidean signature, this is the Euclidean Green's function or *propagator*. Its properties are well understood on $\mathbb{R}^d$, where

$$C(x-y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{p^2 + m^2}\, e^{-ip\cdot(x-y)}\, dp\ . \tag{1.7}$$

This integral is elementary for $d = 1, 3$, while otherwise (1.7) has an explicit formula in terms of Hankel functions, but the short-distance singularity of $C(x-y)$ is always given by the singularity of the Coulomb potential and as $|x-y| \to \infty$, $C(x-y)$ has an exponential decay governed by the mass, and modified by a dimension-dependent polynomial.

Property 1 is a nondegeneracy condition for the family of random variables Φ. If Φ were, for example, a constant map then in all path integrals, the factor $\exp(-\text{Action})$ could be decoupled from the measure and no nontrivial behavior is possible. Property 1 also gives a natural dense domain on which to define unbounded operators.

Some of the motivations for the inclusion of Property 2 have been discussed previously. Here is another. Suppose we wish to prove that $\varphi(h)$ is a bounded perturbation of the free Hamiltonian H_0 for a scalar field on $\mathbb{R}^d$. The first-order perturbation is

$$-\langle \Omega_1, H_0 \Omega_1 \rangle = -\frac{1}{2} \int \frac{|\hat{h}(\vec{p})|^2}{\omega(\vec{p})^2} d\vec{p} \tag{1.8}$$

where we used $\Omega_1 = -H_0^{-1}\varphi(h)\Omega$. Existence of (1.8) is equivalent to $h \in H_{-1}(\mathbb{R}^d)$, so this is a natural condition for $\mathbb{R}^d$. It is therefore a good starting point for the generalization to curved manifolds.

Property 3 is simply the definition of free quantum field theory. It is equivalent to the following formula for the generating function of the measure:

$$\int e^{i\Phi(f)} d\mu = \exp\left(-\frac{1}{2} \langle f, \, Cf \rangle \right) .$$

Definition 1.3 (Standard Domain)**.** Let E denote the linear span of $\{e^{i\Phi(f)} : f \in \mathcal{T}\}$, which is dense in $\mathcal{E} = L^2(d\mu)$. Also let E_Ω denote the linear span of $\{e^{i\Phi(f)} : f \in \mathcal{T}, \operatorname{supp}(f) \subset \Omega\}$.

Definition 1.3 refers to subspaces of $\mathcal{E}$ generated by functions supported in an open set. This includes empty products, so $1 \in \mathcal{E}_\Omega$ for any Ω. Of particular importance for Euclidean field theory is the positive-time subspace

$$\mathcal{E}_+ \ := \ \mathcal{E}_{\Omega_+},$$

where the notation Ω_+ refers to the Schottky decomposition (1.1).

Lemma 1.2 (Sobolev Continuity)**.** *For the free covariance $C = (-\Delta + m^2)^{-1}$, the mapping*

$$\{f_1, \ldots, f_n\} \ \longmapsto \ A(\Phi) := \Phi(f_1) \ldots \Phi(f_n) \in \mathcal{E}$$

is a continuous function from $(H^{-1})^n \to \mathcal{E}$, where we take the product of the Sobolev topologies on $(H^{-1})^n$.

Proof. Since Φ is linear, it is sufficient to show that $\|A(\Phi)\|_{\mathcal{E}}$ is bounded by const. $\prod_i \|f_i\|_{-1}$. As a consequence of the Gaussian property of the measure $d\mu_C$, one needs only bound the linear case. But

$$\|\Phi(f)\|_{\mathcal{E}} = \left| \int \left(\Phi(f)\Phi(f) \right) \, d\mu_C \right|^{1/2} = \|f\|_{-1} \, . \tag{1.9}$$

□

The natural topology on $G = \operatorname{Iso}(M)$ is the compact-open topology, which (for this example) is the same as the topology of uniform convergence on compact sets. With this topology G is a Lie group acting on M as a Lie transformation group. This is used in Section 1.7 when we treat continuity with respect to ψ.

1.5. **Operator induced by a diffeomorphism.** We will consider the effect which diffeomorphisms of the underlying spacetime manifold have on the Hilbert space operators which arise in the quantization of a classical field theory. For $f \in C^\infty(M)$ and $\psi : M \to M$ an isometry, define

$$f^\psi \;\equiv\; (\psi^{-1})^* f = f \circ \psi^{-1}.$$

Under the assumption that $\det(d\psi) = 1$, the operation $f \to f^\psi$ extends to a bounded operator on $H^{\pm 1}(M)$ or $L^2(M)$ (see theorem 1.7).

Definition 1.4 (Operator Induced by a Diffeomorphism)**.** For a monomial $A(\Phi) = \Phi(f_1) \ldots \Phi(f_n) \in \mathcal{E}$, we define

$$\Gamma(\psi)A \;:=\; \Phi({f_1}^\psi) \ldots \Phi({f_n}^\psi)\,.$$

This extends linearly and by limits to the completion, which is $\mathcal{E}$. We refer to $\Gamma(\psi)$ as the operator on $\mathcal{E}$ *induced by the diffeomorphism* ψ.

We shall see later that $\Gamma(\iota)$ is bounded if $\iota : M \to M$ is an isometry.

Lemma 1.3. Γ *is a faithful (i.e. 1-1) representation of the group* $\mathrm{Diff}(M)$ *on* $\mathcal{E}$*.*

Proof. Consider $A(\Phi) = \Phi(a)$ as a representative element of E, and let $\psi, \eta \in \mathrm{Diff}(M)$. Then

$$\Gamma(\psi \circ \eta)A = \Phi((\psi \circ \eta)^{-1*}a) = \Phi(\psi^{-1*}\eta^{-1*}a) = \Gamma(\psi)\Gamma(\eta)A\,.$$

The remaining statements are obvious. □

The operators $\Gamma(\psi)$ are not necessarily bounded in the topology of $\mathcal{E}$. An exception is the case $\psi \in \mathrm{Iso}(M)$ which implies $\|\Gamma(\psi)\|_{\mathcal{E}} = 1$. Thus in general $\Gamma(\psi)$ must be considered as an unbounded operator defined on the dense domain $E \subset \mathcal{E}$, which we have dubbed the 'standard domain.'

For any open set $\Omega \subset M$, let

$$\mathrm{Diff}(M, \Omega) = \{\psi \in \mathrm{Diff}(M) : \psi(\Omega) \subset \Omega\},$$

Similarly define $\mathrm{Iso}(M, \Omega)$. These are not subgroups of $\mathrm{Diff}(M)$ but they are *semigroups*; i.e. they are closed under products but not inverses, and they do contain the identity.

Lemma 1.4. *The Laplacian determines the metric, i.e. if two Riemannian metrics have the same Laplacian, then they are equal.*

Corollary 1.5 (naturality $\Leftrightarrow$ isometry)**.** *For* $F : M \to N$ *a diffeomorphism,* $F^* \circ \Delta_N = \Delta_M \circ F^*$ *if and only if* F *is an isometry.*

The property $F^* \circ \Delta_N = \Delta_M \circ F^*$ is called *naturality.* Naturality with respect to local isometries is satisfied by all powers of the covariant derivative and any operators obtained from those by performing traces and symmetrizations; see [19] for details. In case $M = N$, we can write the condition from Corollary 1.5 as $[\Delta, F^*] = 0$.

Lemma 1.6 (Theorem III.6.5, [35])**.** *Let A be an unbounded operator on a dense domain D_A in Hilbert space $\mathcal{H}$. Let $R(z, A) = (A - z)^{-1}$, and let $B \in \mathcal{B}(\mathcal{H})$. If $[A, B] = 0$, then $[R(z, A), B] = 0$ for all $z \in \rho(A)$. Conversely, if $[R(\zeta_0, A), B] = 0$ for one single $\zeta_0 \in \rho(A)$, then $[A, B] = 0$.*

Theorem 1.7. *For $\psi : M \to M$ a diffeomorphism,*

$$\begin{aligned} \det(d\psi) = 1 \;&\Leftrightarrow\; (\psi^*)^\dagger = \psi^{-1*} \\ &\Leftrightarrow\; \psi^* \;\; \textit{is unitary on} \;\; L^2(M) \\ &\Leftrightarrow\; \psi \textit{ is volume-preserving} \end{aligned}$$

Further,

$$\begin{aligned} \psi \in \mathrm{Iso}(M) \;&\Leftrightarrow\; \Gamma(\psi) \textit{ is unitary on } \mathcal{E} \\ &\Leftrightarrow\; [\psi^*, \Delta] = 0 \\ &\Leftrightarrow\; [\psi^*, C] = 0 \end{aligned} \tag{1.10}$$

The equivalence (1.10) follows by Lemma 1.6. We remark that Γ is a faithful representation of $\mathrm{Diff}(M)$ on $\mathcal{E}$ which becomes a unitary representation when restricted to $G = \mathrm{Iso}(M)$.

Theorem 1.8 (presheaf property)**.** *Let $\psi : U \to V$ be a diffeomorphism, where U, V are open sets in M. Then*

$$\Gamma(\psi)E_U = E_V,$$

and if $\psi : M \to M$ preserves $\Omega \subset M$, then $\Gamma(\psi)$ preserves the corresponding subspace $\mathcal{E}_\Omega \subset \mathcal{E}$.

Proof. Elements of the form $e^{i\Phi(f)}$ with $\mathrm{supp}(f) \subset U$ generate E_U. By definition, $\Gamma(\psi)\left(e^{i\Phi(f)}\right) = e^{i\Phi(f\circ\psi^{-1})}$ and therefore,

$$\mathrm{supp}(f \circ \psi^{-1}) = \psi(\mathrm{supp}\, f) \subset V \;\Rightarrow\; \Gamma(\psi)\left(e^{i\Phi(f)}\right) \in E_V\,.$$

The remaining assertions are obvious. □

For maps $\psi : U \to V$ which are subset inclusions $U \subseteq V$, theorem 1.8 asserts that the association $U \to \mathcal{E}_U$ is a *presheaf*. It also follows from theorem 1.8 that the mappings $U \to \mathcal{E}_U$ and $\psi \to \Gamma(\psi)$ define a covariant functor from the category of open subsets of M with invertible, smooth maps between them into the category of Hilbert spaces and densely defined operators.

Applying Theorem 1.8 to $\Omega = \Omega_+$, we conclude that if $\psi(\Omega_+) \subset \Omega_+$ then $\Gamma(\psi)$ is positive-time invariant. Operators that do not preserve $\mathcal{E}_+$ do not have quantizations. However, the condition $\psi \in \mathrm{Iso}(M, \Omega_+)$ is not sufficient to ensure that $\Gamma(\psi)$ induces an operator on the quantum field Hilbert space; the condition for this is given later in Theorem 2.1. A more subtle application of Theorem 1.8, discussed in Section 2.4, is to the construction of quantization domains.

1.6. **Operator norm of** $\Gamma(\phi)$**.** Let $\phi \in \mathrm{Diff}(M)$ and $A(\Phi) = \Phi(f) \in \mathcal{E}$. If ϕ is an isometry, then $\Gamma(\phi)$ is *unitary*, and therefore $\|\Gamma(\phi)A\|_{\mathcal{E}} = \|A\|_{\mathcal{E}}$. Otherwise, $\|\Gamma(\phi)A\|^2_{\mathcal{E}} = \langle f^\phi, C(f^\phi)\rangle_M$, and then

$$\frac{\langle f^\phi, C(f^\phi)\rangle}{\langle f, Cf\rangle} = 1 + \frac{\langle \phi^*[C, (\phi^{-1})^*]\rangle_f}{\langle C\rangle_f} \,. \tag{1.11}$$

For a general monomial $A(\Phi) = \prod_{i=1}^n \Phi(f_i)$, we have

$$\|\Gamma(\phi)\| \geq \sup_{\{f_i\}} \frac{\| \prod_{i=1}^n \Phi({f_i}^\phi)\|_{\mathcal{E}}}{\| \prod_{i=1}^n \Phi(f_i)\|_{\mathcal{E}}} \,. \tag{1.12}$$

By equation (8.2.4) of [25], the numerator of (1.12) is given by the sum (over the $(2n)!!$ distinct pairings i_k) of the product:

$$(f^\phi_{i_1}, f^\phi_{i_2})_C \ldots (f^\phi_{i_{2n-1}}, f^\phi_{i_{2n}})_C$$

where $f_{n+i} = f_i$ by convention, and $(f, g)_C = \langle f, Cf\rangle_M$. The denominator of (1.12) is given by the very same sum, but with all of the ϕ's removed. If ϕ is an isometry, then $(f^\phi, g^\phi)_C = (f, g)_C$, so in this case we see immediately that $\|\Gamma(\phi)\|_{\mathcal{E}} = 1$ (our second proof of this).

We have not determined an explicit formula for the operator norm of $\Gamma(\phi)$ in the case that ϕ is a diffeomorphism but not an isometry. However, equation (1.11) shows that in the general case, the operator norm depends upon bounding the operator $\phi^*[C, (\phi^{-1})^*]$. Note

$$\|\phi^* f\|_{L^2(M)} = \int |f \circ \phi|^2 \, d\mathrm{vol}_M = \int |f|^2 \, J_\phi \; d\mathrm{vol}_M$$

where J_ϕ denotes the Jacobian determinant. Therefore ϕ^* is a bounded operator on $L^2(M)$ if and only if $\gamma := \sup_{x\in M} |J_\phi(x)|$ exists, in which case $\|\phi^*\|_{L^2} = \gamma$.

If $|J_\phi|$ satisfies uniform upper and lower bounds,

$$\exists\, c_1, c_2 > 0 \;\text{ s.t. }\; c_1 < \sup_{x\in M} |J_\phi(x)| < c_2. \tag{1.13}$$

then $\phi^*[C, (\phi^{-1})^*]$ is a bounded operator, and the above argument shows that $\Gamma(\phi)$ is bounded on $\mathcal{E}$. A further discussion may be found in Section 1.7.

If M is a compact manifold, then (assuming ϕ is a diffeomorphism) both $|J_\phi|$ and $|J_{\phi^{-1}}| = 1/|J_\phi|$ are continuous, and therefore uniformly bounded. This readily furnishes the required constants in (1.13), and implies that $\Gamma(\phi)$ is bounded for any diffeomorphism ϕ on a compact manifold.

1.7. **Strong continuity.** In this section we continue to study the map $\psi \longrightarrow \Gamma(\psi)$ which takes a diffeomorphism to an operator on $\mathcal{E} = L^2(d\mu)$. We take the compact-open topology on $\mathrm{Diff}(M)$. Note that the compact-open topology is equivalent for metric spaces to the topology of uniform convergence on compact sets. If we restrict attention to the subgroup $\mathrm{Iso}(M)$, then the operators $\Gamma(\psi)$ are elements of $\mathcal{B}(\mathcal{E})$, the space of bounded operators on $\mathcal{E}$. The latter has the weak, strong, and norm topologies. Standard theorems which guarantee existence of generators for one-parameter groups, such as Stone's theorem, are generally based on strong continuity, so we focus on the strong topology for $\mathcal{B}(\mathcal{E})$. One then has the following standard result:

Theorem 1.9. *Let $\{\psi_n\}$ be a sequence of orientation-preserving isometries which converge to ψ in the compact-open topology. Then $\Gamma(\psi_n) \to \Gamma(\psi)$ in the strong operator topology.*

An isometry ψ which is such that $\psi(\Omega_+) \subset \Omega_+$ may or may not give rise to an operator on $\mathcal{H}$. A sufficient condition is that $\Theta\Gamma(\psi)^\dagger\Theta$ preserves $\mathcal{E}_+$, where $\Theta = \Gamma(\theta)$. Under this condition, $\Gamma(\psi)$ defines a quantized operator $\hat{\Gamma}(\psi) : \mathcal{H} \to \mathcal{H}$. If this holds for all the elements of a one-parameter group of isometries ψ_t, then we have a one-parameter group of operators $t \to \hat{\Gamma}(\psi_t)$ on $\mathcal{H}$, and theorem 1.9 ensures the applicability of Stone's theorem. Similar results hold with"group" replaced by "semigroup," and we return to this in detail in a later section.

1.8. **Reflection positivity.**

Definition 1.5 (Reflection Positivity)**.** Let θ denote a time-reflection map in the sense of Definition 1.1, and let $\Theta = \Gamma(\theta)$ be the reflection on $\mathcal{E}$ induced by θ. A measure μ is said to be *reflection positive* if

$$\langle F, F\rangle_{\mathcal{E}} = \int \overline{\Theta(F)}\, F \; d\mu \geq 0 \quad (\forall\ F \in \mathcal{E}_+)\,. \tag{1.14}$$

An operator T on $L^2(M)$ is said to be reflection positive if

$$0 \leq \langle f, \theta T f\rangle_{L^2(M)}$$

whenever $\operatorname{supp} f \subset \Omega_+$.

Reflection positivity for the measure μ is equivalent to the following inequality for operators on $\mathcal{E} = L^2(d\mu)$:

$$0 \leq \Pi_+ \Theta \Pi_+$$

where $\Pi_+ : \mathcal{E} \to \mathcal{E}_+$ is the canonical projection.

A Gaussian measure (mean zero, covariance C) is reflection positive if and only if C is reflection positive in the operator sense. Yet another equivalent condition is that for any finite sequence $\{f_i\}$ of real functions supported in Ω_+, the $n \times n$ matrix $M_{ij} = \exp\langle f_i,\, \theta C f_j\rangle$ has no negative eigenvalues.

For Riemannian Schottky doubles, it is possible to obtain a simple proof of reflection positivity for the free theory, as was pointed out by [13]. This relies on the following lemma.

Lemma 1.10 (Gradient Identity)**.** *Let $M = \Omega_- \cup B \cup \Omega_+$ be static, and let $\hat{\mathbf{n}}$ denote the unit normal to B in Ω_+. Then*

$$\hat{\mathbf{n}} \cdot \operatorname{grad}(\theta f) = -\hat{\mathbf{n}} \cdot \operatorname{grad} f \quad on \quad B. \tag{1.15}$$

Proof. By assumption, B is orthogonal to all orbits $\{\phi_t(p) : t \in \mathbb{R}\}$. Since $\xi \neq 0$ on B, in a neighborhood of B every point p lies on a unique orbit. Choose coordinates (x^μ) on B, and label each point p in this neighborhood with the parameter t of the orbit which starts on Σ

and ends at p, and the coordinates (x^μ) of the starting point. Then clearly $\operatorname{grad}(\theta f)|_B = (-\partial_t f, \partial_\mu f)|_B$ and $\hat{\mathbf{n}} = (1, 0, \ldots, 0)$ in these coordinates, so the result is obvious. $\square$

Lemma 1.10 implies reflection positivity; let f be a real smooth function with support in Ω_+. Then

$$\langle \theta f, Cf \rangle = \int_B (\theta U \operatorname{grad} U - U \operatorname{grad} \theta U) \cdot \hat{\mathbf{n}}\, dS$$

where $U = (-\Delta_M + m^2)^{-1} f$ is the potential of f. By (1.15), this reduces to

$$\langle \theta f, Cf \rangle = 2 \int_B U \operatorname{grad} U \cdot \hat{\mathbf{n}}\, dS = \int_{\Omega_-} \left(g^{ab} (\partial_a U)(\partial_b U) + m^2 U^2 \right) dV \geq 0,$$

where the last equality is by Gauss' theorem.

More recently a different proof of reflection positivity on curved spaces was given by Dimock [16], based on work of Nelson [39]. We give a third proof later in this paper. We summarize the results as the following theorem.

Theorem 1.11 (Reflection Positivity, $m > 0$)**.** *Let $\Delta = \Delta_g$ be the Laplacian on a Riemannian manifold (M, g). Suppose that M is a Schottky double, with isometric involution θ as in eqns.* (1.1) *and* (1.2)*. The Gaussian measure $d\mu_C$ on $\mathcal{E}$ defined by the covariance $C = (-\Delta + m^2)^{-1}$ is reflection positive.*

It follows that $C = (-\Delta + m^2)^{-1}$ is reflection positive. In other words, for all $f \in C_c^\infty(M)$ we have

$$\int_M \left(\overline{\theta f} \, \frac{1}{-\Delta + m^2} f \right) d\mathrm{vol} \;\geq\; 0\,. \tag{1.16}$$

The inequality (1.16) is well-known for $M = \mathbb{R}^d$ (see Proposition 6.2.5, [25]) but not obvious for a general Riemannian manifold.

Reflection positivity for C may be reformulated as the statement

$$0 \leq \Pi_+ \theta C \Pi_+ = \frac{1}{2} \Pi_+ (C_N - C_D) \Pi_+$$

where $\Pi_+ : \mathcal{A}' \to \mathcal{E}_+$ is the projection onto $\mathcal{E}_+$. Here, C_D and C_N are defined by

$$\begin{aligned} C_D &= C - \theta C, \\ C_N &= C + \theta C. \end{aligned}$$

As the subscripts indicate, these two conditions correspond to Dirichlet or Neumann boundary conditions with respect to the boundary Σ. This representation was used by [23] to provide another proof of reflection positivity for classical boundary conditions on flat space.

2. Osterwalder-Schrader Quantization and the Feynman-Kac Formula

The Osterwalder-Schrader construction is a standard feature of quantum field theory. It begins with a "classical" Euclidean Hilbert space $\mathcal{E}$ and leads to the construction of a Hilbert Space $\mathcal{H} = \Pi\mathcal{E}_+$, which is the projection Π of the Euclidean space $\mathcal{E}_+$. It also yields a quantization map $T \mapsto \hat{T}$ from a classical operator T on $\mathcal{E}$ to a quantized operator $\hat{T}$ acting on $\mathcal{H}$. In this section we review this standard construction, dwelling on the quantization of bounded operators T on $\mathcal{E}$ that may yield a bounded or an unbounded quantization $\hat{T}$, as well as the quantization of an unbounded operator T on $\mathcal{E}$. We give a variation of the previously unpublished treatment in [30], adapted to curved space-time.

2.1. **The hilbert space.** Throughout the following section, we assume the existence of a static Killing field ξ and we choose a time-reflection map $\theta : M \to M$ associated to ξ, as described by Def. 1.1. We also assume a reflection-positive measure μ; see Def. 1.5 for the meaning and Section 1.8 for a discussion of the known proofs. Theorem 1.11 shows that the standard free field measure on a curved spacetime gives a reflection positive measure. We suspect the corresponding property is true for measures constructed from well-behaved interacting field theories; in particular, those which are perturbations of free theories.

Define a bilinear form (A, B) on $\mathcal{E}_+$ by

$$(A, B) = \langle \Theta A, B\rangle_{\mathcal{E}} \quad \text{for} \quad A, B \in \mathcal{E}_+ \,. \tag{2.1}$$

This form is sesquilinear,

$$(B, A) = \int \overline{\Theta B}\, A\; d\mu = \Big(\int B\, \overline{\Theta A}\, d\mu\Big)^* = \overline{(A, B)}\,. \tag{2.2}$$

The second equality in (2.2) used self-adjointness of Θ on $\mathcal{E}$, which follows from unitarity and $\Theta^2 = I$. An operator that is both unitary and self-adjoint is a $\mathbb{Z}_2$-grading. If θ is an order-two diffeomorphism but *not* an isometry, then Θ is non-unitary in which case Osterwalder-Schrader quantization is not possible. Therefore, it is essential that $\theta \in \mathrm{Iso}(M)$. The form (2.1) is degenerate, and has an infinite-dimensional kernel which we denote $\mathcal{N}$. Therefore (2.1) determines a nondegenerate inner product $\langle\, ,\, \rangle_{\mathcal{H}}$ on $\mathcal{E}_+/\mathcal{N}$, making the latter a pre-Hilbert space.

Definition 2.1 (Osterwalder-Schrader-Hilbert space)**.** Denote by $\mathcal{H}$ the completion of $\mathcal{E}_+/\mathcal{N}$, with inner product $\langle\, ,\, \rangle_{\mathcal{H}}$. Let $\Pi : \mathcal{E}_+ \to \mathcal{H}$ denote the contraction mapping that takes elements $A \in \mathcal{E}_+$ to $\hat{A} = \Pi A$. In other words there is an exact sequence,

$$0 \longrightarrow \mathcal{N} \xrightarrow{\text{incl.}} \mathcal{E}_+ \xrightarrow{\Pi} \mathcal{H} \longrightarrow 0 \,.$$

2.2. **Quantization of operators.** A sufficient condition for a densely defined operator T with domain $\mathcal{D}_0 \subset \mathcal{E}_+$ to have a quantization $\hat{T}$ is that T is null invariant. In other words $T : \mathcal{D}_0 \cap \mathcal{N} \to \mathcal{N}$. This ensures commutativity of the "Toeplitz-quantization" diagram

$$\begin{array}{ccc} \mathcal{E}_+ & \xrightarrow{T} & \mathcal{E}_+ \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathcal{H} & \xrightarrow{\hat{T}} & \mathcal{H} \end{array}$$

Thus we assume that T is a densely defined, closable operator on $\mathcal{E}$ with domain $\mathcal{D} \subset \mathcal{E}$. We also consider a subdomain $\mathcal{D}_0 \subset \mathcal{D} \cap \mathcal{E}_+$, and assume

$$T\colon \mathcal{D}_0 \to \mathcal{E}_+ \,, \qquad \mathcal{D}_0 \subset \mathcal{D}(T^+) \,, \quad \text{where} \quad T^+ = \Theta T^* \Theta \,, \qquad \text{and} \quad T^+\colon \mathcal{D}_0 \to \mathcal{E}_+ \,. \tag{2.3}$$

Theorem 2.1 (Condition for Quantization)**.** *Condition* (2.3) *ensures that T has a quantization $\hat{T}$ with domain $\mathcal{D}(\hat{T}) = \Pi\,(\mathcal{D}(T^+) \cap \mathcal{E}_+)$. If furthermore this domain is dense, then $\hat{T}^*$ is defined and also $\hat{T}$ has a closure.*

Proof. The first thing to ensure is that $\hat{T}$ is well-defined. Suppose $A \in \mathcal{N} \cap \mathcal{D}_0$. Let $B \in \mathcal{E}_+$ range over a set of vectors in the domain of $\Theta T^* \Theta$ such that the image of this set under Π is dense in $\mathcal{H}$. Then

$$\begin{aligned} 0 &= \langle (\Theta T^* \Theta B)\,\hat{}\,, \hat{A} \rangle_{\mathcal{H}} = \langle T^* \Theta B, A \rangle_{\mathcal{E}} \\ &= \langle \Theta B, TA \rangle_{\mathcal{E}} = \langle \hat{B}, \widehat{TA} \rangle_{\mathcal{H}} \,. \end{aligned}$$

Thus $TA \in \mathcal{N}$, as claimed. □

It is worth noting that Theorem 2.1 can be expressed in a single commutative diagram, and is most easily remembered that way. Precisely, Theorem 2.1 states that if the dotted arrow is well-defined, then so are the two solid arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{incl.}} & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \vdots\,{\scriptstyle \Theta T^* \Theta} & & \downarrow{\scriptstyle \hat{T}} & & \\ 0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{incl.}} & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0 \end{array}$$

Moreover, the two horizontal rows are exact sequences. We will often need to calculate the adjoint of an operator on $\mathcal{H}$. The following Lemma is immediate.

Lemma 2.2 (Adjoints)**.** *The adjoint of $\hat{T}$ on $\mathcal{H}$ is $\hat{T}^\dagger = \widehat{\Theta T^* \Theta}$, where T^* denotes the adjoint of T on $\mathcal{E}$.*

We now discuss some important examples of operators satisfying the hypotheses of Theorem 2.1.

Theorem 2.3 (Self-adjoint operators on $\mathcal{H}$)**.** *Let U be unitary on $\mathcal{E}$, and $U(\mathcal{E}_+) \subset \mathcal{E}_+$. If $U^{-1}\Theta = \Theta U$ then U admits a quantization $\hat{U}$ and $\hat{U}$ is self-adjoint. (We do not assume U^{-1} preserves $\mathcal{E}_+$).*

Proof. $\Theta U^* \Theta = \Theta^2 U = U$ which preserves $\mathcal{E}_+$, so by Theorem 2.1, U has a quantization $\hat{U}$. Self-adjointness of $\hat{U}$ follows immediately from Lemma 2.2. □

The next theorem, which constructs unitaries on $\mathcal{H}$ from those on $\mathcal{E}$, is in a sense harder because it requires that both U and U^{-1} preserve $\mathcal{E}_+$.

Theorem 2.4 (Unitaries on $\mathcal{H}$)**.** *Let U be unitary on $\mathcal{E}$, and $U^{\pm 1}(\mathcal{E}_+) \subset \mathcal{E}_+$. If $[U, \Theta] = 0$ then U admits a quantization $\hat{U}$ and $\hat{U}$ is unitary.*

Proof. $\Theta U^* \Theta = U^* = U^{-1}$ which preserves $\mathcal{E}_+$ by assumption, so U has a quantization defined by $\hat{U}\hat{A} = \widehat{UA}$. Also, $\Theta (U^{-1})^* \Theta = U$ preserves $\mathcal{E}_+$, so U^{-1} also has a quantization. Obviously, the quantization of U^{-1} is the inverse of $\hat{U}$. Lemma 2.2 implies that the adjoint of $\hat{U}$ is the quantization of $\Theta U^* \Theta = U^* = U^{-1}$. □

Examples of operators satisfying the conditions of Theorems 2.3 and 2.4 come naturally from isometries on M with special properties. We now discuss two classes of isometries, which give rise to self-adjoint and unitary operators as above.

Example 2.1 (Reflection-Invariant Isometries)**.** A **reflection invariant** isometry is an element $\psi \in I(M, \Omega_+)$ that commutes with time-reflection. It follows that $[\Gamma(\psi), \Theta] = 0$ and $\Gamma(\psi^{\pm 1})\mathcal{E}_+ \subset \mathcal{E}_+$, hence by Theorem 2.4, $\hat{\Gamma}(\psi) : \mathcal{H} \to \mathcal{H}$ is unitary.

Example 2.2 (Reflected Isometries)**.** Let $\psi \in \mathrm{Iso}(M, \Omega_+)$. We say ψ is a **reflected isometry** if

$$\psi^{-1} \circ \theta = \theta \circ \psi\,. \tag{2.4}$$

Then Theorem 2.3 implies that $\hat{\Gamma}(\psi) : \mathcal{H} \to \mathcal{H}$ exists and is self-adjoint. If ψ satisfies (2.4) then so does ψ^{-1}; hence ψ^{-1} is also a reflected isometry if $\psi^{-1}(\mathcal{E}_+) \subset \mathcal{E}_+$. Then $\Gamma(\psi^{-1})$ has a quantization and $\hat{\Gamma}(\psi^{-1})$ is the inverse of $\hat{\Gamma}(\psi)$.

2.3. **Contraction property.** Section 2.2 discussed the sense in which a densely defined operator T on $\mathcal{E}$ may have a *quantization*, i.e. an operator $\hat{T}$ with dense domain $\hat{\mathcal{D}}_T \subset \mathcal{H}$ which satisfies the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;T\;} & \mathcal{E} \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathcal{H} & \xrightarrow{\;\hat{T}\;} & \mathcal{H} \end{array}$$

with the understanding that all operators are defined on dense domains in the indicated spaces.

In the present section we show that the mapping $T \mapsto \hat{T}$, defined on the subset of $\mathcal{B}(\mathcal{E})$ given by the bounded quantizable operators, is a contraction mapping. The method is called a multiple reflection bound, and is due to Glimm and Jaffe.

Theorem 2.5 (Contraction Property)**.** *Let T be a bounded transformation on $\mathcal{E}$ such that T and $\Theta T^*\Theta$ each preserve $\mathcal{E}_+$. Then*

$$\|\hat{T}\|_{\mathcal{H}} \leq \|T\|_{\mathcal{E}}\,. \tag{2.5}$$

Proof. Define $S = \hat{T}^*\hat{T} = \widehat{\Theta T^* \Theta}\hat{T}$. Then for $A \in \mathcal{E}_+$,

$$\begin{aligned} \|\hat{T}\hat{A}\|_{\mathcal{H}} &= \left\langle \hat{A}, S\hat{A} \right\rangle_{\mathcal{H}}^{1/2} \\ &\leq \|\hat{A}\|_{\mathcal{H}}^{1/2} \left\langle \hat{A}, S^2\hat{A} \right\rangle_{\mathcal{H}}^{1/4} \\ &\leq \|\hat{A}\|_{\mathcal{H}}^{1-2^{-n}} \left\langle \hat{A}, S^{2^n}\hat{A} \right\rangle_{\mathcal{H}}^{2^{-n-1}} \\ &= \|\hat{A}\|_{\mathcal{H}}^{1-2^{-n}} \left\langle \Theta A, (\Theta T^*\Theta T)^{2^n} A \right\rangle_{\mathcal{E}}^{2^{-n-1}}\,. \end{aligned}$$

We use the fact that $\Theta T^* \Theta T$ maps $\mathcal{E}_+$ to $\mathcal{E}_+$. Then as $\|\Theta\|_\mathcal{E} = 1$, and $\| \left(\Theta T^* \Theta T\right)^{2^n} \|_\mathcal{E} \leq \|T\|_\mathcal{E}^{2^{n+1}}$, it follows that

$$\begin{aligned} \|\hat{T}\hat{A}\|_\mathcal{H} &\leq \left((\|T\|_\mathcal{E})^{2^{n+1}} \right)^{2^{-n-1}} \|\hat{A}\|_\mathcal{H}^{1-2^{-n}} \, \|A\|_\mathcal{E}^{2^{-n}} \\ &\leq \|T\|_\mathcal{E} \|\hat{A}\|_\mathcal{H}^{1-2^{-n}} \, \|A\|_\mathcal{E}^{2^{-n}} \, . \end{aligned}$$

Taking the limit $n \to \infty$ we have

$$\|\hat{T}\hat{A}\|_\mathcal{H} \leq \|T\|_\mathcal{E} \|\hat{A}\|_\mathcal{H} \, , \tag{2.6}$$

from which we infer (2.5). ☐

2.4. **Quantization domains.** As the map Π is a contraction, any dense linear subspace of $\mathcal{E}_0 \subset \mathcal{E}_+$ which is dense in $\mathcal{E}_+$ projects to a dense subspace $\Pi\mathcal{E}_0$ of $\mathcal{H}$.

Definition 2.2 (Quantization Domain)**.** A **quantization domain** is a subspace $\Omega \subset \Omega_+$ with the property that $\Pi\left(\mathcal{E}_\Omega\right)$ is dense in $\mathcal{H}$.

Perhaps the simplest quantization domain is a half-space lying at times greater than $T > 0$,

$$\mathcal{O}_{+,T} = \left\{ x \in \mathbb{R}^d : x_0 > T \right\} .$$

Let $\mathcal{D}_{+,T} = E_{\mathcal{O}_{+,T}} = \Gamma(\psi_T) E_+$ where $\psi_T(x,t) = (x, t+T)$; then $\Pi(\mathcal{D}_{+,T})$ is dense in $\mathcal{H}$. Theorem 2.6 generalizes this class of examples to curved spacetimes, and also allows one to replace the simple half-space $\mathcal{O}_{+,T}$ with a more general connected subset of Ω_+. That $\mathcal{O}_{+,T}$ is a quantization domain follows from Theorem 2.6 as an easy special case.

Theorem 2.6. *Suppose $\psi \in \mathrm{Iso}(M, \Omega_+)$ so that $\psi(\Omega_+) \equiv U \subset \Omega_+$. If $[\Gamma(\psi), \Theta] = 0$ or $\Gamma(\psi)\Theta = \Theta\Gamma(\psi^{-1})$ then U is a quantization domain.*

Proof. By Theorem 1.8 (the presheaf property), we have

$$E_U = \Gamma(\psi) E_+ \, . \tag{2.7}$$

Let $\hat{C} \in \mathcal{H}$ be orthogonal to every vector $\hat{A} \in \Pi(\mathcal{E}_U)$. Choose representatives $B \in E_+$ and let $A := \Gamma(\psi)B$. By eqn. (2.7), $A \in E_U$. Then

$$0 = \langle \hat{C}, \hat{A} \rangle_\mathcal{H} = \langle \hat{C}, \Pi(\Gamma(\psi)B) \rangle_\mathcal{H} = \langle \Theta C, \Gamma(\psi) B \rangle_\mathcal{E} \, .$$

Since $\Gamma(\psi)^{-1}$ is unitary on $\mathcal{E}$, apply it to the inner product to yield

$$\langle \Gamma(\psi^{-1}) \Theta C, B \rangle_\mathcal{E} = 0 \quad (\forall \ B \in E_+).$$

Therefore $\Gamma(\psi^{-1})\Theta C$ is orthogonal (in $\mathcal{E}$) to the entire subspace $\mathcal{E}_+$.

First, suppose that $[\Gamma(\psi^{-1}), \Theta] = 0$. Then we infer

$$0 = \langle \Theta \Gamma(\psi^{-1}) C, B \rangle_\mathcal{E} = \langle \hat{\Gamma}(\psi^{-1}) \hat{C}, \hat{B} \rangle_\mathcal{H} \quad (\forall \, \hat{B} \in \Pi(E_+)),$$

i.e. $\hat{C} \in \ker\left(\hat{\Gamma}(\psi^{-1}) \right)$. Therefore,

$$(\Pi(\mathcal{E}_U))^\perp = \ker\left(\hat{\Gamma}(\psi^{-1}) \right) \, . \tag{2.8}$$

Since $[\Gamma(\psi^{-1}), \Theta] = 0$ then Theorem 2.4 implies that $\hat{\Gamma}(\psi)$ is unitary, hence the kernel of $\hat{\Gamma}(\psi^{-1})$ is trivial and $\Pi(\mathcal{E}_U)$ is dense in $\mathcal{H}$. We have thus completed the proof in this case.

Now, assume that $\Gamma(\psi)\Theta = \Theta\Gamma(\psi^{-1})$. Example 2.2 implies that $\hat{\Gamma}(\psi)$ exists and is self-adjoint on $\mathcal{H}$, and moreover (by the same argument used above),

$$(\Pi(\mathcal{E}_U))^\perp = \ker\left(\hat{\Gamma}(\psi)\right) .$$

If $\psi = \psi_t$ where $\{\psi_s\}$ is a one-parameter group of isometries, and if $\hat{\Gamma}(\psi_t)$ is a strongly continuous semigroup then by Stone's theorem, $\hat{\Gamma}(\psi_t) = e^{-tK}$ for K self-adjoint. Since e^{-tK} clearly has zero kernel, the proof is also complete in the second case. □

A particularly interesting open question (to which we have provided only a partial solution) is: which open subsets $U \subset \Omega_+$ are quantization domains?

2.5. **Construction of the hamiltonian and ground state.** In this subsection, we use the machinery developed in the preceding sections to construct the Hamiltonian and ground state of free, Euclidean quantum field theory on a curved spacetime. The flows of the time-translation Killing field $\xi = \partial/\partial t$ give a one-parameter group of isometries $\phi_t : M \to M$, from which we define $U(t) = \Gamma(\phi_t)$.

Theorem 2.7 (Time-translation Semigroup)**.** *Let $\xi = \partial/\partial t$ be the time-translation Killing field on the static spacetime M. Let the associated one-parameter group of isometries be denoted $\phi_t : M \to M$. For $t \geq 0$, $U(t) = \Gamma(\phi_t)$ has a quantization, which we denote $R(t)$. Further, $R(t)$ is a well-defined one-parameter family of self-adjoint operators on $\mathcal{H}$ satisfying the semigroup law.*

Proof. Theorem 1.7 implies that $U(t)$ is unitary on $\mathcal{E}$, and it is clearly a one-parameter group. Also,

$$\phi_t \circ \theta = \theta \circ \phi_{-t}$$

and $U(t)\mathcal{E}_+ \subset \mathcal{E}_+$ for $t \geq 0$, so this is a *reflected isometry*; see Example 2.2. Theorem 2.3 implies $R(t) = \hat{U}(t)$ is a self-adjoint transformation on $\mathcal{H}$ for $t \geq 0$, which satisfies the group law

$$R(t)R(s) = R(t+s) \quad \text{for} \quad t, s \geq 0$$

wherever it is defined. □

Theorem 2.8 (Hamiltonian and Ground State)**.** *$R(t)$ is a strongly continuous contraction semigroup, which leaves invariant the vector $\Omega_0 = \hat{1}$. There exists a densely defined, positive, self-adjoint operator H such that*

$$R(t) = \exp(-tH), \quad \textit{and} \quad H\Omega_0 = 0.$$

Thus Ω_0 is a quantum-mechanical ground state.

Proof. It is immediate that $R(t)\Omega_0 = \Omega_0$, and we have already shown the semigroup property

$$R(t)R(s) = R(t+s) \quad \text{for} \quad t, s \geq 0\,.$$

The contraction property $R(t) \leq I$ follows from a *multiple reflection bound.* The single reflection is

$$\begin{aligned}\|R(t)\hat{A}\| &= \langle R(t)\hat{A}, R(t)\hat{A}\rangle_{\mathcal{H}}^{1/2} = \langle \hat{A}, R(2t)\hat{A}\rangle^{1/2} \\ &\leq \|\hat{A}\|^{1/2}\|R(2t)\hat{A}\|^{1/2} .\end{aligned}$$

After n reflections,

$$\|R(t)\hat{A}\| \leq \|\hat{A}\|^{1-2^{-n}}\|R(2^n t)\hat{A}\|^{2^{-n}} . \tag{2.9}$$

Unitarity of Θ and the Schwartz inequality for $\mathcal{E}$ yields immediately that $\|\hat{A}\|_{\mathcal{H}} \leq \|A\|_{\mathcal{E}}$, i.e. the quantization map $\Pi : \mathcal{E} \to \mathcal{H}$ is a contraction mapping. Therefore

$$\|R(2^n t)\hat{A}\|_{\mathcal{H}} = \|(U(2^n t)A)\hat{}\,\|_{\mathcal{H}} \leq \|U(2^n t)A\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}} .$$

This implies that we can replace the second factor in (2.9) by $\|A\|_{\mathcal{E}}^{2^{-n}}$, and let $n \to \infty$. This shows for $A \in \mathcal{E}_+$, that

$$\|R(t)\hat{A}\|_{\mathcal{H}} \leq \|\hat{A}\|_{\mathcal{H}} .$$

Thus $R(t)$ is a contraction, a property equivalently expressed as $0 \leq R(t) \leq I$.

We now establish strong continuity of the semigroup $R(t)$, which is equivalent by classical functional analysis to the statement that $\exists\, \delta > 0, M \geq 1$ and a dense subspace $D \subset \mathcal{H}$ such that $\|R(t)\| \leq M$ for all $t \in [0, \delta]$ and $\lim_{t\downarrow 0} R(t)x = x$ for all $x \in D$.

Since $R(t)$ is a contraction, the uniform bound just mentioned holds. It is therefore sufficient to prove convergence on vectors $A \in \mathcal{D}_+$ of the form $A = \prod_i \Phi(f_i)$, with $f_j \in \mathcal{S}_+$. For such a vector,

$$\begin{aligned}\|R(t)\hat{A} - \hat{A}\|_{\mathcal{H}}^2 &\leq \|U(t)A - A\|_{\mathcal{E}}^2 \\ &= 2\|\prod_i \Phi(f_i)\|_{\mathcal{E}}^2 - 2\Re\Big\langle \prod_i \Phi(f_i), \prod_j \Phi({f_j}^t)\Big\rangle_{\mathcal{E}} ,\end{aligned}$$

Then as a consequence of continuity in t, it follows that $\|R(t)\hat{A} - \hat{A}\|_{\mathcal{H}} \to 0$ as $t \to 0$. We have used the fact that

$$\int \Phi_{t_1}(f_1)\Phi_{t_2}(f_2)\cdots\Phi_{t_n}(f_n)d\mu \tag{2.10}$$

is a continuous function of $t_1, t_2, \ldots, t_n$.

¿From Stone's theorem [42, 44], we infer that $R(t) = \exp(-tH)$, with H a densely defined self-adjoint operator on $\mathcal{H}$. Positivity of H follows from $R(t) \leq I$, and the identification of Ω_0 as the ground state of H follows from $R(t)\Omega_0 = \Omega_0$. □

The operator H is also the quantum mechanical generator (in the Euclidean picture) of translations in the coordinate direction specified by the Killing field ξ. Whenever that direction is time, we refer to H as the *Hamiltonian.* Note that there is not necessarily a unique ground state, but there is a canonical one selected by $\Omega_0 = \hat{1}$.

2.6. **Feynman-Kac theorem.**

Theorem 2.9 (Feynman-Kac)**.** *Let $\hat{A}, \hat{B} \in \mathcal{H}$, and let H be the Hamiltonian constructed in Theorem 2.8. Each matrix element of the heat kernel e^{-tH} is given by a Euclidean functional integral,*

$$\langle \hat{A}, e^{-tH} \hat{B} \rangle_{\mathcal{H}} = \int \overline{\Theta A}\, U(t) B \; d\mu(\Phi) \,. \tag{2.11}$$

The right-hand side of (2.11) is the Euclidean path integral [17] of quantum field theory. Mark Kac' method [33, 34] for calculating the distribution of the integral $\int_0^T v(X_t)dt$, where v is a function defined on the state space of a Markov process X, forms a rigorous version of Feynman's work, valid at imaginary time.

In the present setup, (2.11) requires no proof, since the functional integral on the right-hand side is how we defined the matrix element on the left-hand side. However, some work is required (even for flat spacetime, $M = \mathbb{R}^d$) to see that the Hilbert space and Hamiltonian given by this procedure take the usual form arising in physics. This is true, and was carried out for $\mathbb{R}^d$ by Osterwalder and Schrader [40] and summarized in [25, Ch. 6].

Since H is positive and self-adjoint, the heat kernels can be analytically continued $t \to it$. We therefore define the *Schrödinger group* acting on $\mathcal{H}$ to be the unitary group

$$R(it) = e^{-itH} \,.$$

Given a time-zero field operator, action of the Schrödinger group then defines the corresponding real-time field.

For flat spacetimes in $d \leq 3$ it is known [25] that Theorem 2.9 has a generalization to non-Gaussian integrals, i.e. interacting quantum field theories:

$$\begin{aligned} \langle \hat{A}, e^{-tH_V} \hat{B} \rangle_{\mathcal{H}} &= \Big\langle \Theta A, \exp\Big(-\int_0^t dt' \int d\vec{x}\; V(\Phi(\vec{x}, t'))\Big) B_t \Big\rangle_{\mathcal{E}} \\ &= \int \overline{\Theta A}\, e^{-S^V_{0,t}}\, B_t \; d\mu(\Phi) \,. \end{aligned} \tag{2.12}$$

We hope to prove a similar formula for curved spacetimes in future work.

2.7. **Quantization of subgroups of the isometry group.** We now continue the discussion, began in Theorem 2.1, of operators on $\mathcal{E}$ which possess well-defined quantizations. Physical intuition dictates that after quantization, a spacetime symmetry with p parameters should be represented by a unitary representation of the appropriate p-dimensional Lie group acting on $\mathcal{H}$. Construction of these additional unitary representations is possible, and due to the intrinsic interest of such a construction, we give further details.

Example 2.1 introduced a class of isometries with well-defined quantizations, called reflection-invariant. We now discuss a subclass of these, the purely spatial isometries. Let (M, g) be our static, Euclidean spacetime with distinguished time-slice Σ and static Killing vector ξ. Assume that Σ has the induced metric. Consider the natural inclusion of the isometry group of Σ into that of M defined by the following procedure.

Let $G = \mathrm{Iso}(M)$, and define the centralizer of θ in the usual way,

$$C_G(\theta) \equiv \{\phi \in G \mid \phi \circ \theta = \theta \circ \phi\}\,.$$

Definition 2.3. An isometry $i : \Sigma \to \Sigma$ extends to $\widetilde{i} \in G$ as follows. Define $\widetilde{i}(p)$ by flowing backward in time along an integral curve of ξ until you are on Σ, then apply i, and then flow forward again by the same amount. Thus $\widetilde{i}(p)$ and p are elements of the same constant-t hypersurface. The isometries $\widetilde{i}$ constructed in this way are called **purely spatial** and form a subgroup $G_{\mathrm{space}} \subset G$.

The classic constructions of interacting quantum field theories, such as [32], often assume that spacetime has the topology of a cylinder $S^1 \times \mathbb{R}$ and the standard metric. In this case, G_{space} is that subgroup of the isometries of the cylinder corresponding to rotations around the central axis.

Note that

$$G_{\mathrm{space}} \subset \mathrm{Iso}(M, \Omega_+) \cap C_G(\theta), \tag{2.13}$$

and although the latter is not a subgroup, this is a compact way of expressing that the elements are positive-time invariant and null-invariant.

Since G_{space} is a Lie subgroup of G, its Lie algebra $\mathfrak{g}_{\mathrm{sp}} = \mathrm{Lie}(G_{\mathrm{space}})$ is a Lie subalgebra of $\mathfrak{K} = \mathrm{Lie}(G)$, the algebra of global Killing fields. Each element of $\mathfrak{g}_{\mathrm{sp}}$ corresponds to a one-parameter subgroup of G_{space}.

Consider the restriction $u = \Gamma|$ of the unitary representation Γ to the subgroup G_{space}. By a standard construction, the derivative Du is a Lie algebra representation of $\mathfrak{g}_{\mathrm{sp}}$ on $\mathcal{E}$, and the following diagram commutes:

$$\begin{array}{ccc} G_{\mathrm{space}} & \xrightarrow{u=\Gamma|} & \mathcal{U}(\mathcal{E}_+) \\ {\scriptstyle \mathrm{Lie}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Lie}} \\ \mathfrak{g}_{\mathrm{sp}} & \xrightarrow[Du]{} & \mathfrak{u}(\mathcal{E}_+) \end{array} \tag{2.14}$$

Note that $\mathcal{U}(\mathcal{E}_+)$ is an infinite-dimensional Lie group, and there are delicate analytic questions involving the domains of the self-adjoint operators in $\mathfrak{u}(\mathcal{E}_+)$ as discussed previously. In the present section we investigate only the algebraic structure; a full account of the detailed functional analysis underlying the interplay between self-adjoint (elliptic) operators and Lie groups is given in the monograph of Robinson [43].

By Theorem 2.4, each one-parameter unitary group $U(t)$ on $\mathcal{E}_+$ coming from a one-parameter subgroup of G_{space} has a well-defined quantization $\hat{U}(t)$ which is a unitary group on $\mathcal{H}$. The methods of Section 1.7 establish strong continuity for these unitary groups, so we may infer existence of their self-adjoint generators.

It is natural to then ask: are the commutation relations in $\mathfrak{g}_{\mathrm{sp}}$ in any way reflected by those of the self-adjoint generators on $\mathcal{H}$? Suppose that $[X, Y] = Z$ for three elements $X, Y, Z \in \mathfrak{g}_{\mathrm{sp}}$. Let $\hat{X} : \mathcal{H} \to \mathcal{H}$ be the quantization of $Du(X)$, and similarly for Y and Z. Our assumptions guarantee that $[Du(X), Du(Y)] = Du(Z)$ is null-invariant, therefore we have

$$[\hat{X}, \hat{Y}] = \hat{Z} \tag{2.15}$$

as operators on $\mathcal{H}$.

Using the physics convention of multiplying all Lie algebras by $\sqrt{-1}$, the Lie algebra $\mathfrak{u}(\mathcal{E}_+)$ appearing in the above diagram consists precisely of the Hermitian operators on $\mathcal{E}_+$. For purely spatial symmetries, Lemma 2.2 implies that $\hat{X} = \widehat{Du}(X)$ is self-adjoint on $\mathcal{H}$. This, together with (2.15), implies that the self-adjoint generators satisfy the same commutation relations as the Killing fields which generated them.

As noted following (2.14), there are delicate analytic issues governing when these ideas are applicable. A discussion of the domains of some self-adjoint operators obtained by this procedure was given in Section 2.4. This analysis suggests that Osterwalder-Schrader quantization, when applied in the present context, is really a generalization to infinite-dimensional Lie groups of the procedure of taking the derivative of a representation. Thus, it is not surprising that at some level it is functorial. This also adds to its intrinsic mathematical interest.

Let $E(d)$ denote the isometry group of $\mathbb{R}^d$. The quantizations of $\Gamma(\phi)$ with $\phi \in E(d)$ give quantum mechanical generators for symmetry under the Euclidean group. Osterwalder and Schrader [40, 41] showed that invariance of the Schwinger functions under the Euclidean group implies, by analytic continuation, Lorentz invariance of the Wightman functions in the relativistic theory, with an associated mapping of the generators. There is clearly no analogue of this on a general spacetime. On the other hand, when there is a nontrivial G_{space}, as discussed above, and when the analytic continuation exists, then the generators will also continue to symmetries of the real-time theory.

3. Variation of the Metric

3.1. **Metric dependence of matrix elements in field theory.** Having a quantization procedure for curved spacetimes, we are in a position to obtain analytic control over how the resulting quantization depends on the metric. Let λ denote parametric dependence of the metric. Each value of λ defines a Hilbert space $\mathcal{H}_\lambda$, a semigroup $e^{-\beta H_\lambda}$, and canonical time-zero fields ϕ, π.

Consider the spacetime $M = \mathbb{R} \times M'$, where M' is a Riemannian manifold with metric $g_{\mu\nu} = g_{\mu\nu}(\lambda)$. Assume that the metric on $\mathbb{R} \times M'$ is simply the direct sum of $g_{\mu\nu}$ with the flat metric on $\mathbb{R}$,

$$ds^2_\lambda = dt^2 + g_{\mu\nu}(\lambda) dx^\mu dx^\nu \,. \tag{3.1}$$

Assume that λ is a real parameter on which $g_{\mu\nu}(\lambda)$ depends smoothly for all $\lambda \in I$, where $I \subset \mathbb{R}$ is a connected interval. The simplest example of such a variation is a rescaling $g_{\mu\nu}(\lambda) = \lambda h_{\mu\nu}$. For brevity, we say that families of the form (3.1) have a *stable time direction.*

Consider Osterwalder-Schrader quantization with respect to the static Killing field $\partial/\partial t$. For all $p \in M$, there exists a unique way of writing

$$p = (t_p, m_p), \quad t_p \in \mathbb{R},\ m_p \in M' \,.$$

As before, $\Omega_+ = \{p \in M \mid t_p > 0\}$, and $\mathcal{E} = L^2(Q, \mathcal{O}, d\mu)$ where $(Q, \mathcal{O}, d\mu)$ is a probability measure space with covariance given by the $H^{-1}(M)$ inner product, which as we know depends on the Laplacian. Consider E_+ to be the subspace generated by

$$\left\{e^{i\Phi(f)} \mid \operatorname{supp}(f) \subset \Omega_+\right\}$$

with completion $\mathcal{E}_+ = \overline{E}_+$. We also define E to be the (incomplete) linear span of $e^{i\Phi(f)}$ for $f \in H^{-1}(M)$.

As λ varies, the linear spans E and E_+ are the same *as linear spaces*, but not as Hilbert spaces because the inner product

$$\langle A, B\rangle_{\mathcal{E}}$$

depends on the metric through the covariance. Therefore, the completions $\mathcal{E}_\lambda$ and $\mathcal{E}_{+,\lambda}$ depend on λ. Elements of the form $\exp(i\Phi(f))$ have canonical representatives in $\mathcal{E}(\lambda)$ for all λ. Each $\mathcal{E}(\lambda)$ is the completion with respect to a different Hilbert-space inner product, however, so the additional elements introduced by the completion have no simple relation.

The fortunate upshot of this discussion is that we may without ambiguity fix elements $A, B \in E$ and notice that their inner product $\langle A, B\rangle_{\mathcal{E},\lambda}$ is an analytic function in λ. This follows from the fact that we may take

$$A = \,:e^{-i\Phi(f)}:_{C(\lambda)}, \qquad \text{and} \qquad B = \,:e^{-i\Phi(g)}:_{C(\lambda)}$$

as typical elements of $\mathcal{E}$, and then $\langle A, B\rangle_{\mathcal{E},\lambda} = e^{\langle f, C(\lambda) g\rangle}$, so inner products in $\mathcal{E}$ depend analytically on $C(\lambda)$.

The null space $\mathcal{N}_\lambda$ also depends on the metric, as we discuss presently. The subspace of $\mathcal{N}_\lambda$ corresponding to monomials in the field is canonically isomorphic to the space of test functions f such that[1]

$$\int_M f(\vec{x}, -t)\left(\frac{1}{-\Delta_{g(\lambda)} + m^2}\, f\right)(\vec{x}, t)\sqrt{\det g(\lambda, \vec{x})}\, d^d x = 0\,. \tag{3.2}$$

Note that all of the quantities in the integrand (3.2) which depend on λ do so smoothly.

Note

$$\lambda \to g_{\mu\nu}(\lambda) \to (-\Delta_{g(\lambda)} + m^2)^{-1} \in \mathcal{B}(L^2(M)) \tag{3.3}$$

is a smooth map into the Frechet manifold of bounded operators on $L^2(M)$. Continuity of (3.3) follows from [25, Theorem A.5.4, p. 146]. Strong operator continuity of the heat kernel $\lambda \to e^{-t\Delta_{g(\lambda)}}$ with uniform convergence on bounded intervals then follows from [25, Theorem A.5.5].

Note that the $\mathcal{E}_\lambda$ for different λ are all L^2-spaces of the same underlying topological space and σ-algebra, namely the dual of the test function space. When we change λ, the measure follows a certain path in the space of all Gaussian measures. This change in the measure can be controlled through operator estimates on the covariance. For example [25, 9.1.33, p. 208] we have:

$$\frac{d}{d\lambda}\int A\; d\phi_{C(\lambda)} = \frac{1}{2}\int \Delta_{dC/d\lambda} A\; d\phi_{C(\lambda)}\,.$$

[1]For integrals such as this one, we can factorize the Laplacian as in Sec. 4.

In particular, if $C(\lambda)$ is smooth then so is $\int A \; d\phi_{C(\lambda)}$.

Theorem 3.1 (Smoothness of matrix elements)**.** *Assume that $g(\lambda)_{ij}$ depends smoothly on λ for λ in some compact set $K \subset \mathbb{C}$. Then for any two elements $A(\Phi) = \, :\exp(i\Phi(f)):$ and $B(\Phi) = \, :\exp(i\Phi(h)):$, the map*

$$\lambda \to \langle \hat{A}, R_\lambda(t) \hat{B} \rangle_{\mathcal{H}(\lambda)}$$

is also smooth.

Proof.

$$\begin{aligned}
\langle \hat{A}, R_\lambda(t) \hat{B} \rangle_{\mathcal{H}(\lambda)} &= \int \, : e^{i\Phi(\theta f)} : \, : e^{i\Phi(\phi_{\lambda,t}^{-1 *} h)} : \, d\mu_{C_\lambda} \\
&= \exp \langle \theta f, (C_\lambda h) \circ \phi_{\lambda,t}^{-1} \rangle \\
&= \exp \int f(x,-s) g(y, s'-t) C_\lambda(x-y, s-s') \sqrt{\det g(\lambda)} \; dx \, dy \, ds \, ds'
\end{aligned}$$

where $\phi_{\lambda,t}$ is the time t map of the Killing field $\partial/\partial t$ on the spacetime M_λ. Differentiating under the integral sign is clearly valid here and gives the desired result. ☐

As an application, consider the spacetime $M = \mathbb{R}^{d+1}$, with the metric

$$ds^2 = dt^2 + g(\lambda)_{ij} dx^i dx^j, \quad i,j = 1 \dots d\,.$$

Assume that $g(\lambda)_{ij}$ depends analytically on $\lambda \in \mathbb{C}$, and to order zero it is the flat metric δ_{ij}. Theorem 3.1 implies that the matrix elements of H have a well-defined series expansion about $\lambda = 0$, and we know that precisely at $\lambda = 0$ they take their usual flat-space values.

3.2. **Stably symmetric variations.** We continue to consider variations of the metric with a stable time direction, as in equation (3.1):

$$ds_\lambda^2 = dt^2 + g_{\mu\nu}(\lambda) dx^\mu dx^\nu\,.$$

One important aspect of the quantization that is generally *not* λ-invariant is the symmetry structure of the Riemannian manifold. As before we assume $M = \mathbb{R} \times M'$, where M' is a Riemannian manifold with metric $g_{\mu\nu}(\lambda)$. In this section we study a special case in which the perturbation does not break the symmetry. Let $\mathfrak{K}_\lambda$ denote the algebra of global Killing fields on $(M', g(\lambda))$. In certain very special cases we may have the following.

Definition 3.1 (Stably Symmetric Families)**.** The family of metrics $\lambda \to g(\lambda)$ is said to be **stably symmetric** if all of the following conditions hold:

(a) The number $n \equiv \dim \mathfrak{K}_\lambda$ does not depend on λ,
(b) there exist Killing fields $\xi_i(\lambda)$, $i = 1 \dots n$ such that $\forall\ \lambda$, the set $\{\xi_i(\lambda)\}_{i=1\dots n}$ forms a basis of $\mathfrak{K}_\lambda$, and
(c) $\lambda \to \xi_i(\lambda)$ is smooth $\forall\ i$.

These conditions could be equivalently reformulated to say that the space of Killing fields $\mathfrak{K}_\lambda$ for this class of spaces is a rank n vector bundle over $\mathbb{R}$ (or some subinterval thereof) and we have chosen a complete set $\xi_i(\lambda)$, $i = 1 \dots n$ of smooth sections.

Example 3.1 (constant-curvature hyperbolic metric)**.** Let X be an open subset of $\mathbb{C}$. The curvature of the conformal metric $ds_X = \alpha(z)|dz|$ is given by

$$R(z) = -\frac{4}{\alpha(z)^2}\,\partial\overline{\partial}\log\alpha(z)\,.$$

The most general constant-curvature hyperbolic metric on $\mathbb{H}$ has arc length given by

$$ds = \frac{c}{\Im(z)}|dz| \tag{3.4}$$

and curvature $-c^{-2}$. Consider the spacetime $\mathbb{R}\times\mathbb{H}(c)$ where $\mathbb{H}(c)$ is the upper half-plane with metric (3.4). Variation of the curvature parameter c preserves the number of Killing vectors, and in general satisfies all of the assumptions for a stable symmetry given in Definition 3.1.

For each i, λ, the Killing field $\xi_i(\lambda)$ gives rise to a one-parameter group of isometries on M, which we denote by $\phi_{i,\lambda,x} \in \mathrm{Iso}(M)$, where $x \in \mathbb{R}$ is the flow parameter. These flows act on the "space" part of the manifold for each fixed time; they are *purely spatial isometries* in the sense of Definition 2.3. Therefore, the map

$$T_i(\lambda, x) = \Gamma(\phi_{i,\lambda,x}) : \mathcal{E} \longrightarrow \mathcal{E}\,.$$

is automatically both positive-time invariant and null-invariant, so we infer that it has a quantization

$$\hat{T}_i(\lambda, x) : \mathcal{H} \longrightarrow \mathcal{H}\,. \tag{3.5}$$

Further, we infer by Theorem 2.4 that the operators (3.5) are all unitary. This is a rigorous proof, in the setting of curved space, that spatial symmetries are implemented by unitary groups on $\mathcal{H}$, the physical Hilbert space.

None of the following constructions depend on i, so for the moment we fix i and suppress it in the notation.

Since each $T(\lambda, x)$ depends on a Killing field ξ, the first step is to determine how the Killing fields vary as a function of the metric. Since the Killing fields are solutions to a first-order partial differential equation, one possible method of attack could proceed by exploiting known regularity properties of solutions to that equation. If one were to pursue that, some simplification may be possible due to the fact that a Killing field is completely determined by its first-order data at a point. We obtain a more direct proof.

The T operators depend on the Killing field through its associated one-parameter flow. For each fixed λ, the construction gives a one-parameter subgroup (in particular, a curve) in G_{space}. If we vary $\lambda \in [a, b]$, we have a free homotopy between two paths in G_{space}. Each cross-section of this homotopy, such as $\lambda \to \phi_{\lambda,x}(p)$ with the pair (x, p) held fixed, describes a continuous path in a particular spatial section of M.

Theorem 3.2. *With the above assumptions and with x fixed, we assert that*

$$\hat{T}(\lambda, x) : \mathcal{H} \longrightarrow \mathcal{H}$$

is a strongly continuous function of λ.

Proof. First we remark that $\lambda \to \phi_{\lambda,x}$ is continuous in the compact-open topology. The latter follows from standard regularity theorems for solutions of ODEs, since we have assumed $\lambda \to \xi(\lambda)$ is smooth, and $\phi_{\lambda,x}(p)$ is the solution curve of the differential operator $\xi(\lambda)_p$. Theorem 1.9, implies that $\Gamma(\phi_{\lambda,x})$ (as an element of the unitary group of $\mathcal{E}$) is strongly continuous with respect to λ. By assumption, $\Gamma(\phi_{\lambda,x})$ is a quantizable operator. By theorem 2.5, the embedding of bounded operators on $\mathcal{E}$ into $\mathcal{B}(\mathcal{H})$ is norm-continuous, so the proof is complete. □

4. Sharp-time Localization

The goal of this section is to establish an analogue of [25, Theorem 6.2.6] for quantization in curved space, and to show that the Hilbert space of Euclidean quantum field theory may be expressed in terms of data local to the slice Σ.

Definition 4.1 (quantizable static spacetime)**.** A *quantizable static spacetime* is a complete, connected Riemannian manifold with a globally defined (smooth) Killing field ξ which is orthogonal to a distinguished codimension one hypersurface $\Sigma \subset M$, such that the orbits of ξ are complete and each orbit intersects Σ exactly once.

Under the assumptions for a quantizable static spacetime, but with Lorentz signature, Ishibashi and Wald [29] have shown that the Klein-Gordon equation gives sensible classical dynamics, for sufficiently nice initial data.

These assumptions guarantee that we are in the situation of Definition 1.1 (a). Choose adapted coordinates (t, x^α) such that

$$\Sigma = \{t = 0\}.$$

In this situation, the time-reflection isometry $\theta : M \to M$ is defined by mapping (t, x^α) to the corresponding point $(-t, x^\alpha)$.

The main difficulty comes when trying to prove the analogue of [25, (6.2.16)] in the curved space case, which would imply that the restriction to $\mathcal{E}_0$ of the quantization map is surjective. The proof given in [25] relies on the formula (6.2.15) from Prop. 6.2.5, and it is the latter formula that we must generalize.

4.1. Localization on flat spacetime.

First, we review the relevant construction in $\mathbb{R}^d$; further details are to be found in [25]. The Euclidean propagator on $\mathbb{R}^d$ is given explicitly by the momentum representation

$$C(x; y) = C(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{p^2 + m^2} \, e^{-ip\cdot(x-y)} \, dp \, ,$$

for $x, p \in \mathbb{R}^d$. Let $f = f(\vec{x})$ denote a function on $\mathbb{R}^{d-1}$, and define

$$f_t(\vec{x}, t') = f(\vec{x})\delta(t - t') \, .$$

Theorem 4.1 (Flat-space Localization, (6.2.15) from [25])**.** *Let $M = \mathbb{R}^d$ with the standard Euclidean metric. Then*

$$\langle f_t, C g_s \rangle_{L^2(\mathbb{R}^d)} = \langle f, S g \rangle_{L^2(\mathbb{R}^{d-1})}$$

where S is the operator with momentum-space kernel

$$\frac{1}{2\mu(\vec{p})}\, e^{(t-s)\mu(\vec{p})}, \quad \textit{where} \quad \mu(\vec{p}) = \left(\vec{p}^{\,2} + m^2\right)^{1/2} .$$

Proof. The desired inner product is calculated as

$$\begin{aligned}\langle f_t, C g_s\rangle_{L^2(\mathbb{R}^d)} &= \int d\vec{x}\, d\vec{y}\; f(\vec{x}) C(\vec{x}-\vec{y}, t-s) g(\vec{y}) \\ &= \int d\vec{x} d\vec{y}\, f(\vec{x}) g(\vec{y}) \frac{1}{(2\pi)^d} \int d\vec{p}\; e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int \left(\frac{e^{ip_0(t-s)}}{p_0^2 + \vec{p}^{\,2} + m^2} \right) dp_0 \,. \end{aligned} \tag{4.1}$$

By completing the contour in the complex p_0 plane and using the residue theorem, we have

$$\int \frac{e^{ip_0(t-s)}}{p_0^2 + \vec{p}^{\,2} + m^2}\; dp_0 = \frac{\pi}{\mu(\vec{p})}\, e^{(t-s)\mu(\vec{p})} \,. \tag{4.2}$$

Plugging (4.2) into (4.1) gives the desired result. □

4.2. **Splitting the Laplacian on static spacetimes.** Consider a quantizable, static, space-time M, defined in Definition 4.1. Use Latin indices a, b, etc. to run from 0 to $d-1$ and Greek indices $\mu, \nu = 1 \ldots d-1$. We also denote the spatial coordinates by

$$\vec{x} = (x^1, \ldots, x^{d-1}) = (x^\mu) \,,$$

and set $t = x^0$. Write g in manifestly static form,

$$g_{ab} = \begin{pmatrix} F(x^\alpha) & 0 \;\ldots\; 0 \\ 0 & \\ \vdots & G_{\mu\nu}(x^\alpha) \\ 0 & \end{pmatrix} , \quad \text{with inverse} \quad g^{ab} = \begin{pmatrix} \frac{1}{F(x^\alpha)} & 0 \;\ldots\; 0 \\ 0 & \\ \vdots & G^{\mu\nu}(x^\alpha) \\ 0 & \end{pmatrix} . \tag{4.3}$$

It is then clear that

$$\mathcal{G} := \det(g_{ab}) = FG, \;\text{ where } G = \det(G_{\mu\nu}) \,. \tag{4.4}$$

It follows that $g^{0\nu} = g^{\mu 0} = 0$, and $g^{00} = F^{-1} = {g_{00}}^{-1}$, does not depend upon time. The Laplacian on curved space is $\Delta f = \mathcal{G}^{-1/2}\partial_a \left(\mathcal{G}^{1/2}\, g^{ab}\partial_b f\right)$, so the Laplacian on M is

$$\Delta_M \;=\; \frac{1}{F}\partial_t^2 + Q, \quad \text{where} \tag{4.5}$$

$$Q \;:=\; \frac{1}{\sqrt{\mathcal{G}}}\partial_\mu \left(\sqrt{\mathcal{G}}\, G^{\mu\nu}\, \partial_\nu\right) . \tag{4.6}$$

It will be desirable for us to relate Q to the Laplacian Δ_Σ computed with respect to the induced metric on Σ. Applying the product rule to (4.5) yields

$$Q = \frac{1}{2}\partial_\mu(\ln F)\, G^{\mu\nu}\partial_\nu + \Delta_\Sigma \,. \tag{4.7}$$

Note that a formula generalizing (4.7) to "warped products" appears in Bertola et.al. [6].

In order that the operator $\mu = (-Q+m^2)^{1/2}$ (which arises in sharp-time localization) exists for all $m^2 > 0$, we require that $-Q$ is a positive, self-adjoint operator on an appropriately-defined Hilbert space. The correct Hilbert space is

$$\mathcal{K}_\Sigma \ := \ L^2(\Sigma, \sqrt{\mathcal{G}}\, dx) \ . \tag{4.8}$$

Here $\sqrt{\mathcal{G}}\, dx$ denotes the Borel measure on Σ which has the indicated form in each local coordinate system, and $\mathcal{G} = FG$ as in eq. (4.4).

Spectral theory of the operator $-Q$ considered on $\mathcal{K}_\Sigma$ is mathematically equivalent to that of the "wave operator" A defined by Wald [46, 47] and Wald and Ishibashi [29]. In those references, the Klein-Gordon equation has the form $(\partial_t^2 + A)\phi = 0$. The relation between Wald's notation and ours is that $Q = -(1/F)A - m^2$, and Wald's function V is our $F^{1/2}$. As pointed out by Wald, we have the following,

Theorem 4.2 (**Q is symmetric and negative**). *Let (M, g_{ab}) be a quantizable static space-time. Then $-Q$ is a symmetric, positive operator on the domain $C_c^\infty(\Sigma) \subset \mathcal{K}_\Sigma$.*

Proof. By formula (4.7), Q can be expressed purely in terms of vector fields on Σ, hence Q gives a well-defined operator on $C_c^\infty(\Sigma)$. The adapted coordinate system (x, t) used above, in which $\xi = \partial/\partial t$ is only guaranteed to be valid in a neighborhood of Σ. Therefore choose a tube $\tilde{M}$ in M corresponding to $t \in (-\epsilon, \epsilon)$ on which the adapted coordinate system is valid. Define

$$\tilde{Q} = \Delta_M - \frac{1}{F}\partial_t^2$$

acting on $\tilde{\mathcal{D}} = C_c^\infty(\tilde{M})$. It is well-known [18] that Δ_M is essentially self-adjoint on

$$\tilde{\mathcal{D}} \subset L^2(\tilde{M}, dg),$$

where dg denotes the natural volume measure with respect to the metric g_{ab}. Locally, $dg = \sqrt{\mathcal{G}}\, dxdt$. Since $\mathcal{G}$ does not depend on t, we infer that ∂_t^2 is essentially self-adjoint on $\tilde{\mathcal{D}} \subset L^2(\tilde{M}, dg)$, hence so is $\tilde{Q}$. Moreover, if $f \in \tilde{\mathcal{D}}$ does not depend on t, then $\tilde{Q}f = Qf$. Such f may be equivalently described as a function on Σ, and therefore, there is no boundary term for Q on $C_c^\infty(\Sigma)$.

It remains to show $-Q \geq 0$ on $C_c^\infty(\Sigma) \subset \mathcal{K}_\Sigma$. Using (4.6), the associated quadratic form is

$$\begin{aligned} \langle f, (-Q)f \rangle_{\mathcal{K}_\Sigma} \ &= \ -\int \overline{f}\, \frac{1}{\sqrt{\mathcal{G}}} \partial_\mu \left(\sqrt{\mathcal{G}}\, G^{\mu\nu} \partial_\nu f \right) \ \sqrt{\mathcal{G}}\, dx \\ &= \ \int \|\nabla f\|_G^2 \ \sqrt{\mathcal{G}}\, dx \geq 0 \, . \end{aligned}$$

where we used integration by parts to go from the first line to the second. □

4.3. **Hyperbolic space.** It is instructive to calculate Q in the explicit example of $\mathbb{H}^d$, often called "Euclidean AdS" in the physics literature. The metric is $ds^2 = r^{-2} \sum_{i=0}^{d-1} dx_i^2$, where $r = x_{d-1}$. See Appendix A for a detailed discussion of the Green's function for this example, and its analytic continuation.

The hyperbolic Laplacian in d dimensions is [5]

$$\Delta_{\mathbb{H}^d} = (2-d) r \frac{\partial}{\partial r} + r^2 \Delta_{\mathbb{R}^d} \,. \tag{4.9}$$

Any vector field $\partial/\partial x_i$ where $i \neq d-1$ is a static Killing field. We have set up the coordinates so that it is convenient to define $t = x_0$ as before, and we can quantize in the t direction.

Comparing (4.6) with (4.9), we find for $\mathbb{H}^d$, that $F = r^{-2}$ and

$$Q = (2-d) r \frac{\partial}{\partial r} + r^2 \sum_{i=1}^{d-1} \frac{\partial^2}{\partial x_i^2} = -r \frac{\partial}{\partial r} + \Delta_{\mathbb{H}^{d-1}} \,, \tag{4.10}$$

which matches (4.7) perfectly.

4.4. **Curved space localization.** To generalize Theorem 4.1 to curved space, choose an adapted coordinate system (i.e. one in which the metric takes the manifestly static form) near Σ. We denote these coordinates by $\vec{x}, t$. Let $f = f(\vec{x})$ denote a function on the slice Σ. Define

$$f_t(\vec{x}, t') = f(\vec{x}) \delta(t - t'),$$

a distribution on the patch of M covered by this coordinate chart. For the moment, we assume that this coordinate patch is the region of interest. By equation (4.6), we infer that the kernel $\mathcal{C}$ of the operator $C = (-\Delta + m^2)^{-1}$ is time-translation invariant, so that we may write

$$\mathcal{C}(x, y) = \mathcal{C}(\vec{x}, \vec{y}, x_0 - y_0) \,.$$

In order to apply spectral theory to Q, we choose a self-adjoint extension of the symmetric operator constructed by theorem 4.2. For definiteness, we may choose the Friedrichs extension, but any ambiguity inherent in the choice of a self-adjoint extension will not enter into the following analysis. We denote the self-adjoint extension also by Q, which is an unbounded operator on the Hilbert space $\mathcal{K}_\Sigma = L^2(\Sigma, \sqrt{\mathcal{G}}\, dx)$ defined above. The following is a generalization of Theorem 4.1 to curved space.

Theorem 4.3 (sharp-time localization of the covariance)**.** *Let M be a quantizable static spacetime (definition 4.1), and further that $-Q + m^2 > 0$. Then:*

$$\langle f_t, C g_s \rangle_M = \left\langle f, \ \left(F^{1/2} \frac{e^{-|t-s|\omega}}{2\omega} F^{1/2} \right) g \right\rangle_{\mathcal{K}_\Sigma} \,, \tag{4.11}$$

where $\mu = (-Q + m^2)^{1/2}$ and $\omega = \left(\sqrt{F} \mu^2 \sqrt{F} \right)^{1/2}$. Hence C is reflection positive on $L^2(M)$.

Proof. Because M was assumed to be a quantizable static spacetime, $F = \langle \xi, \xi \rangle_\Sigma \geq 0$. Moreover, if $F(p) = 0$ then $\xi_p = 0$, for any $p \in \Sigma$. A non-trivial Killing field cannot vanish on an open set, so the zero-set of F has measure zero in Σ. From this we infer that multiplication by the function F^{-1} defines a (possibly-unbounded) but densely-defined self-adjoint multiplication operator on $\mathcal{K}_\Sigma$.

For simplicity of notation, assume f is real-valued. Perform a partial Fourier transform with respect to the time variable:

$$\langle f_t, Cg_s\rangle_M = \int f(\vec{x}) \left(\frac{1}{2\pi} \int dE \frac{e^{iE(t-s)}}{F^{-1}E^2 - Q + m^2} g \right)(\vec{x}) \sqrt{\mathcal{G}}\, dx\,. \tag{4.12}$$

Define $\mu := \left(-Q + m^2\right)^{1/2}$, where the square root is defined through the spectral calculus on $\mathcal{K}_\Sigma$. As a consequence of theorem 4.2, μ and ω are positive, self-adjoint operators on $\mathcal{K}_\Sigma$. The integrand of (4.12) contains the operator:

$$\frac{e^{iE(t-s)}}{F^{-1}E^2 + \mu^2} = \frac{e^{iE(t-s)}}{F^{-1/2}\left(E^2 + F^{1/2}\mu^2 F^{1/2}\right)F^{-1/2}} = F^{1/2}\frac{e^{iE(t-s)}}{E^2 + \omega^2}F^{1/2}\,.$$

We next establish that ω is invertible. By assumption, $\mu^2 > \epsilon I$, where $\epsilon > 0$. Hence

$$\omega^2 = \sqrt{F}\mu^2\sqrt{F} > \epsilon F$$

and therefore,

$$\omega^{-2} < \left(\sqrt{F}\mu^2\sqrt{F}\right)^{-1} < \frac{1}{\epsilon F}\,.$$

Since $1/F$ is a densely defined operator on $\mathcal{K}_\Sigma$, it follows that ω^2 (hence ω) is invertible.

For $\lambda > 0$,

$$\int \frac{e^{iE\tau}}{E^2 + \lambda^2}\, dE = \frac{\pi e^{-|\tau|\lambda}}{\lambda}\,. \tag{4.13}$$

Decompose the operator ω according to its spectral resolution, with $\omega = \int \lambda dP_\lambda$ and $I = \int dP_\lambda$ the corresponding resolution of the identity, and apply (4.13) in this decomposition to conclude

$$\int \frac{e^{iE(t-s)}}{F^{-1}E^2 + \mu^2}\, dE = F^{1/2}\frac{\pi e^{-|t-s|\omega}}{\omega}F^{1/2} \tag{4.14}$$

Inserting (4.14) into (4.12) gives

$$\begin{aligned} \langle f_t, Cg_s\rangle_M &= \int_\Sigma \left(F^{1/2}f\right)(\vec{x}) \left(\frac{e^{-|t-s|\omega}}{2\omega}(F^{1/2}g)\right)(\vec{x}) \sqrt{\mathcal{G}}\, d\vec{x} \\ &= \left\langle f,\ F^{1/2}\frac{e^{-|t-s|\omega}}{2\omega}F^{1/2}g\right\rangle_{\mathcal{K}_\Sigma}, \end{aligned} \tag{4.15}$$

also demonstrating reflection positivity. □

The operator ω^2 may be calculated explicitly if the metric is known, and is generally not much more complicated than Q. For example, using the conventions of sec. 4.3, one may calculate ω^2 for $\mathbb{H}^d$:

$$\omega^2 = -\sum_{i=1}^{d-1} \partial_i^2 + d\, r^{-1}\partial_r + (m^2 - d)r^{-2}\,.$$

For $d = 2$, the eigenvalue problem $\omega^2 f = \lambda f$ becomes a second-order ODE which is equivalent to Bessel's equation. The two linearly-independent solutions are $r^{3/2}J_{\frac{1}{2}\sqrt{4m^2+1}}(r\sqrt{\lambda})$ and $r^{3/2}Y_{\frac{1}{2}\sqrt{4m^2+1}}(r\sqrt{\lambda})$. The spectrum of ω^2 on $\mathbb{H}^2$ is then $[0, +\infty)$.

Theorem 4.4 (Analogue of [25], Thm. 6.2.6)**.** *Let M be a quantizable static spacetime. Then the vectors* $\exp(i\Phi(f_0))$ *lie in* $\mathcal{E}_+$*, and quantization maps the span of these vectors isometrically onto* $\mathcal{H}$*.*

Proof. Recall that given a function f on Σ, we obtain a distribution f_t supported at time t as follows:

$$f_t(x, t') = f(x)\delta(t - t').$$

It may appear that this is not well-defined because it depends on a coordinate. However, given a static Killing vector, the global time coordinate is fixed up to an overall shift by a constant, which we have determined by the choice of an orthogonal hypersurface where $t = 0$. Thus a pair (p, t) where $p \in \Sigma$ and $t \in \mathbb{R}$ uniquely specify a point in M.

Since $\mathcal{E}_+$ is the *closure* of the set E_+ of vectors $\exp(i\Phi(f))$ with $\mathrm{supp}(f) \subset \Omega_+$, it follows that any sequence in $\mathcal{E}_+$ which converges in the topology of $\mathcal{E}$ has its limit in $\mathcal{E}_+$. The L^2 norm in $\mathcal{E}$,

$$\int \left|e^{i\Phi(f)} - e^{i\Phi(g)}\right|^2 d\Phi_C = 2(1 - e^{-\frac{1}{2}\|f-g\|_{-1}}),$$

is controlled in terms of the norm $\| \, \|_{-1}$ on Sobolev space, which is the space of test functions. This will give us the first part of the theorem.

If $t > 0$, then there exists a sequence of smooth test functions $\{g_n\}$ with compact, positive-time support such that

$$\lim_{n\to\infty} g_n = f_t$$

in the Sobolev topology, hence $\exp(i\Phi(f_t)) \in \mathcal{E}_+$. Define the *time-$t$ subspace* $\mathcal{E}_t \subset \mathcal{E}_+$ to be the subspace generated by vectors of the form $\exp(i\Phi(f_t))$. By taking the $t \to 0$ limit, we see that $\exp(i\Phi(f_0)) \in \mathcal{E}_+$ and the first part is proved.

It is straightforward that the quantization map $\Pi(A) \equiv \hat{A}$ is isometric when restricted to vectors of the form $\exp(i\Phi(f_0))$, since the time-reflection θ acts trivially on these vectors. It remains to see that the restriction to such vectors is *onto* $\mathcal{H}$. Define the Wick-ordered exponential by

$$: \exp \Phi(f) : = \exp[\Phi(f) - \frac{1}{2}\langle f, Cf\rangle]. \tag{4.16}$$

Then we wish to prove

$$(\mathcal{E}_0)\hat{} \supset \Big(\bigcup_{t>0} \mathcal{E}_t\Big)\hat{} \,. \tag{4.17}$$

First, let us see why (4.17), if true, finishes the proof. We must show that $\bigcup_{t>0} \mathcal{E}_t$ is dense in $\mathcal{E}_+$. Of course, $\mathcal{E}_+$ is spanned by polynomials in functionals of the form

$$A(\Phi) = \int \Phi(x,t) f(x,t) \sqrt{\mathcal{G}}\, dx dt \,.$$

Write the t integral as a Riemann sum:

$$A(\Phi) = \lim_{N\to\infty} \sum_{i=1}^{N} (\delta t)_i \int \Phi(x,t_i) f_i(x) \sqrt{\mathcal{G}}\, dx \tag{4.18}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} (\delta t)_i \Phi\Big((f_i)_{t_i}\Big) \tag{4.19}$$

where $f_i(x) = f(x,t_i)$. Eqn. (4.18) represents $\Phi(f)$ as a limit of polynomials in elements $\Phi(f_{t_i})$ where $f_{t_i} \in \mathcal{E}_{t_i}$. Thus $\bigcup_{t>0} \mathcal{E}_t$ is dense in $\mathcal{E}_+$. Then (4.17) implies $(\mathcal{E}_0)\hat{}$ is also dense in $\mathcal{E}_+$.

Equation (4.17) is proved by means of the following identity:

$$\langle \hat{A}, : \exp(i\Phi(f_t)) : \hat{}\,\rangle_{\mathcal{H}} = \langle \hat{A}, : \exp(i\Phi({f^t}_0)) : \hat{}\,\rangle_{\mathcal{H}} \tag{4.20}$$

where

$$f^t \;:=\; (F^{-1/2} e^{-t\omega} F^{1/2}) f, \tag{4.21}$$

and f is a function on Σ. Note that f^t is still a function on Σ. Thus

$${f^t}_0(p,t') = \delta(t')(F^{-1/2} e^{-t\omega} F^{1/2} f)(p) \quad \text{for} \quad p \in \Sigma\,.$$

To prove (4.20), we first suppose $A = \,: e^{i\Phi(g_s)} :$ where $g \in \mathcal{T}_\Sigma$ and $s > 0$. Then

$$\begin{aligned} \langle \hat{A}, : \exp(i\Phi(f_t)) : \hat{}\,\rangle_{\mathcal{H}} &= \langle : e^{i\Phi(\theta g_s)} :, : e^{i\Phi(f_t)} : \rangle_{\mathcal{E}} \\ &= \exp \langle \theta g_s, C f_t \rangle_M \\ &= \exp \big\langle g, F^{1/2} \frac{e^{-(t+s)\omega}}{2\omega} F^{1/2} f \big\rangle_{\mathcal{K}_\Sigma} \end{aligned} \tag{4.22}$$

where we have used localization (Theorem 4.3) in the last line.

Computing the right side of (4.20) gives

$$\begin{aligned} \big\langle : e^{i\Phi(\theta g_s)} :, : e^{i\Phi({f^t}_0)} : \big\rangle_{\mathcal{E}} &= \exp \big\langle \theta g_s,\, C({f^t}_0) \big\rangle_M \\ &= \exp \big\langle g, F^{1/2} \frac{e^{-s\omega}}{2\omega} F^{1/2} f^t \big\rangle_{\mathcal{K}_\Sigma} \\ &= \exp \big\langle g, F^{1/2} \frac{e^{-(t+s)\omega}}{2\omega} F^{1/2} f \big\rangle_{\mathcal{K}_\Sigma} = (4.22). \end{aligned}$$

We conclude that eqns. (4.20)-(4.21) hold true for $A = \,: e^{i\Phi(g_s)} :$. We then infer the validity of (4.20) for all A in the span of $\bigcup_{t>0} \mathcal{E}_t$ by linear combinations and limits.

Equation (4.20) is remarkable. It says that for every vector v in a set that is dense in $\mathcal{H}$, there exists $v' \in (\mathcal{E}_0)\hat{}$ such that $L(v) = L(v')$ for any linear functional L on $\mathcal{H}$. If $v \neq v'$ then we could find some linear functional to separate them, so they are equal. Therefore $(\mathcal{E}_0)\hat{}$ is a dense set, completing the proof of Theorem 4.4. $\square$

Theorem 4.4 implies that the physical Hilbert space is isometrically isomorphic to $\mathcal{E}_0$, and to an L^2 space of the Gaussian measure with covariance which can be found by the $t, s \to 0$

limit of (4.22), to be:

$$\mathcal{H} = L^2\left(\left(\mathcal{T}_{d-1}\right)',\ d\phi_{\mathfrak{C}}\right)\ , \quad \text{where} \quad \mathfrak{C} = F^{1/2}\frac{1}{2\omega}F^{1/2}\,. \tag{4.23}$$

Compare with [25], eqn. (6.3.1). By assumption, 0 lies in the resolvent set of ω, implying that $\mathfrak{C}$ is a bounded, self-adjoint operator on $\mathcal{K}_\Sigma$.

4.5. **Fock representations.** To obtain a Fock representation of the *time-zero fields* we mimic the construction of [25, § 6.3] with the covariance (4.23).

To simplify the constructions in this section, we assume the form $ds^2 = dt^2 + G_{\mu\nu}dx^\mu dx^\nu$ and $F = 1$. Then $Q = \Delta_\Sigma$, the Laplacian on the time-zero slice, and $\mu = (-\Delta_\Sigma + m^2)^{1/2}$ is known to exist. The set of functions $h \in L^2(\Sigma)$ such that $\mu^p h \in L^2(\Sigma)$ is precisely the Sobolev space $H_p(\Sigma)$, which is also the set of h such that $\mathfrak{C}^{-p}h \in L^2$. Sobolev spaces satisfy the reverse inclusion relation $p \geq q \Rightarrow H_q \subseteq H_p$. Also $\mathfrak{C}^q f \in H_p \Leftrightarrow f \in H_{q-p}$.

This allows us to determine the natural space of test functions for the definition of the Fock representations:

$$\begin{aligned} a(f) &= \frac{1}{2}\phi\left(\mathfrak{C}^{-1/2}f\right) + i\pi\left(\mathfrak{C}^{1/2}f\right) \\ a^*(f) &= \frac{1}{2}\phi\left(\mathfrak{C}^{-1/2}f\right) - i\pi\left(\mathfrak{C}^{1/2}f\right)\ . \end{aligned}$$

In particular, if the natural domain of ϕ is H_{-1} as discussed following eqn. (1.8), then f must lie in the space where $\mathfrak{C}^{-1/2}f \in H_{-1}$, i.e. $f \in H_{1/2}$.

5. Conclusions and Outlook

5.1. **Summary of conclusions obtained.** The framework of Euclidean quantum field theory [40, 41] has been successfully generalized to static spacetimes, which enjoy many special features. We quickly recall the central results of the paper.

Static Riemannian spacetimes admit a splitting $M = \Omega_- \cup \Sigma \cup \Omega_+$, where Σ is orthogonal to a Killing vector $\partial/\partial t$. There is an isometry θ such that $\theta^2 = 1$, $\theta\Omega_\pm = \Omega_\mp$ and $\theta\Sigma = \Sigma$ pointwise. There is a coordinate system adapted to this structure, for which the metric is $ds^2 = F\,dt^2 + h_{\mu\nu}dx^\mu dx^\nu$, $1 \leq \mu, \nu \leq d-1$ and $\Sigma = \{t = 0\}$. For this class of spacetimes, the embedding within a complex 4-manifold with a Euclidean section is guaranteed, and in such a way that Einstein's equation is preserved (Sec. 1.2), thus analytic continuation to real time makes sense. Then our fundamental assumptions are laid out carefully (Sec. 1.4) and the choice of test function space as $\mathcal{T} = H^{-1}(M)$ is justified. We introduce the covariance $C = (-\Delta + m^2)^{-1}$ and give a standard dense domain $E = \mathrm{Span}\{e^{i\Phi(f)}, f \in \mathcal{T}\}$ in $\mathcal{E} = L^2(d\mu_C)$. We show that $\{f_1, \ldots, f_n\} \longmapsto \Phi(f_1)\ldots\Phi(f_n)$ is a continuous function from $(H^{-1})^n \to \mathcal{E}$.

Section 1.5 introduces $\Gamma(\phi)$, and gives many fundamental properties: Γ is a faithful (i.e. 1-1) representation of the group $\mathrm{Diff}(M)$ on $\mathcal{E}$. It turns out that $\psi \in \mathrm{Iso}(M) \Leftrightarrow [\psi^*, C] = 0$ $\Leftrightarrow \Gamma(\psi)$ is unitary. Also, Γ is covariant in the sense that if $\phi : U \to V$ is a diffeomorphism, then $\Gamma(\phi)E_U = E_V$. The operator norm of $\Gamma(\phi)$ turns out (Sec. 1.6) to depend on how well one can bound the Jacobian of $\phi \in \mathrm{Diff}(M)$. If ψ is an isometry then $\|\Gamma(\psi)\| = 1$. Since we will often have a continuous family of isometries, it's nice to know that $\Gamma(\psi)$ is strongly continuous (Sec. 1.7) in the variable ψ. Finally in Section 1.8 we get around to reflection positivity, which allows construction of the physical Hilbert space. We discuss three different proofs of reflection positivity on curved spacetimes, only one of which is new.

Section 2 describes the quantization procedure that is the fundamental object of study. Define a bilinear form $(A,B) = \langle \Theta A, B\rangle_{\mathcal{E}}$ which is only useful when restricted to $\mathcal{E}_+$, and then the physical Hilbert space $\mathcal{H}$ is defined to be the quotient of $\mathcal{E}_+$ by the (infinite dimensional) form kernel $\mathcal{N}$, completed. Unfortunately, not every operator $T : \mathcal{E} \to \mathcal{E}$ determines any kind of operator on $\mathcal{H}$, but those T which are positive-time invariant ($T(\mathcal{E}_+) \subset \mathcal{E}_+$) and null-invariant ($T(\mathcal{N}) \subset \mathcal{N}$) do give such an operator. Since the form kernel is a somewhat unwieldy object, null-invariance can be hard to check directly. We give a sufficient condition for null invariance: assume positive-time invariance of T and $\Theta T^*\Theta$; then T is null-invariant. In other words, if the dotted arrow is well-defined, then so are the two solid arrows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{incl.}} & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & \vdots{\scriptstyle \Theta T^*\Theta} & & \downarrow{\scriptstyle \hat{T}} & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{incl.}} & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0
\end{array}
$$

Moreover, the two horizontal rows are exact sequences. Also $\hat{T}^\dagger = \widehat{\Theta T^*\Theta}$ and $\|\hat{T}\|_{\mathcal{H}} \le \|T\|_{\mathcal{E}}$.

Theorems 2.3 and 2.4 show that unitary operators on $\mathcal{E}$ can give either unitary or self-adjoint operators on $\mathcal{H}$, assuming they have quantizations. Two nice classes of isometries (reflection-invariant or reflected) which implement the assumptions of Thms. 2.3 and 2.4 are identified.

Unbounded operators must have dense domains. Any linear subspace of $\mathcal{E}_+$ which is dense in the closure $\mathrm{cl}(\mathcal{E}_+)$ projects to a dense subspace of $\mathcal{H}$. Moreover, Sec. 2.4 defines "quantization domains" which are subspaces $U \subset \Omega_+$ such that $\Pi(\mathcal{E}_U)$ is dense in $\mathcal{H}$. Assuming that $\psi^{-1}(U) = \Omega_+$ so that $E_U = \Gamma(\psi)E_+$, we found that if

$$[\Gamma(\phi), \Theta] = 0 \quad \text{or } \Gamma(\psi)\Theta = \Theta\Gamma(\psi^{-1})$$

then U is a quantization domain. On flat space one may see directly that $\mathcal{O}_{+,T} = \{x \in \mathbb{R}^d : x_0 > T\}$ is a quantization domain.

With this structure, it's simple to construct the Hilbert space, Hamiltonian, and ground state (Sec. 2.5). Let $\xi = \partial/\partial t$ be the time-translation Killing field with one-parameter group of isometries $\phi_t : M \to M$. For $t \ge 0$, $U(t) = \Gamma(\phi_t)$ has a quantization, which we denote $R(t)$. This $R(t)$ is a strongly continuous contraction semigroup, which leaves invariant the vector $\Omega_0 = \hat{1}$. There exists a densely defined, positive, self-adjoint operator H such that

$$R(t) = \exp(-tH), \quad \text{and} \quad H\Omega_0 = 0.$$

Thus Ω_0 is a quantum-mechanical ground state. Since H is positive and self-adjoint, the heat kernels can be analytically continued $t \to it$. Further,

$$\langle \hat{A}, e^{-tH}\hat{B}\rangle_{\mathcal{H}} = \int A(\Phi)\big(U(t)B\big)(\Phi)\; d\mu(\Phi)\,.$$

This Feynman-Kac formula is the Euclidean path integral of quantum field theory. Of course, what we'd *really* like is the interacting version, eq. (2.12).

In Definition 2.3, we construct the group $G_{\text{space}} \subset \text{Iso}(M)$ of purely spatial isometries, which *always* have quantizations, defined on $\mathcal{H}$. By Theorem 2.4 these quantizations are unitary. If $\hat{U}(t)$ on $\mathcal{H}$ comes from a one-parameter subgroup of G_{space}, then by Section 1.7 we infer the existence of a self-adjoint generator. This is elegantly expressed by a commutative diagram of Lie groups and algebras:

$$\begin{array}{ccc} G_{\text{space}} & \xrightarrow{u=\Gamma|} & \mathcal{U}(\mathcal{E}_+) \\ {\scriptstyle \text{Lie}}\downarrow & & \downarrow{\scriptstyle \text{Lie}} \\ \mathfrak{g}_{\text{sp}} & \xrightarrow[Du]{} & \mathfrak{u}(\mathcal{E}_+) \end{array}$$

So the self-adjoint generators satisfy the same commutation relations as the Killing fields which generated them.

Having defined a rigorous quantization procedure for curved spacetimes, we are in a position to obtain analytic control over how the resulting quantization depends on the metric. For this we assume the product structure $ds^2_\lambda = dt^2 + g_{\mu\nu}(\lambda)dx^\mu dx^\nu$ where $g_{\mu\nu}(\lambda)$ is a function of λ. We are able to formulate some information about how the form kernel $\mathcal{N}_\lambda$ and the heat kernel $e^{-t\Delta_{g(\lambda)}}$ depend on λ. Finally, we show that matrix elements $\lambda \to \langle \hat{A}, R_\lambda(t)\hat{B}\rangle_{\mathcal{H}(\lambda)}$ are smooth functions of λ. A nice application is to perturbations of a flat metric.

One may also consider variations for which (a) the number n of Killing fields of $g_{\mu\nu}(\lambda)$ does not depend on λ, (b) there exist Killing fields $\xi_i(\lambda)$ that span $\mathfrak{K}_\lambda$ for every λ, and (c) $\lambda \to \xi_i(\lambda)$ is smooth $\forall\ i$. We call this sort of thing "stable symmetry." (note: $\mathfrak{K}_\lambda$ does not include $\partial/\partial t$). Then $T_i(\lambda, x) = \Gamma(\phi_{i,\lambda,x}) : \mathcal{E} \to \mathcal{E}$ is automatically both positive-time invariant and null-invariant, so it has a quantization which is strongly continuous in λ. A beautiful example of a stably symmetric variation is hyperbolic space, with Laplacian $\Delta_{\mathbb{H}^d} = (2-d)r\partial_r + r^2\Delta_E$ where $r = x_{d-1}$ and Δ_E is the Laplacian on $\mathbb{R}^d$. (The heat kernel and the resolvent kernel are easy to compute for $d = 2, 3$ but they get increasingly complicated for $d \geq 4$.)

With a static metric, the Laplacian can be split $\Delta_M = (1/F)\partial_t^2 + Q$ where

$$Q = \mathcal{G}^{-1/2}\partial_\mu(\mathcal{G}^{1/2}\, G^{\mu\nu}\, \partial_\nu) = (2F)^{-1}F_{,\mu}\, G^{\mu\nu}\partial_\nu + \Delta_\Sigma.$$

We then prove self-adjointness and positivity of $-Q$ on the Hilbert space $\mathcal{K}_\Sigma := L^2(\Sigma, \sqrt{\mathcal{G}}\, dx)$, where $\mathcal{G} = FG$. This self-adjointness and positivity is important due to the localization formula:

$$\langle f_t, Cg_s\rangle_M = \left\langle f,\ \Big(F^{1/2}\frac{e^{-|t-s|\omega}}{2\omega}F^{1/2}\Big)g\right\rangle_{\mathcal{K}_\Sigma},$$

The localization formula allows us to infer that the vectors $\exp(i\Phi(f_0))$ lie in $\mathcal{E}_+$, and quantization maps the span of these vectors isometrically onto $\mathcal{H}$. This implies a characterization of $\mathcal{H}$ in terms of data local to the time zero slice, otherwise known as the Schrödinger representation. However, unless $-Q + m^2 \geq 0$, this procedure will not work as μ will be ill-defined.

5.2. **Future research.** Dimock [15] constructed an interacting $\mathcal{P}(\varphi)_2$ model with variable coefficients, with interaction density $\rho(t, x) : \varphi(x)^4 :$, and points out that a Riemannian $(\varphi^4)_2$ theory may be reduced to a Euclidean $(\varphi^4)_2$ theory with variable coefficients. However, the main constructions of [15] apply to the Lorentzian case and for curved spacetimes no analytic continuation between them is known. Establishing such an analytic continuation is clearly a priority. In order to do that, one would have to establish an estimate known in constructive field theory as the *phi bound* [21]. Work on this is in progress.

Also, in the present paper we have not treated the case of a non-linear field, though all of the groundwork is in place. Such construction would necessarily involve a generalization of the Feynman-Kac integral (2.12) to curved space, which would have far-reaching implications. Then one would like to establish properties of the small-energy spectrum of the Hamiltonians for these theories. For many of these questions, and due to intrinsic interest, we require generalizations of the cluster expansion and phase cell localization methods [22, 26].

Another important direction is to isolate specific spacetimes suggested by physics which have more symmetry or other special properties, and then to extend the methods of constructive field theory to obtain mathematically rigorous proofs of such properties. Several interesting studies along these lines exist [8, 29] but there is much more to be done. A complete, rigorous understanding of the the holographically dual theory on the boundary of AdS suggested by Maldacena [1, 27, 38, 49] may be within reach of present methods.

More speculative, but no less exciting, is the proposition that a deeper understanding of the connection between statistical mechanics and QFT on curved spacetimes could lead in the context of black holes to new interpretations of their statistical properties.

In broad terms, constructive field theory on flat spacetimes has been developed over four decades and comprises thousands of published journal articles. Every statement in each of those articles is either: (i) an artifact of the zero curvature and high symmetry of $\mathbb{R}^d$ or $\mathbb{T}^d$ or (ii) generalizable to curved spaces with less symmetry. The present paper shows that the Osterwalder-Schrader construction and many of its consequences are in class (ii). For each construction in class (ii), investigation is likely to yield non-trivial connections between geometry, analysis, and physics. There is clearly enough here to occupy researchers for an additional four decades.

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Appendix A. Euclidean Anti-de Sitter and its Analytic Continuation

The Green's function G on a general curved manifold is the inverse of the corresponding positive transformation, so it satisfies

$$(\Delta - \mu^2)G = -g^{-1/2}\delta \,, \tag{A.1}$$

where $G(p,q)$ is a function of two spacetime points. By convention Δ acts on G in the first variable, and δ denotes the Dirac distribution of the geodesic distance $d = d(p,q)$. Translation invariance implies that G only depends on p and q through $d(p,q)$. We note that solutions of the homogeneous equation $(\Delta - \mu^2)\phi = 0$ may be recovered from the Green's function. Conversely, we may deduce the Green's function by solving the homogeneous equation for $d > 0$ and enforcing the singularity at $d = 0$.

The equation (A.1) for the Green's function takes a simple form in geodesic polar coordinates on $\mathbb{H}^n$ with $r = d =$ geodesic distance; the Green's function has no dependence on the angular variables and the radial equation yields

$$\left(\partial_r^2 + (n-1)\coth(r)\partial_r - \mu^2\right) G(r) = -\delta(r)\,. \tag{A.2}$$

We find it convenient to write the homogeneous equation in terms of the coordinate $u = \cosh(r)$. When $u \neq 1$, (A.2) becomes

$$(\Delta - \mu^2)G(u) = -(1-u^2)G''(u) + nuG'(u) - \mu^2 G(u) = 0\,. \tag{A.3}$$

For $n = 2$ and $\mu^2 = \nu(\nu+1)$, eqn. (A.3) is equivalent to Legendre's differential equation:

$$(1-u^2)Q_\nu''(u) - 2uQ_\nu'(u) + \nu(\nu+1)Q_\nu(u) = 0\,. \tag{A.4}$$

Note that (A.4) has two independent solutions for each ν, called Legendre's P and Q functions, but the Q function is selected because it has the correct singularity at $r = 0$. Thus

$$G_2(r;\mu^2) = \frac{1}{2\pi}Q_\nu(\cosh r), \quad \text{where} \quad \nu = -\frac{1}{2} + \left(\mu^2 + \frac{1}{4}\right)^{1/2}\,. \tag{A.5}$$

The case $\mu^2 = 0$ is particularly simple; there the Legendre function becomes elementary:

$$G_2(r;0) = -\frac{1}{2\pi}\ln\left(\tanh\frac{r}{2}\right) = \frac{1}{2\pi}Q_0(\cosh r)\,. \tag{A.6}$$

For $n = 3$, one has

$$G_3(r;\mu^2) = \frac{1}{4\pi}\frac{e^{\pm r\sqrt{\mu^2+1}}}{\sinh(r)}\,. \tag{A.7}$$

Finally, we note that the analytic continuation of (A.5) gives the Wightman function on AdS_2. The real-time theory on Anti-de Sitter, including its Wightman functions, were discussed by Bros et al. [8]. In particular, our equation (A.5) analytically continues to their equation (6.8).

Given a complete set of modes, one may also calculate the Feynman propagator by using the relation $iG_F(x,x') = \langle 0 \,|\, T\{\phi(x)\phi(x')\} \,|\, 0 \rangle$ and performing the mode sum explicitly as in [10]; the answer may be seen to be related to the above by analytic continuation. Here,

T denotes an AdS-invariant time-ordering operator. A good general reference is the classic paper [4].

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