Annals of Physics **278**, 10–61 (1999)
Article ID aphy.1999.5918, available online at http://www.idealibrary.com on IDEAL®

Twist Positivity[1]

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Received October 9, 1998

We study a heat kernel $e^{-\beta H}$ defined by a self-adjoint Hamiltonian H acting on a Hilbert space $\mathfrak{H}$, and a unitary representation $U(g)$ of a symmetry group G of H, normalized so that the ground vector of H is invariant under $U(g)$. The triple $\{H, U(g), \mathfrak{H}\}$ defines a twisted partition function $\mathfrak{Z}_g$ and a twisted Gibbs expectation $\langle\cdot\rangle_g$, $\mathfrak{Z}_g = \mathrm{Tr}_{\mathfrak{H}}(U(g^{-1})\, e^{-\beta H})$ and $\langle\cdot\rangle_g = \mathrm{Tr}_{\mathfrak{H}}(U(g^{-1}) \cdot e^{-\beta H}) / \mathrm{Tr}_{\mathfrak{H}}(U(g^{-1})\, e^{-\beta H})$. We say that $\{H, U(g), \mathfrak{H}\}$ is *twist positive* if $\mathfrak{Z}_g > 0$. We say that $\{H, U(g), \mathfrak{H}\}$ has a Feynman–Kac representation with a twist $U(g)$, if one can construct a function space and a probability measure $d\mu_g$ on that space yielding (in the usual sense on products of coordinates) $\langle\cdot\rangle_g = \int \cdot\, d\mu_g$. Bosonic quantum mechanics provides a class of specific examples that we discuss. We also consider a complex bosonic quantum field $\varphi(x)$ defined on a spatial s-torus $\mathbb{T}^s$ and with a translation-invariant Hamiltonian. This system has an $(s+1)$-parameter abelian twist group $\mathbb{T}^s \times \mathbb{R}$ that is twist positive and that has a Feynman–Kac representation. Given $\tau \in \mathbb{T}^s$ and $\theta \in \mathbb{R}$, the corresponding paths are random fields $\Phi(x, t)$ that satisfy the *twist relation* $\Phi(x, t+\beta) = e^{i\Omega\theta}\Phi(x-\tau, t)$. We also utilize the twist symmetry to understand some properties of "zero-mass" limits, when the twist τ, θ lies in the complement of a set Υ_{sing} of singular twists.

Contents

[1] Work supported in part by the Department of Energy under Grant DE-FG02-94ER-25228 and by the National Science Foundations under Grant DMS-94-24344.

0003-4916/99 $30.00

I. INTRODUCTION

We study a bosonic quantum mechanical system defined on a Hilbert space $\mathfrak{H}$. We assume that H is a self-adjoint Hamiltonian with a unique ground state Ω_{vacuum} and that the heat kernel $e^{-\beta H}$ is trace class. Let G denote a symmetry group of H, and $U(g)$ a continuous unitary representation of G on $\mathfrak{H}$, so $U(g)H = HU(g)$. Thus Ω_{vacuum} is an eigenvector of $U(g)$; normalize the phase of $U(g)$ so that

$$U(g)\,\Omega_{\text{vacuum}} = \Omega_{\text{vacuum}}. \tag{I.1}$$

This triple defines a twisted partition function $\mathfrak{Z}_g$ and a twisted Gibbs expectation $\langle\cdot\rangle_g$,

$$\mathfrak{Z}_g = \mathrm{Tr}_{\mathfrak{H}}(U(g^{-1})\,e^{-\beta H}) \qquad \text{and} \qquad \langle\cdot\rangle_g = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(g^{-1})\cdot e^{-\beta H})}{\mathrm{Tr}_{\mathfrak{H}}(U(g^{-1})\,e^{-\beta H})}. \tag{I.2}$$

Define $\{H, U(g), \mathfrak{H}\}$ to be *twist positive* if for all $g \in G$ and for all $\beta > 0$,

$$\mathfrak{Z}_g > 0. \tag{I.3}$$

Clearly $\mathfrak{Z}_{\mathrm{Id}} > 0$. Note that $\lim_{\beta\to\infty}(\mathfrak{Z}_g/\mathfrak{Z}_{\mathrm{Id}}) = \langle \Omega_{\text{vacuum}}, U(g^{-1})\,\Omega_{\text{vacuum}}\rangle$, showing (I.1) to be necessary.

A Feynman–Kac measure is a countably-additive, Borel probability measure $d\mu_g$. We say that $\{H, U(g), \mathfrak{H}\}$ has a Feynman–Kac representation with a twist $U(g)$, if one can construct a function space and a Feynman–Kac measure $d\mu_g$ on that space yielding

$$\langle\cdot\rangle_g = \int \cdot\, d\mu_g \tag{I.4}$$

in the sense that Gibbs expectations of time-ordered products of coordinates equal the integral of the same product of paths. We give examples in Propositions III.3 and (V.34) for quantum mechanics and in Propositions VI.6 and VI.8 for quantum fields.

Once we establish the existence of a measure, we can use classical inequalities and harmonic analysis to study particular integrals, and therefore to gain quantitative insights concerning $\mathfrak{Z}_g$ or $\langle\cdot\rangle_g$. Hence if we have a measure, then we may benefit from these basic tools of mathematical physics. Such inequalities are especially useful in understanding non-Gaussian expectations that cannot be evaluated in closed form.[2]

[2] In problems with fermions, it is common to define a functional on a Grassmann algebra, either using Berezin integration or using expectations on a Fock space. In specific examples of interest, these methods often do not yield positive functionals. Then the resulting theory is restricted to the Gaussian case, where the functionals yield Pfaffians and determinants. In the case of boson–fermion systems the purely bosonic part of the system may have a Feynman–Kac measure. This measure, multiplied by a Pfaffian or a determinant, can be studied using the positivity properties of the bosonic measure. This method of studying boson–fermion systems is common in constructive quantum field theory. Of course one can combine the measures we discuss in this paper with Pfaffians or determinants arising from fermionic degrees of freedom.

Twist positivity is necessary for $\mathfrak{Z}_g$ to have a Feynman–Kac representation, but it is neither necessary nor sufficient for $\langle \cdot \rangle_g$ to have a Feynman–Kac representation.[3] The fact remains that in the examples that we study, twist positivity holds. Twist positivity also appears in our proof of the existence of the Feynman–Kac measure for $\langle \cdot \rangle_g$. Thus twist positivity appears to be an excellent condition on which to focus, and we make it our title.

Our results apply to expectations that occur in various branches of physics and mathematics, including quantum mechanics, quantum field theory, and related problems in probability theory. We illustrate them with examples, starting in the simplest cases, and bosonic quantum mechanics provides specific ones. Then we discuss other examples from quantum field theory.[4] Consider a quantum theory with a complex coordinate z acting as a multiplication operator on the Hilbert space $\mathfrak{H} = L^2(\mathbb{C})$. Given a positive constant $\Omega > 0$, define the elementary $U(1)$ twist group with parameter θ and period $2\pi/\Omega$ by

$$z \to e^{i\Omega\theta} z. \tag{I.5}$$

Assume that the unitary operators $U(\theta)$ acting on $\mathfrak{H}$ implement this twist, so $U(\theta)\, z = e^{i\Omega\theta} z\, U(\theta)$. Furthermore, assume that the Hamiltonian H of the system is twist-invariant, and has a trace-class heat kernel. We consider the Gibbs functional $\langle \cdot \rangle_\theta$ with θ replacing g.

In the quantum mechanics case, the measure $d\mu_\theta$ is concentrated on continuous paths. Høegh-Krohn discovered that for $\theta = 0$, notably when $\langle \cdot \rangle_0$ is a Gibbs state, a Feynman–Kac representation arises from paths $\omega_0(t)$ that are periodic in time [3],

$$\omega_0(t+\beta) = \omega_0(t). \tag{I.6}$$

Our present results generalize this picture. In the simplest case of quantum theory with the twist (I.5), the twisted Feynman–Kac representation (I.4) arises from paths that are twisted periodic; namely they satisfy the twist relation

$$\omega_\theta(t+\beta) = e^{i\Omega\theta} \omega_\theta(t). \tag{I.7}$$

In the case of a vector-valued coordinate $z \in \mathbb{C}^n$, we may use a different rate of twisting in each coordinate direction. Thus we replace (I.5) by the relation

$$z_j \to U(\theta)\, z_j\, U(\theta)^* = e^{i\Omega_j\theta} z_j, \tag{I.8}$$

with individual periods of twisting $2\pi/\Omega_j > 0$, for each $1 \leqslant j \leqslant n$. We call the $\Omega = \{\Omega_j\}$ *weights*. In place of (I.7), our vector-valued paths satisfy the twist relation

$$\omega_{j,\theta}(t+\beta) = e^{i\Omega_j\theta} \omega_{j,\theta}(t), \qquad \text{for} \quad 1 \leqslant j \leqslant n. \tag{I.9}$$

[3] A necessary condition for $\langle \cdot \rangle_g$ to have a Feynman–Kac representation is: the pair correlation operator C_g, defined in (II.63) and in (VI.34), should be a linear transformation with positive spectrum. If $\langle \cdot \rangle_g$ is a Gaussian functional, this condition is also sufficient.

[4] Surprisingly, we have not found these observations in the literature, though some of them are "folk theorems" in physics, where their mathematical status is not clarified.

The existence of a minimal strictly positive constant h such that $h\Omega_j \in \mathbb{Z}$ for all $1 \leqslant j \leqslant n$ shows that the ratios of weights are rational. In that case, we say that the weights $\{\Omega_j\}$ are rationally commensurate. The group $U(\theta)$ then has a period $2\pi h$.

In Sections II–IV we study a twist-invariant oscillator with Hamiltonian $H = H_0$ and frequency $m > 0$. We work out this example in great detail, because our quantum mechanics results for other potentials rely on a detailed understanding of the oscillator. We give an elementary argument that the twisted Gibbs functional for the oscillator is Gaussian, and we give the Feynman–Kac measure $d\mu_{m,\beta,\theta}$. This measure is Gaussian and it is twist-invariant. The covariance of the measure is the pair correlation function of the oscillator. In Theorem III.1 we identify the covariance as the resolvent (i.e., the Green's function) of a twisted Laplacian. Also we show that this pair correlation function extends as a function of the time difference $t-s$ into the complex plane, to be complex periodic. The imaginary part of the period determines the twist angle θ.

Once we have derived properties of the twist-invariant oscillator, we study perturbed Hamiltonians H of the form

$$H = H_0 + V. \tag{I.10}$$

Here V is a suitable real, twist-invariant function that is bounded from below. We detail our assumptions on V in Definition V.1. In order to make this situation more concrete, let us illustrate some types of acceptable potential functions V.

V-i. The absolute square $V(z, \bar{z}) = |W(z)|^2$ of a holomorphic, homogeneous polynomial $W(z)$ is acceptable.

V-ii. A sum of acceptable potentials V's is an acceptable potential.

One such acceptable sum is the square of the gradient of a holomorphic, homogeneous polynomial W,

$$V(z, \bar{z}) = \sum_{j=1}^{n} \left| \frac{\partial W(z)}{\partial z_j} \right|^2. \tag{I.11}$$

Such V's arise as the bosonic potential in quantum theory with $N = 2$ supersymmetry, where $W(z)$ is called the superpotential. (The polynomial W may be quasihomogeneous rather than homogeneous; see Section V.)

The twisted Gibbs expectations that arise from the Hamiltonians $H = H_0 + V$ are non-Gaussian. They possess Feynman–Kac representations, and the measures defining these representations are non-Gaussian. The measures leading to (I.4) have the form

$$d\mu^V_{m,\beta,\theta} = \frac{e^{-\int_0^\beta V(\omega_\theta(s), \overline{\omega_\theta(s)})\, ds}\, d\mu_{m,\beta,\theta}}{\int e^{-\int_0^\beta V(\omega_\theta(s), \overline{\omega_\theta(s)})\, ds}\, d\mu_{m,\beta,\theta}}. \tag{I.12}$$

The normalizing factor for these measures is in fact the ratio of two traces, and we define this ratio to be the relative twisted partition function of $H=H_0+V$ relative to H_0,

$$\mathfrak{Z}^{V}_{m,\beta,\theta}=\frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta(H_0+V)})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H_0})}=\int e^{-\int_0^\beta V(\omega_\theta(s),\,\overline{\omega_\theta(s)})\,ds}\,d\mu_{m,\beta,\theta}. \tag{I.13}$$

We find that these traces are positive for all θ, and for all $\beta>0$, namely

$$0<\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H}),\qquad \text{and so}\qquad 0<\mathfrak{Z}^{V}_{m,\beta,\theta}. \tag{I.14}$$

Furthermore, inspecting these representations shows that if V grows at least quadratically in $|z|$ as $|z|\to\infty$, then

$$0<\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta(H_0+\lambda^2 V)})\leqslant\left(\frac{M}{\beta(m+\lambda)}\right)^{2n},\qquad \text{for}\quad 0\leqslant\lambda\leqslant 1, \tag{I.15}$$

where M is a constant depending only on V.

Functional integration is a central tool for quantum field theory, for string theory, and for statistical mechanics. The study of zero-mass measures poses special difficulties; they occur both in the Hilbert space formulation of field theory and in the formulation by functional integration. We clarify certain aspects of zero-mass limits by using twisted expectations in the complex case. A paper of Witten [9] suggests some properties of this sort, the understanding of which led us to this investigation. We plan to use this framework in another paper that provides quantum mechanics and field theory examples to illustrate the phenomena in [6].

Define the singular set of twisting angles

$$\Upsilon_{\mathrm{sing}}=\{\theta\colon e^{i\Omega_j\theta}=1,\ \text{for any } 1\leqslant j\leqslant n\}. \tag{I.16}$$

If the weights are rationally commensurate, the set Υ_{sing} contains a finite number of points for θ in a period interval. We prove that for fixed β and for fixed $\theta\notin\Upsilon_{\mathrm{sing}}$ the twisted zero-mass measure exists as a weak limit. As $m\to 0$,

$$d\mu^{V}_{m,\beta,\theta}\xrightarrow{\mathrm{w}} d\mu^{V}_{0,\beta,\theta}. \tag{I.17}$$

For each β, θ, the limiting measure $d\mu^{V}_{0,\beta,\theta}$ is a countably additive, Borel probability measure. These results improve on the canonical bounds (I.15), and in this case the twisted trace $\mathrm{Tr}_{\mathfrak{H}}(U(\theta)\, e^{-\beta(H_0+\lambda^2 V)})$ converges for $\theta\notin\Upsilon_{\mathrm{sing}}$ when $m+\lambda\to 0$ with β, θ fixed.

In addition our method applies to quantum fields $\varphi(x)$, that may also have vector components, $\varphi(x)=\{\varphi_j(x)\}$. We take the spatial variable x in an s-torus $\mathbb{T}^s$. We take the spatial periods equal to $\ell_1>0$, where $1\leqslant i\leqslant s$, and let the spatial volume be $\mathrm{Vol}=\prod_{i=1}^{s}\ell_i$. Then the Fourier decomposition of the field has the form

$$\varphi_j(x)=\frac{1}{\sqrt{\mathrm{Vol}}}\left(z_j+\sum_{k\neq 0}\hat{\varphi}_j(k)\, e^{-ikx}\right), \tag{I.18}$$

where $k = \{k_i\}$ ranges over the lattice $k_i \in (2\pi/\ell_i)\,\mathbb{Z}$ dual to $\mathbb{T}^s$. If $\varphi(x)$ is a complex field, the constant Fourier modes z_j are just the complex coordinates in quantum theory considered above. Let H denote a twist-invariant, translation-invariant Hamiltonian for the field. We assume that the Hamiltonian operator H, the momentum operator P, and the symmetry group $U(\theta) = e^{i\theta J}$ mutually commute. We also assume that H has a unique ground state Ω_{vacuum} so Ω_{vacuum} is an eigenstate of $U(\theta)$ and of the momentum. We add a constant to J and to P so

$$U(\theta)\,\Omega_{\text{vacuum}} = \Omega_{\text{vacuum}} \qquad \text{and} \qquad P\Omega_{\text{vacuum}} = 0. \tag{I.19}$$

We consider $U(\theta)$ implementing in a manner similar to that above,

$$\varphi_j(x) \to U(\theta)\,\varphi_j(x)\,U(\theta)^* = e^{i\Omega_j\theta}\varphi_j(x). \tag{I.20}$$

With these assumptions, we can interpret the translation group as an additional twist, yielding the $(s+1)$-parameter twist group

$$U(\tau, \theta) = e^{i\tau P + i\theta J} = e^{i\tau P}U(\theta) = U(\theta)\,e^{i\tau P}. \tag{I.21}$$

Here $U(\tau, 0) = e^{i\tau P}$, with $\tau \in \mathbb{T}^s$. In terms of components, $\tau P = \sum_{i=1}^{s} \tau_i P_i$. Then

$$U(\tau, \theta)\,\varphi_j(x)\,U(\tau, \theta)^* = e^{i\Omega_j\theta}\varphi_j(x-\tau). \tag{I.22}$$

Twist positivity is the statement

$$\mathfrak{Z}_{m,\beta,\tau,\theta} = \operatorname{Tr}_{\mathfrak{H}}(U(\tau,\theta)^*\,e^{-\beta H}) = \operatorname{Tr}_{\mathfrak{H}}(U(\theta)^*\,e^{-i\tau P-\beta H}) > 0. \tag{I.23}$$

The limit $\beta \to \infty$ shows that (I.19) is necessary. Define the twisted Gibbs functional

$$\langle\,\cdot\,\rangle_{m,\beta,\tau,\theta} = \frac{\operatorname{Tr}_{\mathfrak{H}}(U(\theta)^*\cdot e^{-i\tau P-\beta H})}{\operatorname{Tr}_{\mathfrak{H}}(U(\theta)^*\,e^{-i\tau P-\beta H})}. \tag{I.24}$$

We first establish that twist positivity holds for the massive free field, with $H = H_0$. This allows us to prove that the massive free-field pair correlation operator has a strictly positive spectrum. Furthermore, we show that the twisted free-field Gibbs functional is Gaussian. These two properties lead to a Feynman–Kac representation for the corresponding free-field functional by a measure $d\mu_{m,\beta,\tau,\theta}$.

The probability interpretation for the quantum field case arises from paths called random fields and is defined on a space-time $\mathscr{C} = \mathbb{T}^s \times [0, \beta]$. Since the random fields satisfy a twist relation depending on both τ and on θ, we denote them $\Phi_{\tau,\theta}(x, t)$ with components $\Phi_{\tau,\theta,j}(x, t)$, for $1 \leqslant j \leqslant n$. After averaging with a smooth function of the coordinate x, the random fields give paths that are continuous functions of the time. Furthermore, the translation group acts continuously on C^∞ functions on $\mathscr{C}$, also in the Fréchet topology, so translations also act continuously on random fields. We find in Section VI that the appropriate twist relation for random fields is

$$\Phi_{\tau,\theta,j}(x, t+\beta) = e^{i\Omega_j\theta}\Phi_{\tau,\theta,j}(x-\tau, t). \tag{I.25}$$

There is a Gaussian Feynman–Kac representation for the massive free-field Hamiltonian H_0 with the above twist. We denote the Gaussian measure by $d\mu_{m,\beta,\tau,\theta}(\Phi_{\tau,\theta}(\cdot))$. We show that the covariance of this measure $C_{\tau,\theta}$ equals the resolvent of a twisted Laplace operator $\Delta_{\tau,\theta}$,

$$C_{\tau,\theta} = (-\Delta_{\tau,\theta} + m^2)^{-1}. \tag{I.26}$$

We also construct non-Gaussian measures $d\mu^V_{m,\beta,\tau,\theta}(\Phi_{\tau,\theta}(\cdot))$ for interacting fields with certain Hamiltonians $H = H_0 + V$. These measures provide Feynman–Kac representations for twisted Gibbs functionals (I.24). We regularize these theories to avoid discussing renormalization at this time. The measures have the form

$$d\mu^V_{m,\beta,\tau,\theta}(\Phi_{\tau,\theta}(\cdot)) = \frac{e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\theta,\chi}(y,s), \overline{\Phi_{\tau,\theta,\chi}(y,s)})\,ds\,dy}\, d\mu_{m,\beta,\tau,\theta}}{\int e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\theta,\chi}(y,s), \overline{\Phi_{\tau,\theta,\chi}(y,s)})\,ds\,dy}\, d\mu_{m,\beta,\tau,\theta}}. \tag{I.27}$$

Here χ indicates that the random field $\Phi_{\tau,\theta,\chi}$ has a regularization, as does the potential in $H = H_0 + V$. In Section VI, we establish a result concerning the $m \to 0$ limit for non-singular twisting for the complex quantum field.

In Section VI we also study real fields $\varphi(x)$. These fields still have the s-parameter group arising from translations that is twist positive. The corresponding random fields depend on the parameters τ, but not on θ, and we denote them by $\Phi_\tau(x, t)$. Given $\tau \in \mathbb{T}^s$, these random fields satisfy the twist relation

$$\Phi_\tau(x, t+\beta) = \Phi_\tau(x-\tau, t). \tag{I.28}$$

They also lead to a twist-positive partition function and a Feynman–Kac representation.

II. THE TWIST-INVARIANT OSCILLATOR

We define a twist-invariant harmonic oscillator as a harmonic oscillator with a complex coordinate and a rotationally symmetric potential, equal to $m^2 |z|^2$. We now study the complex oscillator in detail.

II.1. *Canonical Field Coordinates*

We assume that the frequency (mass) of the oscillator is strictly positive, $m > 0$, and use the normalization suggested by field theory, in which the coordinate z is proportional to $m^{-1/2}$ and linear in dimension-less creation and annihilation operators.

Assume z take values in $\mathbb{C}$, and let $\mathfrak{H} = L^2(\mathbb{C}, dz)$ denote the Hilbert space of L^2 functions with respect to Lebesgue measure on $\mathbb{C}$. Let $\partial = \partial/\partial z$ and let $\bar\partial = \partial/\partial \bar z$, so $\bar\partial = -\partial^*$. We introduce independent annihilation operators a_+ and a_-, and write

$$a_+ = \frac{1}{\sqrt{2}}\left(\sqrt{m}\,\bar z + \frac{1}{\sqrt{m}}\partial\right) \quad \text{and} \quad a_- = \frac{1}{\sqrt{2}}\left(\sqrt{m}\,z + \frac{1}{\sqrt{m}}\bar\partial\right). \tag{II.1}$$

Let $a^{\#}$ denote either a or a^*. As a consequence of II.1, we find the canonical commutation relations (CCR) for the operators $a_{\pm}$ and their adjoints,

$$[a_{\pm}, a^*_{\pm}] = I \qquad \text{and} \qquad [a_{\pm}, a^{\#}_{\mp}] = 0. \tag{II.2}$$

Also, (II.1) inverts to yield

$$z = \frac{1}{\sqrt{2m}} (a^*_{+} + a_{-}), \qquad \partial = \sqrt{\frac{m}{2}} (a_{+} - a^*_{-}). \tag{II.3}$$

The Hamiltonian $H = H_0$ has the form

$$\begin{aligned} H_0 &= \partial^* \partial + m^2 \, |z|^2 - mI = -\partial \bar{\partial} + m^2 \, |z|^2 - mI \\ &= m(a^*_{+} a_{+} + a^*_{-} a_{-}), \end{aligned} \tag{II.4}$$

where we subtract the constant m to ensure that the minimum eigenvalue of H_0 equals 0. Let

$$N_{\pm} = a^*_{\pm} a_{\pm} \tag{II.5}$$

denote two commuting number operators. They have non-negative, integer spectrum, and we denote the eigenvalues by n_{+}, n_{-}, respectively. We may also write

$$H_0 = m(N_{+} + N_{-}). \tag{II.6}$$

The generator J of the symmetry $U(\theta) = e^{iJ\theta}$ has a simple form, either in terms of the coordinates or in terms of $N_{\pm}$, namely

$$J = \Omega(z\partial - \bar{z}\bar{\partial}) = \Omega(N_{+} - N_{-}). \tag{II.7}$$

Then

$$U(\theta) \, z U(\theta)^* = e^{i\Omega\theta} z \qquad \text{and} \qquad U(\theta) \, \partial U(\theta)^* = e^{-i\Omega\theta} \partial. \tag{II.8}$$

These representations of H and J in terms of $N_{\pm}$ show that they commute as self-adjoint operators. Consider their joint spectrum Σ; we represent a point in Σ as

$$h = m(n_{+} + n_{-}), \qquad j = \Omega(n_{+} - n_{-}). \tag{II.9}$$

Each point is specified by a unique pair of non-negative integers $\{n_{+}, n_{-}\}$ that are eigenvalues of $N_{\pm}$, and each point in the joint spectrum has multiplicity one. Let $|n_{+}, n_{-}\rangle$ label an orthonormal basis of simultaneous eigenvectors.

Let us introduce the complex constant γ, lying in the interior of the unit disc. This parameter encodes most of the dependence on m, β, θ, and Ω. Let

$$\gamma = e^{-m\beta + i\Omega\theta}. \tag{II.10}$$

Then we denote our twisted partition function by $\mathfrak{Z}_\gamma$ and our twisted expectation by $\langle \cdot \rangle_\gamma$, where

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \, e^{-\beta H_0}) \qquad \text{and} \qquad \langle \cdot \rangle_\gamma = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \cdot e^{-\beta H})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \, e^{-\beta H})}. \tag{II.11}$$

II.2. *The Oscillator Twisted Partition Function*

We now compute the oscillator partition function.

PROPOSITION II.1. *Let $H = H_0$ and suppose that $|\gamma| < 1$. Then*

$$\mathfrak{Z}_\gamma = \frac{1}{|1-\gamma|^2} = \frac{e^{m\beta}}{4\,|\sin(\Omega\theta/2 + im\beta/2)|^2} > 0. \tag{II.12}$$

Proof. We evaluate $\mathfrak{Z}_\gamma$ by taking the trace in the basis $|n_+, n_-\rangle$ of simultaneous eigenstates of $N_\pm$, introduced in Section II.1. The result is that

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \, e^{-\beta H_0}) = \sum_{n_+, n_- = 0}^{\infty} \gamma^{n_+} \bar{\gamma}^{n_-} = |1-\gamma|^{-2}. \tag{II.13}$$

II.3. *The Pair Correlation Function and Time-Ordered Products*

Let T denote an operator with domain $\mathscr{D}(T)$. Define the imaginary time propagation $T(t)$ of T by the general Hamiltonian H to be

$$T(t) = e^{-tH} T e^{tH}. \tag{II.14}$$

For $t > 0$, we obtain an operator with domain $e^{-tH}\mathscr{D}(T)$. In general $T(t)^*$ and $(T^*)(t)$ are equal only for $t = 0$. For commuting $T_1, ..., T_n$, denote the time-ordered product of $T_1(t_1), ..., T_n(t_n)$ by $(T_1(t_1) \cdots T_n(t_n))_+$, with the definition

$$(T_1(t_1) \cdots T_n(t_n))_+ = T_{i_1}(t_{i_1}) \, T_{i_2}(t_{i_2}) \cdots T_{i_n}(t_{i_n}), \qquad \text{where} \quad t_{i_1} \leqslant t_{i_2} \leqslant \cdots \leqslant t_{i_n}. \tag{II.15}$$

Define the *pair correlation function* by

$$C_\gamma(t, s) = \langle (\bar{z}(t) \, z(s))_+ \rangle_\gamma. \tag{II.16}$$

We now make a side technical assumption that $0 \leqslant H$ and that $z(H+I)^{-1}$ is bounded. This is a fact in the case $H = H_0$. As a consequence, we can use cyclicity of the trace in order to establish the following twist condition.

PROPOSITION II.2. *Suppose that $0 \leqslant H$, that $e^{-\beta H}$ has a trace, and that $z(H+I)^{-1}$ is bounded. Then the pair correlation function $C_\gamma(t, s)$ of* (II.16) *is a function of the difference coordinate $\xi = t - s \in [-\beta, \beta]$, and $C_\gamma(\xi)$ satisfies the twist relations*

$$\begin{aligned} C_\gamma(\xi + \beta) &= e^{-i\Omega\theta} C_\gamma(\xi), \qquad \text{if} \quad \xi \leqslant 0, \qquad \text{and} \\ C_\gamma(\xi - \beta) &= e^{i\Omega\theta} C_\gamma(\xi), \qquad \text{if} \quad \xi \geqslant 0. \end{aligned} \tag{II.17}$$

Proof. We use the definition of the pair correlation function and cyclicity of the trace. As $z(H+I)^{-1}$ is bounded, we can also cyclically permute factors of z and of $\bar{z}$ in the heat-kernel regularized trace. Note that $\bar{z}(t+t') = e^{-t'H}\bar{z}(t)\, e^{t'H}$. Then for $0 \leqslant t \leqslant s \leqslant \beta$ and $0 \leqslant t+a \leqslant s+a \leqslant \beta$, we have

$$\begin{aligned} C_\gamma(t+a, s+a) &= \langle \bar{z}(t+a)\, z(s+a) \rangle_\gamma = \langle e^{-aH}\bar{z}(t)\, z(s)\, e^{aH} \rangle_\gamma \\ &= \langle \bar{z}(t)\, z(s) \rangle_\gamma = C_\gamma(t, s). \end{aligned} \tag{II.18}$$

For $0 \leqslant s \leqslant t \leqslant \beta$ and $0 \leqslant s+a \leqslant t+a \leqslant \beta$ the same result holds. Hence $C_\gamma(t, s) = C_\gamma(\xi)$ is a function of the time difference.

We establish the twist property in a similar fashion. For $0 \leqslant t \leqslant s \leqslant \beta$, we have

$$\begin{aligned} C_\gamma(t-s+\beta) &= \langle (\bar{z}(t-s+\beta)\, z(0))_+ \rangle_\gamma = \langle z(0)\, \bar{z}(t-s+\beta) \rangle_\gamma \\ &= e^{-i\Omega\theta} \langle \bar{z}(t-s+\beta)\, z(\beta) \rangle_\gamma = e^{-i\Omega\theta} C_\gamma(t-s). \end{aligned} \tag{II.19}$$

For the case $0 \leqslant s \leqslant t \leqslant \beta$, write

$$\begin{aligned} C_\gamma(t-s-\beta) &= \langle (\bar{z}(t-s)\, z(\beta))_+ \rangle_\gamma = \langle \bar{z}(t-s)\, z(\beta) \rangle_\gamma \\ &= e^{i\Omega\theta} \langle z(0)\, \bar{z}(t-s) \rangle_\gamma = e^{i\Omega\theta} \langle (\bar{z}(t-s)\, z(0))_+ \rangle_\gamma \\ &= e^{i\Omega\theta} C_\gamma(t-s). \end{aligned} \tag{II.20}$$

II.4. *Holonomy Moves*

In this brief section we explain the idea of *holonomy moves*, namely the elementary steps in the holonomy expansion that we introduced in [7]. We explore here an elementary special case of a holonomy move. This case arises if we assume that there exists a complex number $s = s(m, \beta, \theta)$ such that S satisfies the commutation relation

$$S e^{-\beta H} U(\theta)^* = s e^{-\beta H} U(\theta)^*\, S. \tag{II.21}$$

Let $\langle ST \rangle_\gamma$ denote the twisted expectation of the product of S and T.

An S-holonomy move is an identity that involves moving S through the trace, and cyclically back to its original position. In particular, we may move S in a clockwise fashion and exploit the relation (II.21) to permute S cyclically in the trace. We find that

$$\langle ST \rangle_\gamma = \langle TS \rangle_\gamma + \langle [S, T] \rangle_\gamma = s \langle ST \rangle_\gamma + \langle [S, T] \rangle_\gamma. \tag{II.22}$$

As long as $s \neq 1$, this *S-holonomy move* generates the identity between expectations,

$$\langle ST \rangle_\gamma = \frac{1}{(1-s)} \langle [S, T] \rangle_\gamma. \tag{II.23}$$

The same identity can be written as

$$\langle ST \rangle_\gamma = \frac{-s^{-1}}{(1-s^{-1})} \langle [S, T] \rangle_\gamma. \tag{II.24}$$

We might also interpret the latter identity as coming from counter-clockwise movement of S through the expectation, which generates the relation

$$\langle ST \rangle_\gamma = s^{-1} \langle TS \rangle_\gamma = s^{-1} \langle ST \rangle_\gamma + s^{-1} \langle [T, S] \rangle_\gamma. \tag{II.25}$$

We use the S-holonomy identity (II.23),(II.24) below.

II.5. *The Twisted Oscillator Gibbs Functional Is Gaussian*

In this section we show that the twisted Gibbs expectation is Gaussian. Our proof is an elementary application of two holonomy moves. Let $\{(t_1, s_{i_1}), (t_2, s_{i_2}), ..., (t_n, s_{i_n})\}$ denote one of the $n!$ pairings of $\{t_1, ..., t_n\}$ with $\{s_1, ..., s_n\}$.

PROPOSITION II.3. *Let $H = H_0$ and $|\gamma| < 1$. Then the pair correlation function $C_\gamma(t, s) = C_\gamma(\xi)$ equals*

$$C_\gamma(\xi) = \frac{1}{2m} \frac{\gamma}{(1-\gamma)} e^{-m\xi} + \frac{1}{2m} \frac{\bar{\gamma}}{(1-\bar{\gamma})} e^{m\xi} + \frac{1}{2m} e^{-m|\xi|}, \tag{II.26}$$

and

$$0 < C_\gamma(0) = \frac{1}{2m} \left(\frac{1-|\gamma|^2}{|1-\gamma|^2} \right). \tag{II.27}$$

Furthermore, the general expectation (II.11) *of time-ordered products of coordinates satisfies the Gaussian relation*

$$\begin{aligned} &\langle (\bar{z}(t_1) \cdots \bar{z}(t_n)\, z(s_1) \cdots z(s_r))_+ \rangle_\gamma \\ &\quad = \delta_{nr} \sum_{\text{pairings}} C_\gamma(t_1 - s_{i_1}) \cdots C_\gamma(t_n - s_{i_n}). \end{aligned} \tag{II.28}$$

Remark. In particular, if $t_1 = t_2 = \cdots = t_n = s_1 = \cdots = s_n$, then $\langle (\bar{z}z)^n \rangle_\gamma = n!\, C_\gamma(0)^n$.

Proof. Let us begin by verifying (II.28). In the course of this proof we also show (II.26). Hence (II.27) results from the evaluation

$$C_\gamma(0) = \frac{1}{2m} \left(\frac{\bar{\gamma}}{1-\bar{\gamma}} + \frac{\gamma}{1-\gamma} + 1 \right) = \frac{1}{2m} \left(\frac{1-|\gamma|^2}{|1-\gamma|^2} \right). \tag{II.29}$$

We argue that (II.28) vanishes except for $n=r$. As $U(\theta)$ commutes with H, the expectation $\langle\cdot\rangle_\gamma$ is invariant under $U(\theta)$ in the sense that

$$\langle T\rangle_\gamma = \langle U(\theta)\, TU(\theta)^*\rangle_\gamma. \tag{II.30}$$

We take for T the time-ordered product, $T=(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_r))_+$. Then we have

$$U(\theta)\, TU(\theta)^* = e^{i\Omega\theta(r-n)}T \tag{II.31}$$

and

$$\langle T\rangle_\gamma = e^{i\Omega\theta(r-n)}\langle T\rangle_\gamma \tag{II.32}$$

for all θ. In the case $n\neq r$, this can only be true for $\langle T\rangle_\gamma=0$. Thus we restrict attention to the case $n=r$.

It is no loss of generality to relabel the times so that the smallest time is either s_1 or t_1. Let us first consider the case $0\leqslant s_1\leqslant t_1\leqslant\beta$. Define

$$T=e^{s_1H}(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_2)\cdots z(s_n))_+\, e^{-s_1H}. \tag{II.33}$$

Furthermore, let T_i, for $1\leqslant i\leqslant n$, denote T with the factor $\bar{z}(t_i)$ omitted. We rewrite the expectation (II.28) in terms of the operators T and T_j. Since H and $U(\theta)$ commute, by cyclicity of the trace we can write

$$\langle(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_n))_+\rangle_\gamma = \langle zT\rangle_\gamma. \tag{II.34}$$

We claim that this expectation satisfies the *Gaussian recursion relation*

$$\langle(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_n))_+\rangle_\gamma = \langle zT\rangle_\gamma = \sum_{j=1}^{n} C_\gamma(t_j-s_1)\langle T_j\rangle_\gamma. \tag{II.35}$$

The second case, on the other hand, is $0\leqslant t_1\leqslant s_1\leqslant\beta$. In this case we define

$$S=e^{t_1H}(\bar{z}(t_2)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_n))_+\, e^{-t_1H}, \tag{II.36}$$

and we have

$$\langle(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_n))_+\rangle_\gamma = \langle\bar{z}S\rangle_\gamma. \tag{II.37}$$

Define S_i, for $1\leqslant i\leqslant n$, as the operator S with the factor $z(s_i)$ omitted. In this case, we claim that this expectation satisfies the *conjugate Gaussian recursion relation*

$$\langle(\bar{z}(t_1)\cdots\bar{z}(t_n)\, z(s_1)\cdots z(s_n))_+\rangle_\gamma = \langle\bar{z}S\rangle_\gamma = \sum_{j=1}^{n} C_\gamma(t_1-s_j)\langle S_j\rangle_\gamma. \tag{II.38}$$

The identity (II.28) then follows by iteration of the recursion relations (II.35) and (II.38).

We now prove (II.35) and (II.38), and thereby complete the proof of the proposition. Begin the proof of (II.35) by rewriting (II.34) in a form in which we can apply a holonomy move (II.23), with $a^{\#}$ equal to an annihilitation operator or a creation operator. Using (II.3), we have

$$\langle (\bar{z}(t_1) \cdots \bar{z}(t_n)\, z(s_1) \cdots z(s_n))_+ \rangle_\gamma = \frac{1}{\sqrt{2m}} \langle (a_+^* + a_-)\, T \rangle_\gamma . \tag{II.39}$$

Let us consider the two terms separately.

In performing a holonomy move, we need to evaluate the commutators between a_- and T or between a_+^* and T. One can commute $a_\pm$ with z or $\bar{z}$ using

$$[a_-, \bar{z}] = (2m)^{-1/2} = -[a_+^*, \bar{z}] \quad \text{and} \quad [a_+^*, z] = [a_-, z] = 0. \tag{II.40}$$

These identities follow from (II.3) and the canonical commutation relations (II.2). In order to commute $a_\pm$ with e^{-tH} or $U(\theta)$, use the basic identities that are a consequence of (II.4),

$$a_\pm e^{-tH} = e^{-tm} e^{-tH} a_\pm \quad \text{and} \quad a_\pm U(\theta)^* = e^{\mp i\Omega\theta} U(\theta)^* a_\pm . \tag{II.41}$$

Thus

$$\begin{aligned} a_- e^{-\beta H} U(\theta)^* &= \gamma e^{-\beta H} U(\theta)^* a_- \quad \text{and} \\ a_+ e^{-\beta H} U(\theta)^* &= \bar{\gamma} e^{-\beta H} U(\theta)^* a_+ . \end{aligned} \tag{II.42}$$

The relation (II.42) shows that a_- and a_+ satisfy the hypothesis (II.21) with $s = e^{-m\beta/i\Omega\theta} = \gamma$ and $s = \bar{\gamma}$, respectively. Furthermore, by substituting $U(-\theta)$ for $U(\theta)$ and taking adjoints, we obtain similar relations for $a_\pm^*$,

$$\begin{aligned} a_+^* e^{-\beta H} U(\theta)^* &= \gamma^{-1} e^{-\beta H} U(\theta)^* a_+^* \quad \text{and} \\ a_-^* e^{-\beta H} U(\theta)^* &= \bar{\gamma}^{-1} e^{-\beta H} U(\theta)^* a_-^* . \end{aligned} \tag{II.43}$$

Hence $a_\pm$ and their adjoints may be used to define holonomy moves of the form (II.23) or the form (II.24).

Let us begin by expanding the second term in (II.39). We perform an a_--holonomy move and substitute the identity (II.23). Using (II.40) and (II.41) we have

$$[a_-, T] = \sum_{j=1}^{n} \frac{1}{\sqrt{2m}} e^{-m(t_j - s_1)} T_j . \tag{II.44}$$

Therefore

$$\frac{1}{\sqrt{2m}} \langle a_- T \rangle_\gamma = \frac{1}{2m} \frac{1}{(1-\gamma)} \sum_{j=1}^{n} e^{-m(t_j - s_1)} \langle T_j \rangle_\gamma . \tag{II.45}$$

On the other hand, for the first term in (II.39), we choose to perform an a_+^*-holonomy move. In this case, we refer to (II.43), which gives the representation (II.21) with $s^{-1}=\bar{\gamma}$. Use (II.40) to obtain

$$[a_+^*, T] = -\sum_{j=1}^{n} \frac{1}{\sqrt{2m}} e^{m(t_j - s_1)} T_j. \tag{II.46}$$

Then we write the alternative form (II.24) of the a_+-holonomy move as

$$\frac{1}{\sqrt{2m}} \langle a_+^* T \rangle_\gamma = \frac{1}{2m} \frac{\bar{\gamma}}{(1-\bar{\gamma})} \sum_{j=1}^{n} e^{m(t_j - s_1)} \langle T_j \rangle_\gamma. \tag{II.47}$$

Add (II.45) and (II.47), so that (II.34)–(II.39) become

$$\langle zT \rangle_\gamma = \sum_{j=1}^{n} \frac{1}{2m} \left(\frac{\bar{\gamma}}{(1-\bar{\gamma})} e^{m(t_j - s_1)} + \frac{1}{(1-\gamma)} e^{-m(t_j - s_1)} \right) \langle T_j \rangle_\gamma. \tag{II.48}$$

Note that in the case $n=1$, $T_1 = I$ and $\langle zT \rangle_\Gamma = C_\gamma(t_1 - s_1)$. Thus we have also proved that (in the case $0 \leqslant s_1 \leqslant t_1 \leqslant \beta$)

$$C_\gamma(t_1 - s_1) = \frac{1}{2m} \left(\frac{\bar{\gamma}}{(1-\bar{\gamma})} e^{m(t_1 - s_1)} + \frac{1}{(1-\gamma)} e^{-m(t_1 - s_1)} \right), \tag{II.49}$$

as claimed in (II.26). Insert the identity (II.49) into (II.48), to yield the desired recursion relation (II.35).

Next, let us treat the case $0 \leqslant t_1 \leqslant s_1 \leqslant \beta$ and establish (II.38). In place of (II.39), we use

$$\langle (\bar{z}(t_1) \cdots \bar{z}(t_n)\, z(s_1) \cdots z(s_n))_+ \rangle_\gamma = \frac{1}{\sqrt{2m}} \langle (a_+ + a_-^*)\, S \rangle_\gamma, \tag{II.50}$$

with S given by (II.36). Consider first the a_+-term, and perform an a_+-holonony; observe that (II.53) ensures $s=\bar{\gamma}$. Furthermore, in place of (II.44), we use

$$[a_+, S] = \sum_{j=1}^{n} \frac{1}{\sqrt{2m}} e^{-m(s_j - t_1)} S_j. \tag{II.51}$$

Thus

$$\frac{1}{\sqrt{2m}} \langle a_+ S \rangle_\gamma = \frac{1}{2m} \frac{1}{(1-\bar{\gamma})} \sum_{j=1}^{n} e^{-m(s_j - t_1)} \langle S_j \rangle_\gamma. \tag{II.52}$$

Similarly, we obtain

$$\frac{1}{\sqrt{2m}} \langle a_-^* S \rangle_\gamma = \frac{1}{2m} \frac{\gamma}{(1-\gamma)} \sum_{j=1}^{n} e^{m(s_j - t_1)} \langle S_j \rangle_\gamma. \tag{II.53}$$

Adding (II.52) and (II.53) we infer

$$\langle \bar{z}S \rangle_\gamma = \sum_{j=1}^{n} \frac{1}{2m} \left(\frac{1}{(1-\bar{\gamma})} e^{-m(s_j - t_1)} + \frac{\gamma}{(1-\gamma)} e^{m(s_j - t_1)} \right) \langle S_j \rangle_\gamma. \tag{II.54}$$

In the case $n=1$, note that $S_1 = I$, so that (II.54) reduces to

$$C_\gamma(t_1 - s_1) = \frac{1}{2m} \left(\frac{1}{(1-\bar{\gamma})} e^{-m(s_1 - t_1)} + \frac{\gamma}{(1-\gamma)} e^{m(s_1 - t_1)} \right), \tag{II.55}$$

thus establishing (II.26) in the case $0 \leqslant t_1 \leqslant s_1 \leqslant \beta$. Inserting this identity into (II.54) completes the proof of (II.38) and of the proposition.

II.6. Fourier Expansions

We encounter functions $f(t)$ on the interval $[0, \beta]$ that are continuous and satisfy the twisted periodicity condition

$$f(\beta) = e^{-i\Omega\theta} f(0). \tag{II.56}$$

According to Proposition II.2, the kernels $C_\gamma(t)$ and $C_\gamma(t-\beta)$ have this property. We expand such functions in a Fourier representation of the form

$$f(t) = \sum_{E \in (2\pi/\beta)\,\mathbb{Z}} \hat{f}(E)\, e^{iEt} e^{-i\Omega\theta t/\beta}. \tag{II.57}$$

In fact, on the interval $[0, \beta]$, linear combinations of the Fourier basis functions $\{e^{iEt}\}$ with $\beta E \in 2\pi\mathbb{Z}$ are a dense subset of $L^2([0, \beta], dt)$. Furthermore the smooth function of t, $e^{-i\Omega\theta t/\beta}$, defines a multiplication operator on $L^2([0, \beta])$. The modulus of this function is one, so the multiplication operator is unitary. Thus the functions $\{e^{iEt} e^{-i\Omega\theta t/\beta}\}$ also span L^2. Functions $f(t)$ on $[0, \beta]$, which have pointwise convergent Fourier representations (II.57), both satisfy the boundary condition (II.56) and extend (using this representation) to functions on the line $\mathbb{R}$ that satisfy the twist relation

$$f(t+\beta) = e^{-i\Omega\theta} f(t). \tag{II.58}$$

We may also rewrite this relation as follows. Define the set

$$K_\gamma = \{E: \beta E = 2\pi\mathbb{Z} - \Omega\theta\}. \tag{II.59}$$

Then

$$f(t) = \sum_{E \in K_\gamma} \hat{f}(E)\, e^{iEt}, \tag{II.60}$$

with

$$\beta \hat{f}(E) = \int_0^\beta f(t)\, e^{-iEt}\, dt, \tag{II.61}$$

and

$$\int_0^\beta |f(t)|^2 \, dt = \beta \sum_{E \in K_\gamma} |\hat{f}(E)|^2. \tag{II.62}$$

II.7. *The Oscillator Pair Correlation Operator*

Regard the pair correlation function $C_\gamma(t, s)$ as the integral kernel of an operator C_γ on $L^2([0, \beta], dt)$, so

$$(C_\gamma f)(t) = \int_0^\beta C_\gamma(t, s) \, f(s) \, ds. \tag{II.63}$$

Denote by C_γ the *pair correlation operator.*

PROPOSITION II.4. *The operator C_γ is self-adjoint, strictly positive, and bounded. In fact,*

$$0 < C_\gamma = C_\gamma^* \leqslant M, \tag{II.64}$$

where the upper bound M equals

$$M = \max \left\{ \left(\left(\frac{\Omega\theta}{\beta} \right)^2 + m^2 \right)^{-1}, \left(\left(\frac{2\pi - \Omega\theta}{\beta} \right)^2 + m^2 \right)^{-1} \right\}. \tag{II.65}$$

In particular, M is singular only if $\theta \in \{0, 2\pi/\Omega\} = \Upsilon_{\mathrm{sing}}$ and also $m \to 0$.

Proof. The function (II.26) satisfies

$$C_\gamma(s, t) = C_{\bar{\gamma}}(t, s) = \overline{C_\gamma(t, s)}. \tag{II.66}$$

Thus C_γ is a hermitian operator. Furthermore, $C_\gamma(t, s)$ is a bounded function on the compact set $[0, \beta] \times [0, \beta]$, so the operator C_γ is Hilbert–Schmidt. In particular C_γ is bounded and self-adjoint. In order to show that C_γ is strictly positive and to compute the upper bound M, we compute the spectrum of C_γ.

We compute the Fourier series for $C_\gamma(\xi)$. The variable ξ lies in the interval $[-\beta, \beta]$ and $C_\gamma(\xi)$ satisfies the twist relation (II.17). Thus

$$C_\gamma(\xi) = \sum_{E \in K_\gamma} \hat{C}_\gamma(E) \, e^{iE\xi} \tag{II.67}$$

with the consequence that

$$\begin{aligned} \langle f, C_\gamma g \rangle &= \int_0^\beta dt \int_0^\beta ds \, \overline{f(t)} \, C_\gamma(t - s) \, g(s) \\ &= \beta^2 \sum_{E \in K_\gamma} \hat{C}_\gamma(E) \, \overline{\hat{f}(E)} \, \hat{g}(E). \end{aligned} \tag{II.68}$$

This identity gives a diagonalization of C_γ, so the spectrum of C_γ is the set of values of

$$\beta \hat{C}_\gamma(E) = \int_0^\beta C_\gamma(\xi)\, e^{-iE\xi}\, d\xi, \tag{II.69}$$

as E ranges over K_γ, and for such E we have

$$\begin{aligned} \frac{1}{1-\gamma} \int_0^\beta e^{-(m+iE)\,\xi}\, d\xi &= \frac{1}{m+iE} \qquad \text{and} \\ \frac{\bar{\gamma}}{1-\bar{\gamma}} \int_0^\beta e^{(m-iE)\,\xi}\, d\xi &= \frac{1}{m-iE}. \end{aligned} \tag{II.70}$$

Hence we infer from (II.26) that

$$\beta \hat{C}_\gamma(E) = \frac{1}{2m} \left(\frac{1}{m+iE} + \frac{1}{m-iE} \right) = \frac{1}{E^2+m^2}. \tag{II.71}$$

The positivity of C_γ follows, as well as the value of $M = \|C_\gamma\|$ in (II.65), namely

$$M = \sup_{E \in K_\gamma} (E^2+m^2)^{-1}. \tag{II.72}$$

Hence the proof of the proposition is complete.

We summarize the last calculation by the statement

PROPOSITION II.5. *The pair correlation function $C_\gamma(t, s)$ has a Fourier series*

$$C_\gamma(t, s) = \frac{1}{\beta} \sum_{E \in K_\gamma} \frac{1}{E^2+m^2}\, e^{iE(t-s)}, \tag{II.73}$$

where K_γ is defined in (II.59).

II.8. *The Oscillator Pair Correlation Function Has a Complex Period*

The pair correlation function $C_\gamma(t, s) = C_\gamma(t-s)$ is a function of the difference variable $\xi = t-s$. We now see that $C_\gamma(\xi)$ has a natural extension to all complex ξ, and this function has a complex period. Let $\Re(\xi)$ and $\Im(\xi)$ denote the real and imaginary parts of ξ, respectively. This extension is neither holomorphic nor anti-holomorphic, but in each strip $n\beta < \Re(\xi) < (n+1)\,\beta$, it is the sum of a holomorphic and an antiholomorphic part. Furthermore, it is continuous, single-valued, and periodic, with a complex period η. The extension also obeys the reflection principle

$$C_\gamma(\xi) = \overline{C_\gamma(-\xi)}. \tag{II.74}$$

For ξ real, the limiting case $\gamma=0$ (obtained from $C_\gamma(\xi)$ as $\beta\to\infty$) equals $(1/2m)\, e^{-m\,|\xi|}$. Let us define the extension of $C_0(\xi)$ to all $\xi\in\mathbb{C}$ by

$$\mathfrak{C}_0(\xi)=\begin{cases}\dfrac{1}{2m}\, e^{m\xi} & \text{if}\quad \Re(\xi)\leqslant 0\\[2ex] \dfrac{1}{2m}\, e^{-m\bar{\xi}} & \text{if}\quad 0\leqslant\Re(\xi).\end{cases} \tag{II.75}$$

Clearly, (II.75) satisfies (II.74). Introduce the complex periodic function

$$\mathfrak{C}_\gamma(\xi)=\sum_{k\in\mathbb{Z}}\mathfrak{C}_0(\xi+k\eta), \tag{II.76}$$

defined for all complex ξ. We choose the period η of $\mathfrak{C}_\gamma(\xi)$ to equal

$$\eta=\beta+i\,\frac{\Omega\theta}{m},\qquad \text{so}\quad \gamma=e^{-m\eta}. \tag{II.77}$$

We now show that the sum (II.76) agrees with the function (II.26) in the domain of the definition of the latter, and thereby defines a natural extension of (II.26) to the entire complex ξ plane.

Proposition II.6. *Let $\mathfrak{C}_\gamma(\xi)$ and η be given by* (II.76) *and* (II.77), *respectively, with $|\gamma|<1$. Then*

(a) *Given $n\in\mathbb{Z}$, let ξ lie in the strip $n\beta\leqslant\Re(\xi)\leqslant(n+1)\,\beta$. We also write $\xi=n\beta+\xi_1$. Then*

$$\mathfrak{C}_\gamma(\xi)=\frac{e^{-in\Omega\theta}}{2m}\left[e^{-m\bar{\xi}_1}\left(\frac{1}{1-\gamma}\right)+e^{m\xi_1}\left(\frac{\bar{\gamma}}{1-\bar{\gamma}}\right)\right]. \tag{II.78}$$

(b) *For real ξ in the interval $-\beta\leqslant\xi\leqslant\beta$, the function $\mathfrak{C}_\gamma(\xi)$ in* (II.78) *agrees with* (II.26).

(c) *The function $\mathfrak{C}_\gamma(\xi)$ in* (II.78) *obeys reflection symmetry* (II.74), *and also satisfies the periodicity relations*

$$\mathfrak{C}_\gamma(\xi+\eta)=\mathfrak{C}_\gamma(\xi)\qquad \textit{and}\qquad \mathfrak{C}_\gamma(\xi+\beta)=e^{-i\Omega\theta}\mathfrak{C}_\gamma(\xi), \tag{II.79}$$

for all $\xi\in\mathbb{C}$.

Proof. The representation (II.78) follows from (II.75) and the definition (II.76). We evaluate the sum over translations $k\eta$, $k\in\mathbb{Z}$, by splitting the sum into the two ranges $-\infty<k\leqslant -n-1$ and $-n\leqslant k<\infty$. Note that the expression (II.78) is single valued. In fact, it is the convergent sum of translations of the single-valued function (II.75). The value of (II.78) with $\Re(\xi) = n\beta$ is

$$\frac{e^{-in\Omega\theta}}{2m}\, e^{im\Im(\xi_1)}\left(\frac{\gamma}{1-\gamma}+\frac{1}{1-\bar{\gamma}}\right)=\frac{e^{-in\Omega\theta}}{2m}\, e^{im\Im(\xi_1)}\left(\frac{1-|\gamma|^2}{|1-\gamma|^2}\right). \tag{II.80}$$

This could be obtained either from the formula (II.78) applied with ξ_1 purely imaginary, or with $\Re(\xi_1) = \beta$. Thus we have verified part (a) of the proposition.

The proof of statement (b) of the proposition merely involves the inspection of the function defined by (II.26). Compare it with (II.78) in the case $-\beta \leqslant \xi \leqslant \beta$ (namely for $n = -1$ yielding $-\beta \leqslant \xi \leqslant 0$ and for $n = 0$ yielding $0 \leqslant \xi \leqslant \beta$). We easily conclude that the two functions agree.

In order to verify the reflection symmetry (II.74), we use this symmetry for $\mathfrak{C}$. Then

$$\begin{aligned}\mathfrak{C}_\gamma(-\xi) &= \sum_{n\in\mathbb{Z}} \mathfrak{C}_0(-\xi+n\eta) = \sum_{n\in\mathbb{Z}} \mathfrak{C}_0(-\xi-n\eta)\\ &= \sum_{n\in\mathbb{Z}} \overline{\mathfrak{C}_0(\xi+n\eta)} = \overline{\mathfrak{C}_\gamma(\xi)},\end{aligned} \tag{II.81}$$

establishing the desired identity. Also, the fact that $\mathfrak{C}_\gamma(\xi)$ is periodic with period η is a consequence of the definition (II.76). Then, for all ξ,

$$\mathfrak{C}_\gamma(\xi+\beta) = \mathfrak{C}_\gamma(\xi+\beta-\eta) = \mathfrak{C}_\gamma\left(\xi - i\frac{\Omega\theta}{m}\right). \tag{II.82}$$

But ξ and $\xi - i\frac{\Omega\theta}{m}$ have the same real part. Hence using (II.78) to evaluate (II.82), we find that translation by $-i\Omega\theta/m$ does not affect n. It only affects the exponential factors in (II.78), and each is multiplied by $e^{-i\Omega\theta}$. This ensures that $\mathfrak{C}_\gamma(\xi+\beta) = e^{-\Omega\theta}\mathfrak{C}_\gamma(\xi)$. This verifies statement (c). We have now checked all parts of Proposition II.6.

III. THE GAUSSIAN PATH-SPACE MEASURE

III.1. *The Twisted Laplacian and Its Green's Operator*

Let $\mathscr{S}_\theta([0,\beta])$ denote the space of C^∞ functions on $[0,\beta]$ that have Fourier representations of the form

$$f(t) = \sum_{E\in K_\gamma} \hat{f}(E)\, e^{i(E-\Omega\theta/\beta)\,t}, \tag{III.1}$$

where the coefficients $\hat{f}(E)$ are rapidly decreasing in E. Such functions are C^∞ and satisfy the twist relation

$$f(t+\beta) = e^{-i\Omega\theta} f(t). \tag{III.2}$$

Endow $\mathscr{S}_\theta([0, \beta])$ with the usual Fréchet topology, given by the countable set of norms

$$\|f\|_n = \sup_{E \in (2\pi/\beta)\,\mathbb{Z}} (1+E^2)^n\, |\hat{f}(E)|, \tag{III.3}$$

and as such $\mathscr{S}_\theta([0, \beta])$ is a nuclear space. The operator d/dt acting on $L^2([0, \beta])$ with domain $\mathscr{S}_\theta([0, \beta])$ is skew-symmetric. Let D_θ denote its closure. The operator $-iD_\theta$ is self-adjoint.[5] Also, the twisted Laplace operator D_θ^2 is the self-adjoint closure of d^2/dt^2 on the domain $\mathscr{S}_\theta([0, \beta])$. The resolvent of the twisted Laplace operator is called the twisted Green's operator. We designate it

$$G_\theta = (-D_\theta^2 + m^2)^{-1}, \tag{III.4}$$

and call $m > 0$ the *mass* using the usual name from physics. As an operator on $\mathscr{S}_\theta([0, \beta])$, or as an operator on $L^2([0, \beta])$, the twisted Green's operator G_θ has an integral kernel $G_\theta(t, s)$ that we denote the twisted Green's function. The twisted Laplace operator is translation invariant, and hence so is the twisted Green's operator. In terms of the Green's function, $G_\theta(t, s) = G_\theta(t - s)$.

THEOREM III.1. *The Green's operator G_θ equals the pair correlation operator C_γ defined in* (II.63),

$$G_\theta = C_\gamma = (-D_\theta^2 + m^2)^{-1}. \tag{III.5}$$

Proof. Identifying the operators C_γ and G_θ is equivalent to identifying their integral kernels, or the Fourier series of these kernels. We computed the Fourier series for the kernel of C_γ in Proposition II.5, and clearly this agrees with the Fourier series for the kernel of G_θ. Thus the operators agree. In addition, the orthonormal basis of functions

$$e_E(t) = \beta^{-1/2} e^{iEt} \in \mathscr{S}_\theta([0, \beta]), \qquad E \in K_\gamma, \tag{III.6}$$

are eigenfunctions of the operator C_γ, corresponding to the eigenvalues $(E^2 + m^2)^{-1}$.

III.2. *The Measure*

Consider the Schwartz space $\mathscr{S}_\theta$ with the topology given by the countable set of norms (III.3). Let $\omega_\theta \in \mathscr{S}'_\theta$ denote a path in the space of distributions dual to $\mathscr{S}_\theta$. Continuous functions $\omega_\theta(t)$ pair with elements of $\mathscr{S}_\theta$ through the bilinear relation

$$\omega_\theta(f) = \int_0^\beta \omega_\theta(t)\, f(t)\, dt, \tag{III.7}$$

[5] Note that d/dt defined on the domain $C_0^\infty([0, \beta])$ of smooth, compactly supported functions, has deficiency indices $(1, 1)$, so d/dt has a one-parameter family of skew-adjoint extensions. Here θ parameterizes this family, as Wightman discussed in his 1964 Cargèse lectures [8].

and this pairing extends by continuity to $\mathscr{S}'_\theta$. The pair correlation operator C_γ maps $\mathscr{S}'_\theta$ continuously into itself. Therefore there is an adjoint map C_γ^+ that is a continuous linear transformation of $\mathscr{S}_\theta$ into itself, defined by

$$(C_\gamma^+ \omega_\theta)(f) = \omega_\theta(C_\gamma f). \tag{III.8}$$

The kernel $C_\gamma^+(t, s) = C_\gamma^+(t-s)$ of C_γ^+ is the complex conjugate of the kernel of C_γ, so using the twist relation of Proposition II.2, we infer

$$C_\gamma^+(t-s+\beta) = e^{i\Omega\theta} C_\gamma^+(t-s). \tag{III.9}$$

As a consequence, continuous elements ω_θ satisfy a twist relation dual to the twist relation (II.58) for functions, namely

$$\omega_\theta(t+\beta) = e^{i\Omega\theta} \omega_\theta(t). \tag{III.10}$$

The identification of C_γ in Theorem III.1 ensures that if

$$|\gamma| \leqslant 1, \qquad \gamma \neq 1, \tag{III.11}$$

then C_γ is a continuous map of $\mathscr{S}_\theta$ to $\mathscr{S}_\theta$ in the Schwartz space topology. Furthermore, acting on $L^2([0, \beta])$, the operator C_γ is strictly positive and has a bounded operator norm. Such a C_γ is an appropriate covariance operator for a Gaussian probability measure on the dual space $\mathscr{S}'_\theta$ of generalized functions. See, for example, Chapter IV of [1], Sections A.3–A.6 of [2].

DEFINITION III.2. Let $|\gamma| \leqslant 1$ and $\gamma \neq 1$. Let $d\mu_\gamma = d\mu_\gamma(\omega_\theta(\cdot))$ denote the Gaussian probability measure on $\mathscr{S}'_\theta$ with vanishing first moments

$$\int \omega_\theta(f)\, d\mu_\gamma = 0 = \int \overline{\omega_\theta(f)}\, d\mu_\gamma, \tag{III.12}$$

and with second moments

$$\int \omega_\theta(f)\, \omega_\theta(g)\, d\mu_\gamma = 0 \qquad \text{and} \qquad \int \overline{\omega_\theta(f)}\, \omega_\theta(g)\, d\mu_\gamma = \langle f, C_\gamma g \rangle_{L^2}. \tag{III.13}$$

The Gaussian recursion relation for moments of $d\mu_\gamma$, along with (III.13), ensure that the moments

$$\int_{\mathscr{S}'_\theta} \overline{\omega_\theta(t_1)\, \omega_\theta(t_2) \cdots \omega_\theta(t_n)}\, \omega_\theta(s_1)\, \omega_\theta(s_2) \cdots \omega_\theta(s_r)\, d\mu_\gamma \tag{III.14}$$

vanish unless $n = r$. As usual, the measure $d\mu_\gamma$ is concentrated on the set of Hölder-continuous functions on $[0, \beta]$ with exponent less than $\frac{1}{2}$. We now prove that the twisted Gibbs functional $\langle \cdot \rangle_\gamma$ of (II.11), applied to coordinates, equals the moments of the measure $d\mu_\gamma$.

PROPOSITION III.3 (Gaussian Feynman–Kac Identity). *Let* $H = H_0$ *and* $|\gamma| < 1$. *Consider the expectation*

$$\langle \cdot \rangle_\gamma = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \cdot e^{-\beta H_0})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* e^{-\beta H_0})}. \tag{III.15}$$

Then expectations of time-ordered products of coordinates, defined in (II.15), *are moments of* $d\mu_\gamma$,

$$\langle (\bar{z}(t_1)\, \bar{z}(t_2) \cdots \bar{z}(t_n)\, z(s_1)\, z(s_2) \cdots z(s_r))_+ \rangle_\gamma$$

$$= \int_{\mathscr{S}'_\theta} \overline{\omega_\theta(t_1)\, \omega_\theta(t_2) \cdots \omega_\theta(t_n)}\, \omega_\theta(s_1)\, \omega_\theta(s_2) \cdots \omega_\theta(s_r)\, d\mu_\gamma. \tag{III.16}$$

Proof. According to Proposition II.3, the left side of (III.16) is a Gaussian functional, and it vanishes unless $n = r$. The right side of (III.16) is Gaussian by definition. Likewise, in Definition III.2 the first moments of the measure $d\mu_\gamma(\omega_\theta(\cdot))$ are defined to vanish. Hence it is sufficient to identify the $n = r = 1$ expectation on the left side of (III.16) with the corresponding second moment on the right side of (III.16). But the $n = 1$ expectations agree by definition, so the proof is complete.

Remark 1. The twisted expectation (III.16) of time-ordered products of coordinates agrees with the moments of the measure $d\mu_\gamma$. So it is natural to use the same notation $\langle \cdot \rangle_\gamma$ for an integral in $\mathscr{S}'_\theta$ and also for the twisted Gibbs functional on the Hilbert space $L^2(\mathbb{C})$. This notation should cause no confusion, so we write

$$\langle F \rangle_\gamma = \int_{\mathscr{S}'_\theta} f(\omega_\theta, \overline{\omega_\theta})\, d\mu_\gamma. \tag{III.17}$$

Remark 2. The measures $d\mu$ are associated with distinct notions of positivity. The first notion is ordinary *positivity as a measure* stating that the integral of a positive function is positive,

$$\int |A|^2\, d\mu_{\beta,\,\theta} \geqslant 0. \tag{III.18}$$

The second notion of positivity expresses the fact that the Gibbs state at zero twist must be positive when evaluated on a positive operator,

$$\langle \hat{A}^* \hat{A} \rangle_{\beta,\,0} \geqslant 0. \tag{III.19}$$

This condition is called *reflection positivity* when it is expressed in terms of the measure. Suppose that the operator $\hat{A}$ denotes a time-ordered product of the coordinates $z(t_j)$ and $\bar{z}(t_j)$, where $\beta/2 \leqslant t_j \leqslant \beta$. Let A denote the same product of $\omega_\theta(t_j)$'s and $\overline{\omega_\theta(t_j)}$'s. Define a linear time reflection operator Θ on paths so that it reflects the time of a path about the midpoint of the time interval $[0, \beta]$. In particular, let $(\Theta\omega_\theta)(t)=\omega_\theta(\beta-t)$. Extend this definition to functions of paths as $(\Theta A)(\omega_\theta)=A(\Theta\omega_\theta)$. Then (III.19) is equivalent to the statement that

$$\langle \hat{A}^*\hat{A}\rangle_{\beta,0}=\int \overline{(\Theta A)(\omega_\theta)}\, A(\omega_\theta)\, d\mu_{\beta,0}\geqslant 0, \tag{III.20}$$

where $\overline{A(\omega_\theta)}$ denotes complex conjugation of the function $A(\omega_\theta)$. For $\theta=0$, the measure $d\mu_{\beta,0}(\omega_{\theta_0})$ satisfies reflection positivity. The reflection positivity of a Gaussian measure is equivalent to reflection positivity of its covariance.[6]

III.3. *Gaussian Integration by Parts and Mass Renormalization*

The measure $d\mu_\gamma$ with covariance $C_\gamma(t,s)$ satisfies an integration by parts formula. Let us consider the path ω_θ and the complex conjugate path $\overline{\omega_\theta}$ as varying independently, and let $F(\omega_\theta,\overline{\omega_\theta})$ denote a functional on $\mathscr{S}'_\theta$. We say that F is differentiable if for $0<t<\beta$ the limits

$$\frac{\partial F}{\partial\omega_\theta(t)}=\lim_{\varepsilon\to 0}\frac{F(\omega_\theta+\varepsilon\delta_t,\overline{\omega_\theta})-F(\omega_\theta,\overline{\omega_\theta})}{\varepsilon}\quad\text{and}$$
$$\frac{\partial F}{\partial\overline{\omega_\theta}(t)}=\lim_{\varepsilon\to 0}\frac{F(\omega_\theta,\overline{\omega_\theta}+\varepsilon\delta_t)-F(\omega_\theta,\overline{\omega_\theta})}{\varepsilon} \tag{III.21}$$

exist, where δ_t denotes the Dirac measure translated by t. We may assume that the limit defines an element of $\mathscr{S}'_\theta$. Since C_γ acts on $\mathscr{S}_\theta$, it also acts on $\mathscr{S}'_\theta$, and we denote this action by

$$\omega_\theta(\cdot)\to(C_\gamma\omega_\theta)(\cdot)=\int_0^\beta C_\gamma(\cdot,t)\,\omega_\theta(t)\,dt. \tag{III.22}$$

Proposition III.4 (Integration by Parts). *Let F denote a differentiable, polynomially bounded function, and let $|\gamma|<1$. Then*

$$\langle\omega_\theta(s)\,F\rangle_\gamma=\int_0^\beta C_\gamma(s,t)\left\langle\frac{\partial F}{\partial\overline{\omega_\theta}(t)}\right\rangle_\gamma dt\qquad\textit{and}$$
$$\langle\overline{\omega_\theta}(t)\,F\rangle_\gamma=\int_0^\beta C_\gamma(s,t)\left\langle\frac{\partial F}{\partial\omega_\theta(s)}\right\rangle_\gamma ds. \tag{III.23}$$

Remark. These identities follow immediately from the properties of Gaussian integrals. The two relations (III.23) are related to each other by complex conjugation. The continuity of the expectations $\langle\partial F/\partial\overline{\omega_\theta}(t)\rangle_\gamma$, etc., at the endpoints of the interval $t\in[0,\beta]$ ensures that the integration can be extended to the endpoint. This is the case if ω_θ is averaged with smooth test functions. In terms of the Feynman–Kac expectations, a special case of integration by parts corresponds to the recursion

[6] For a discussion of reflection positivity in more detail, see [2].

relation (II.35) for the trace functional. In this case we require Dirac measures as test functions. Take $F(\omega_\theta, \overline{\omega_\theta}) = \omega_\theta(\delta_{s_2 - s_1}) \cdots \omega_\theta(\delta_{s_n - s_1})\, \overline{\omega_\theta}(\delta_{t_1 - s_1})\, \overline{\omega_\theta}(\delta_{t_n - s_1})$. Then

$$\langle zT \rangle_\gamma = \langle \omega_\theta(0)\, F \rangle_\gamma = \int_0^\beta C_\gamma(0, t) \left\langle \frac{\partial F}{\partial \overline{\omega_\theta}(t)} \right\rangle_\gamma dt \tag{III.24}$$

generates the relation (II.35).

Given $\varepsilon \geqslant 0$, introduce the measure

$$d\mu_{\gamma, \varepsilon} = \frac{1}{\mathfrak{Z}(\gamma, \varepsilon)}\, e^{-\varepsilon^2 \int_0^\beta |\omega_\theta(s)|^2\, ds}\, d\mu_\gamma, \tag{III.25}$$

where the partition function is

$$\mathfrak{Z}(\gamma, \varepsilon) = \langle e^{-\varepsilon^2 \int_0^\beta |\omega_\theta(s)|^2\, ds} \rangle_\gamma = \int_{\mathscr{S}'_\theta} e^{-\varepsilon^2 \int_0^\beta |\omega_\theta(s)|^2\, ds}\, \mu_\gamma. \tag{III.26}$$

Also define the expectation

$$\langle F \rangle_{\gamma, \varepsilon} = \int_{\mathscr{S}'_\theta} F\, d\mu_{\gamma, \varepsilon}. \tag{III.27}$$

PROPOSITION III.5 (Mass Renormalization). *Let* $\gamma = e^{-m\beta + i\Omega\theta}$, *where* $|\gamma| < 1$, *and let* $\varepsilon > 0$. *Also let* $m' = \sqrt{m^2 + \varepsilon^2}$ *and* $\gamma' = e^{-m'\beta + i\Omega\theta}$. *Then*

$$d\mu_{\gamma, \varepsilon} = d\mu_{\gamma'}. \tag{III.28}$$

Furthermore,

$$\mathfrak{Z}(\gamma, \varepsilon) = \frac{|1 - \gamma|^2}{|1 - \gamma'|^2}\, e^{\beta(m - m')}. \tag{III.29}$$

Proof. The measures $d\mu_{\gamma, \varepsilon}$ and $d\mu_{\gamma'}$ are both Gaussian probability measures with mean zero. Therefore the measures agree if and only if they have the same covariance matrix. We use the integration by parts identity (III.23) to prove this fact. Start from this identity applied to $\langle \overline{\omega_\theta(t)}\, \omega_\theta(s) \rangle_{\gamma, \varepsilon'}$ yielding

$$\langle \overline{\omega_\theta(t)}\, \omega_\theta(s) \rangle_{\gamma, \varepsilon} = C_\gamma(t, s) - \varepsilon^2 \int_0^\beta C_\gamma(t, u) \langle \overline{\omega_\theta(u)}\, \omega_\theta(s) \rangle_{\gamma, \varepsilon}\, du. \tag{III.30}$$

This can also be written

$$(I + \varepsilon^2 C_\gamma) \langle \overline{\omega_\theta(\cdot)}\, \omega_\theta(s) \rangle_{\gamma, \varepsilon} = C_\gamma(\cdot, s), \tag{III.31}$$

with $C_\gamma = (-D_\theta^2 + m^2)^{-1}$. The solution to this linear equation is the covariance $C_{\gamma,\varepsilon}$ of $d\mu_{\gamma,\varepsilon}$ with the kernel $C_{\gamma,\varepsilon}(t,s) = \int_{\mathscr{S}'_\theta} \overline{\omega_\theta(t)}\, \omega_\theta(s)\, d\mu_{\gamma,\varepsilon} = \langle \overline{\omega_\theta(t)}\, \omega_\theta(s) \rangle_{\gamma,\varepsilon}$. Then

$$\begin{aligned} C_{\gamma,\varepsilon} &= (I + \varepsilon^2 C_\gamma)^{-1} C_\gamma = (C_\gamma^{-1} + \varepsilon^2)^{-1} \\ &= (-D_\theta^2 + m^2 + \varepsilon^2)^{-1} = C_{\gamma'}, \end{aligned} \tag{III.32}$$

as claimed.

As a consequence of the mass renormalization formula, and with H given in (II.4), we have the identity

$$\mathfrak{Z}(\gamma,\varepsilon) = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta(H + \varepsilon^2 |z|^2)})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H})}. \tag{III.33}$$

Furthermore by Proposition II.1,

$$\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta(-\partial\bar\partial + m'^2 |z|^2 - m')}) = \frac{1}{|1-\gamma'|^2}, \tag{III.34}$$

so (III.29) follows.

III.4. *Gaussian Normal Ordering of Operator Products*

Each Gaussian functional $\langle \cdot \rangle_\gamma$ on coordinates $z(t)$ and $\bar z(s)$ defines a "normal ordering" of polynomials in the coordinates. If this functional is the expectation on the zero-particle state in $\mathscr{H}$, namely for the case $q = 0$, the normal-ordered product coincides with the Wick-ordered product. We denote the normal ordering of $\bar z(t)^r z(t)^s$ by $\vdots \bar z(t)^r z(t)^s \vdots_\gamma$. Define

$$c_\gamma = C_\gamma(0) > 0, \tag{III.35}$$

where the positivity is a consequence of (II.27). Let

$$\vdots \bar z(t)^r z(t)^s \vdots_\gamma = \sum_{j=0}^{\min\{r,s\}} (-1)^j \binom{r}{j}\binom{s}{j} j!\, c_\gamma^j \bar z(t)^{r-j} z(t)^{s-j}. \tag{III.36}$$

In particular, a few of these normal-ordered monomials are

$$\vdots z(t)^n \vdots_\gamma = z(t)^n, \qquad \vdots \bar z(t)^n \vdots_\gamma = \bar z(t)^n, \tag{III.37}$$

$$\vdots |z(t)|^2 \vdots_\gamma = |z(t)|^2 - c_\gamma, \tag{III.38}$$

$$\vdots |z(t)|^4 \vdots_\gamma = |z(t)|^4 - 4c_\gamma |z(t)|^2 + 2c_\gamma^2, \tag{III.39}$$

$$\vdots |z(t)|^6 \vdots_\gamma = |z(t)|^6 - 9c_\gamma |z(t)|^4 + 18c_\gamma^2 |z(t)|^2 - 6c_\gamma^3. \tag{III.40}$$

Note that $\binom{r}{j}^2 j! = (\binom{r}{j} j!)^2 \frac{1}{j!}$, so another expression for the normal-ordered monomial is

$$\vdots \bar z^r z^s \vdots_\gamma = e^{-c_\gamma \partial\bar\partial} \bar z^r z^s. \tag{III.41}$$

The latter definition preserves its form when extended linearly to a general class of functions of z and $\bar{z}$ at equal time. For $\gamma \neq 0$, the constant c_γ has a limit as $m \to 0$ with θ fixed (see below); on the other hand, the no-particle expectation of $|z|^2$ diverges. For this reason, we conclude that the qualitative behavior of normal-ordered monomials depends on whether $\gamma = 0$. We are interested in certain aspects of the infrared ($\gamma \to 0$) limit.

A principal property of these normal-ordered monomials is their orthogonality in the sense that

$$\langle \vdots \bar{z}(t)^r z(t)^s \vdots_\gamma \vdots \bar{z}(t')^{r'} z(t')^{s'} \vdots_\gamma \rangle_\gamma = \delta_{rs'} \delta_{sr'} r!\, s!\, C_\gamma(t-t')^r\, C_\gamma(t'-t)^s, \quad \text{(III.42)}$$

and in particular

$$\langle \vdots \bar{z}(t)^r z(t)^s \vdots_\gamma \rangle_\gamma = 0. \quad \text{(III.43)}$$

Also for $r = s$ and $r' = s'$,

$$\langle \vdots |z(t)|^{2r} \vdots_\gamma \vdots |z(t')|^{2r'} \vdots_\gamma \rangle_\gamma = \delta_{rr'} r!^2\, |C_\gamma(t-t')|^{2r}. \quad \text{(III.44)}$$

III.5. *Gaussian Normal Ordering on Path Space*

There is an equivalent notion of Gaussian normal ordering of functions of paths $\omega_\theta(t)$ on $\mathscr{S}'_\theta$. In this case, normal ordering of a monomial with respect to the measure $d\mu_\gamma$ arises as the orthogonalization of the monomial to polynomials of lower degree in ω_θ or $\overline{\omega_\theta}$. This leads to polynomials in ω_θ and $\overline{\omega_\theta}$. In particular, with the definition

$$\Delta_{\omega\theta} = \int_{0 \leqslant t,\, s \leqslant \beta} C_\gamma(t-s)\, \frac{\partial}{\partial \omega_\theta(t)}\, \frac{\partial}{\partial \overline{\omega_\theta}(s)}\, dt\, ds, \quad \text{(III.45)}$$

we can define

$$\vdots F(\omega_\theta, \overline{\omega_\theta}) \vdots_\gamma = e^{-\Delta_{\omega\theta}} F(\omega_\theta, \overline{\omega_\theta}). \quad \text{(III.46)}$$

In particular, a few of these normal-ordered monomials are

$$\vdots \omega_\theta(t)^n \vdots_\gamma = \omega_\theta(t)^n, \qquad \vdots \overline{\omega_\theta}(t)^n \vdots_\gamma = \overline{\omega_\theta}(t)^n, \quad \text{(III.47)}$$

$$\vdots |\omega_\theta(t)|^2 \vdots_\gamma = |\omega_\theta(t)|^2 - c_\gamma, \quad \text{(III.48)}$$

$$\vdots |\omega_\theta(t)|^4 \vdots_\gamma = |\omega_\theta(t)|^4 - 4c_\gamma\, |\omega_\theta(t)|^2 + 2c_\gamma^2, \quad \text{(III.49)}$$

$$\vdots |\omega_\theta(t)|^6 \vdots_\gamma = |\omega_\theta(t)|^6 - 9c_\gamma\, |\omega_\theta(t)|^4 + 18c_\gamma^2\, |\omega_\theta(t)|^2 - 6c_\gamma^3, \quad \text{(III.50)}$$

etc. The integrals of these polynomials satisfy

$$\langle \vdots \overline{\omega_\theta}(t)^r\, \omega_\theta(t)^s \vdots_\gamma \vdots \overline{\omega_\theta}(t')^{r'}\, \omega_\theta(t')^{s'} \vdots_\gamma \rangle_\gamma$$
$$= \delta_{rs'} \delta_{sr'} r!\, s!\, C_\gamma(t-t')^r\, C_\gamma(t'-t)^s. \quad \text{(III.51)}$$

III.6. *The Zero-Mass Limit* ($m \to 0$)

We consider here the $m \to 0$ limit with the inverse temperature β fixed. In terms of γ this entails the radial limit $|\gamma| \to 1$. Neither the expectation (II.11), nor the coordinates (II.3) are defined in this limit. However, the measure $d\mu_\gamma$ is well-defined. The following result, combined with the identity of Proposition III.3, allows us to extend the functional (II.11) to the case $m = 0$. Recall the definition of Υ_{sing} in (I.16), as well as the normalizing factor $\mathfrak{Z}_\gamma$ given in (II.11).

PROPOSITION III.6. *Let* $\gamma = e^{-m\beta + i\Omega\theta}$, *and* $|\gamma| < 1$.

(a) *There is a constant M, independent of m, β, θ such that the partition function* $\mathfrak{Z}_\gamma$ *satisfies*

$$0 < \mathfrak{Z}_\gamma \leqslant \frac{M}{\beta^2 m^2}. \tag{III.52}$$

(b) *If* $\theta \notin \Upsilon_{\text{sing}}$, *then the measures* $d\mu_\gamma(\omega_\theta(\cdot))$ *converge weakly as measures on* $\mathcal{S}'_\theta$ *as* $m \to 0$ *with* β, θ *fixed.*

(c) *The expectations* $\langle (\bar{z}(t_1)\, \bar{z}(t_2) \cdots \bar{z}(t_n)\, z(s_1)\, z(s_2) \cdots z(s_m))_+ \rangle_\gamma$ *converge as* $m \to 0$ *with* β, θ *fixed.*

Remark 1. For a particular phase $e^{i\Omega\theta}$ the nuclear space $\mathcal{S}_\theta$ is independent of m. Thus we can formulate continuity and convergence of the measures $d\mu_\gamma(\omega_\theta(\cdot))$ on a fixed space $\mathcal{S}'_\theta$. If we allow the phase $e^{i\Omega\theta}$ of γ to vary, then we must formulate a more general notion of continuity and convergence involving a family of measures on a family of spaces. In this broader context, convergence of moments, satisfying some uniform bound, provides a natural framework. We investigated such convergence in a related problem some time ago [5], but we do not discuss these questions further here.

Remark 2. Let $m = 0$, and let $\gamma = e^{i\Omega\theta} \neq 1$. By comparing Proposition II.3, Proposition II.5, and (II.29), we evaluate the kernel of C_γ as

$$C_\gamma(t, s) = \frac{\beta}{4 \sin^2(\Omega\theta/2)}. \tag{III.53}$$

In fact, (III.53) is independent of t and s, so

$$\langle f, C_\gamma g \rangle_{L^2} = \frac{\beta^3}{4 \sin^2(\frac{\Omega\theta}{2})} \overline{\hat{f}(0)}\, \hat{g}(0). \tag{III.54}$$

Proof. The bound (III.52) follows from the explicit form of $\mathfrak{Z}_\gamma$ established in Proposition II.1, as well as the bound $|1 - \gamma| > \text{const.}\, \beta m$. The bound of part (a) follows.

For fixed $\Omega\theta$ the space $\mathcal{S}'_\theta$ remains fixed. Gelfand and Vilenkin [1] show in Section IV.4.2 that weak convergence of Gaussian probability measures on a fixed nuclear space is equivalent to convergence of the first and second moments as operators on the nuclear space. In the $m \to 0$ limit that we consider here, the eigenfunctions of C_γ remain fixed, while the eigenvalues converge as $m \to 0$,

$$\beta \hat{C}_\gamma(E) = \frac{1}{E^2 + m^2} \to \frac{1}{E^2}. \tag{III.55}$$

For $\theta \in (0, 2\pi/\Omega)$ and bounded away from the endpoints of the interval, E is bounded away from zero. Hence the operator C_γ is norm convergent as $m \to 0$.

III.7. *The Zero-Temperature Limit* $(\gamma \to 0)$

The massive zero-temperature limit is $\beta \to \infty$ with $m > 0$ fixed, as β denotes the inverse temperature. In terms of γ, this is the limit $\gamma \to 0$. The limiting covariance is just the vacuum expectation value

$$\lim_{\gamma \to 0} C_\gamma(\xi) = C_0(\xi) = \frac{1}{2m} e^{-m|\xi|}, \tag{III.56}$$

and it does not depend on the twist angle θ. Interestingly, the vacuum expectation covariance (III.56) has no zero-mass limit, and this is one aspect of the "infrared" problem alluded to in Section I.

There is a different normalization of the coordinate that corresponds to the standard normalization used in the discussion of the zero-momentum mode of a quantized string. Let

$$z_{\text{bare}} = m^{1/2} z = \frac{1}{\sqrt{2}} (a_+^* + a_-), \tag{III.57}$$

denoting the coordinate similar to (II.3) but without the scaling by $m^{-1/2}$. Then the twisted expectation of the time-ordered product of coordinates z_{bare} and their conjugates do have a zero-mass limit, determined by

$$\lim_{\gamma \to 0} C_\gamma^{\text{bare}}(\xi) = \lim_{\gamma \to 0} \langle (\bar{z}_{\text{bare}}(t)\, z_{\text{bare}}(s))_+ \rangle = C_0^{\text{bare}}(\xi) = \tfrac{1}{2} e^{-m|\xi|}. \tag{III.58}$$

Finally, we remark that the $m \to 0$ and the $\gamma \to 0$ limits cannot be interchanged. The $m \to 0$ limit of the pair correlation function (III.53) has no $\beta \to \infty$ limit. Likewise, the $\gamma \to 0$ (or $\beta \to \infty$) limit (III.56) has no $m \to 0$ limit.

IV. THE *N*-COMPONENT OSCILLATOR

The oscillator treated above has a straight-forward generalization to an n-component oscillator with the coordinate taking values in $\mathbb{C}^n$. Likewise, there is a probability measure on vector-valued paths, whose integrals give rise to twisted Gibbs expectations with respect to the n-component oscillator. We give a brief outline of these properties here.

IV.1. *Complex Operator Coordinates*

Let $z=\{z_1, z_2, ..., z_n\} \in \mathbb{C}^n$ denote this coordinate. We may express z in terms of $2n$ independent annihilation operators $a_{j\pm}$, $j=1, ..., n$, and their adjoints,

$$z_j = \frac{1}{\sqrt{2m}} (a_{j+}^* + a_{j-}) \qquad \text{and} \qquad \partial_j = \sqrt{\frac{m}{2}} (a_{j+} - a_{j-}^*). \tag{IV.1}$$

Introduce the mutually commuting number operators $N_+ = \sum_{j=1}^n a_{j+}^* a_{j+}$ and $N_- = \sum_{j=1}^n a_{j-}^* a_{j-}$. We express the Hamiltonian $H=H_0$ and the twist generator J as

$$H_0 = \sum_{j=1}^n m(N_{j+} + N_{j-}) \qquad \text{and} \qquad J = \sum_{j=1}^n \Omega_j (N_{j+} - N_{j-}). \tag{IV.2}$$

We only consider equal masses here, but we allow for different twist weights Ω_j for the twist in each coordinate direction. Then $U(\theta) = e^{iJ\theta}$ generates the transformation

$$\begin{aligned} z_j \to U(\theta)\, z_j\, U(\theta)^* &= e^{i\Omega_j \theta} z_j, \qquad \text{which we also write as} \\ U(\theta)\, z\, U(\theta)^* &= e^{i\Omega\theta} z. \end{aligned} \tag{IV.3}$$

Likewise,

$$\partial_j \to U(\theta)\, \partial_j\, U(\theta)^* = e^{-i\Omega_j\theta} \partial_j. \tag{IV.4}$$

The corresponding expectations $\langle \cdot \rangle_\gamma$ on $L^2(\mathbb{C}^n, d^n z)$ depend on an n-component parameter $\gamma = \{\gamma_1, ..., \gamma_n\}$, with $\gamma_j = e^{-m\beta + i\Omega_j\theta}$. Because of Proposition III.6a, the partition function satisfies

$$0 < \mathfrak{Z}_\gamma \leqslant \frac{M^n}{m^{2n}}, \tag{IV.5}$$

where M is a constant independent of m, n, β, θ. The expectation of one coordinate vanishes, and the expectation of two coordinates equals

$$C_\gamma(t, s) = \langle \bar{z}(t)\, z(s) \rangle_\gamma, \tag{IV.6}$$

with $C_\gamma(t, s) = C_\gamma(\xi)$ a diagonal matrix, depending on the difference variable $\xi = t - s$. The entries of this matrix are $\{C_\gamma(\xi)_{ij}\}$, where $1 \leqslant i, j \leqslant n$, and they equal

$$C_\gamma(\xi)_{ij} = \langle \bar{z}_i(t)\, z_j(s) \rangle_\gamma = \delta_{ij} C_{\gamma_j}(\xi). \tag{IV.7}$$

Furthermore,

$$C_\gamma(\xi+\beta)_{ij}=\delta_{ij}e^{-i\Omega_j\theta}C_{\gamma_j}(\xi). \tag{IV.8}$$

Define the constant c_j as the jth-component of the covariance on the diagonal, namely

$$c_j=C_{\gamma_j}(0), \tag{IV.9}$$

and let

$$\mathfrak{c}=\sum_{j=1}^{n}c_j. \tag{IV.10}$$

More generally, let

$$\mathfrak{c}^{(p)}=\sum_{j=1}^{n}c_j^p. \tag{IV.11}$$

IV.2. *Normal Ordering*

We revisit the combinatorics of normal ordering in the case of the multi-component oscillator. Let the number of components be fixed at n. Introduce the Laplacian on $L^2(\mathbb{C}^n)$ as

$$\Delta_c=\sum_{j=1}^{n}c_j\,\partial_j\bar{\partial}_j. \tag{IV.12}$$

Then define normal ordering of a function $P(z,\bar{z})$ on $L^2(\mathbb{C}^n)$ by the Laplacian Δ_c acting on P as

$$\vdots P(z,\bar{z})\vdots_\gamma=e^{-\Delta_c}P(z,\bar{z}). \tag{IV.13}$$

As some particular examples, the following polynomials in $|z|^2=\sum_{j=1}^n|z_j|^2$ are normal ordered:

$$\vdots 1\vdots_\gamma=1, \tag{IV.14}$$

$$\vdots|z|^2\vdots_\gamma=|z|^2-\mathfrak{c}, \tag{IV.15}$$

$$\vdots|z|^4\vdots_\gamma=(|z|^2)^2-2\mathfrak{c}\,|z|^2+\mathfrak{c}^2-2\sum_{j=1}^{n}c_j\,|z_j|^2+\mathfrak{c}^{(2)}. \tag{IV.16}$$

If all the γ_j are equal, then $c_j=c$ and these expressions simplify to

$$\vdots|z|^2\vdots_\gamma=|z|^2-nc \tag{IV.17}$$

and

$$\vdots|z|^4\vdots_\gamma=|z|^4-2(n+1)\,c\,|z|^2+n(n+1)\,c^2. \tag{IV.18}$$

These normal-ordered monomials have the orthogonality property

$$\langle :|z(t)|^{2k}:_{\gamma}\, :|z(s)|^{2k'}:_{\gamma}\rangle_{\gamma}=\delta_{k,k'}(k!)^2 \sum_{k_1+k_2+\cdots+k_n=k}\ \prod_{j=1}^{n} |C_{\gamma_j}(t-s)|^{2k_j}, \qquad \text{(IV.19)}$$

where $k_j \in \mathbb{Z}_+$ ranges over non-negative integers. In particular (IV.17) and (IV.18) reduce in the case $n=1$ and $k=1, 2$ to (III.38) and (III.39), and (IV.19) reduces in the case of equal γ_j's to (III.44).

The path space for the n-component oscillator consists of vector-valued paths $\omega_\theta(t)=\{\omega_{j,\theta}\}$, which are parameterized by the set of numbers $\gamma_j=e^{-m\beta+i\Omega_j\theta}$. Thus $\omega_\theta\colon \mathbb{R}\to \mathcal{S}'_\theta\oplus\mathcal{S}'_\theta\oplus\cdots\oplus\mathcal{S}'_\theta$, with the implicit notation that the jth component $\omega_{j,\theta}(t)$ of the path satisfies the periodicity condition

$$\omega_{j,\theta}(t+\beta)=e^{i\Omega_j\theta}\omega_{j,\theta}(t). \qquad \text{(IV.20)}$$

The measure on $\mathcal{S}'_\theta\oplus\mathcal{S}'_\theta\oplus\cdots\oplus\mathcal{S}'_\theta$ is the product measure

$$d\mu_\gamma(\omega_\gamma(\cdot))=\prod_{j=1}^{n} d\mu_{\gamma_j}(\omega_{j,\theta}(\cdot)), \qquad \text{(IV.21)}$$

and integrals with respect to this measure are related to the expectations in the twisted Gibbs functional $\langle\cdot\rangle_\gamma$ by a Feynman–Kac formula similar to (I.4) and (III.16).

V. NON-GAUSSIAN FUNCTIONALS AND NON-GAUSSIAN MEASURES

The twisted Gibbs expectations in Sections II–IV are Gaussian. They arise from harmonic oscillators, and they are characterized by a linear equation of motion. In this section we construct non-Gaussian Feynman–Kac probability measures providing Feynman–Kac representations for various twisted, non-Gaussian functionals. These examples arise from non-linear, twist-positive quantum-mechanical systems $\{H, U(\theta), \mathfrak{H}\}$. The twist-invariant Hamiltonians $H=H_0+V$ are perturbations of the Hamiltonian H_0, the Hamiltonian of a massive, n-component harmonic oscillator. We take the perturbation $V=V(z,\bar z)$ to be a twist-invariant, multiplication operator on $\mathbb{C}^n$. In the subsections that follow, we detail our assumptions on V, establish twist positivity, and establish the Feynman–Kac representation. For simplicity, we study polynomial potentials. We also investigate some aspects of the $m\to 0$ limit.

V.1. *Allowed Potentials*

Definition V.1. (a) The potential function $V(z,\bar z)$ is allowed if it is a real polynomial in z and $\bar z$ that is bounded from below and twist-invariant, namely

$$U(\theta)\ V(z,\bar z)\ U(\theta)^* = V(e^{i\Omega_j\theta}z_j, e^{-i\Omega_j\theta}\bar z_j)=V(z,\bar z). \qquad \text{(V.1)}$$

(b) The potential V is elliptic, if there are positive constants $M_1, M_2 < \infty$ such that

$$|z|^2 \leqslant M_1(V(z, \bar{z}) + M_2), \tag{V.2}$$

where $|z|^2$ denotes $\sum_{j=1}^{n} |z_j|^2$.

(c) A potential V is infrared regular if there are positive constants $M_1, M_2 < \infty$ such that the Laplacian of V satisfies

$$\left| \sum_{j=1}^{n} \frac{\partial^2 V(z, \bar{z})}{\partial z_j \, \partial \bar{z}_j} \right| \leqslant M_1(V(z, \bar{z}) + M_2). \tag{V.3}$$

In the Introduction, we mentioned some examples of acceptable potential functions. Clearly $|z|^2$ is twist invariant, so any polynomial function of $|z|^2$ that is bounded from below is acceptable. Another example mentioned in the Introduction arose from the absolute square of a holomorphic, quasihomogeneous polynomial W. A holomorphic polynomial $W(z)$ is quasihomogeneous with positive weights $\{\Omega_j\}$ for the coordinates $\{z_j\}$ if

$$W(z) = \sum_{j=1}^{n} \Omega_j z_j \frac{\partial W(z)}{\partial z_j}. \tag{V.4}$$

The identity (V.4) is the infinitesimal form of the relation

$$W(\{e^{i\Omega_j\theta} z_j\}) = e^{i\theta} W(\{z_j\}). \tag{V.5}$$

In other words, quasihomogeneity with weights $\{\Omega_j\}$ means that under twisting by $U(\theta)$ (defined with weights $\{\Omega_j\}$),

$$U(\theta) \, W U(\theta)^* = e^{i\theta} W. \tag{V.6}$$

A homogeneous, holomorphic polynomial has equal weights Ω_j, equal to the inverse of the degree of the polynomial. The absolute square $|W(z)|^2$ of a holomorphic, quasihomogeneous polynomial $W(z)$ is twist-invariant. Furthermore, if $W(z)$ is holomorphic and quasihomogeneous, then the kth component of its gradient $\partial W/\partial z_k$ is also quasihomogeneous with weights $\{\Omega_j(1-\Omega_k)^{-1}\}$. In fact

$$U(\theta) \frac{\partial W(z)}{\partial z_k} U(\theta)^* = e^{i(1-\Omega_k)\theta} \frac{\partial W(z)}{\partial z_k}, \tag{V.7}$$

yielding the claimed quasihomogeneity. Thus the absolute square of the gradient of a holomorphic, quasihomogeneous polynomial

$$V(z, \bar{z}) = \sum_{j=1}^{n} \left| \frac{\partial W(z)}{\partial z_j} \right|^2 \tag{V.8}$$

is twist-invariant.

Let us give some examples of holomorphic, quasihomogeneous polynomials $W(z)$ and some potentials $V(z, \bar{z})$. For instance, choose $W(z)$ to be a sum of monomials in the individual coordinates,

$$W(z) = \sum_{j=1}^{n} c_j \frac{z_j^{n_j}}{n_j}, \qquad \text{where} \quad 1 \leqslant n_j \in \mathbb{Z}. \tag{V.9}$$

In this case, W is holomorphic and quasihomogeneous with weights $\Omega_j = 1/n_j$. The squared gradient $V = \sum_{j=1}^{n} |\partial W(z)/\partial z_j|^2$ has the form

$$V(z, \bar{z}) = \sum_{j=1}^{n} |c_j z_j^{n_j - 1}|^2. \tag{V.10}$$

A second example is

$$W(z_1, z_2) = z_1^k + z_1 z_2^l, \tag{V.11}$$

which is quasihomogeneous with weights $\Omega_1 = 1/k$ and $\Omega_2 = (k-1)/kl$. In this case the gradient squared has the form

$$V(z, \bar{z}) = |k z_1^{k-1} + z_2^l|^2 + l^2 \, |z_1|^2 \, |z_2|^{2(l-1)}. \tag{V.12}$$

V.2. *Hamiltonians and the Trotter Product Formula*

Let $\mathscr{D} \subset \mathfrak{H}$ denote the domain of C^∞ functions of z and $\bar{z}$. We say that the Hamiltonian $H = H_0 + V$, with V an allowed potential, is an *allowed Hamiltonian*, and it has the form

$$H = -\partial\bar{\partial} + V_1(z, \bar{z}), \qquad \text{where} \quad V_1(z, \bar{z}) = m^2 \, |z|^2 + V(z, \bar{z}) - nm. \tag{V.13}$$

Since V is bounded below, it follows that V_1 is elliptic. Such Hamiltonians H are known to be essentially self-adjoint, and to have trace-class heat kernels $e^{-\beta H}$, for $\beta > 0$.

The Trotter product representation is a form of the Lie formula for semigroups such as $e^{-\beta H}$ with unbounded generators,

$$e^{-\beta H} = \text{st.} \lim_{N \to \infty} \left(e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N} \right)^N. \tag{V.14}$$

For operators H_0 and V that are self-adjoint and bounded from below, and such that $H = H_0 + V$ is essentially self-adjoint, convergence of the Lie–Trotter product formula (V.14) is standard; see, for example, Theorem A.5.1 of [2]. We require a variation.

Proposition V.2. *Let H be an allowed Hamiltonian. Then the Trotter product representation holds in the form*

$$\lim_{N \to \infty} \operatorname{Tr}_{\mathfrak{H}}\left(U(\theta)^* \left(e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N} \right)^N \right) = \operatorname{Tr}_{\mathfrak{H}}\left(U(\theta)^* \, e^{-\beta H} \right). \tag{V.15}$$

Remark. The important point is that we can interchange the order of the limit over N and the trace as claimed in (V.15). Let us define $T_N = e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N}$. Clearly T_N is self-adjoint and positive. It is no loss of generality to add a constant to V so that $0 \leqslant V$, and we make this assumption in proving the proposition. Therefore

$$0 \leqslant T_N \leqslant e^{-\beta H_0/N}, \tag{V.16}$$

so T_N is bounded from above by the contraction semigroup generated by H_0. Rewrite, (V.14) as

$$e^{-\beta H} = \text{st.} \lim_{N \to \infty} (T_N)^N. \tag{V.17}$$

and (V.15) as

$$\text{Tr}_{\mathfrak{H}}(U(\theta)^* e^{-\beta H}) = \lim_{N \to \infty} \text{Tr}_{\mathfrak{H}}(U(\theta)^* (T_N)^N). \tag{V.18}$$

Proof. Let us denote the jth eigenvalue of T_N by $e^{-\lambda_{N,j}/N}$, where we order the eigenvalues so that $\lambda_{N,j}$ is an increasing function of j. Then the eigenvalues of T_N^N are $e^{-\lambda_{N,j}}$. Likewise, denote the eigenvalues of $e^{-\beta H_0}$ by $e^{-\lambda_j}$, with λ_j increasing. From the inequality (V.16) and the minimax principle, we infer that $\lambda_j \leqslant \lambda_{N,j}$. Furthermore, from the explicit form of the eigenvalues of H_0 with mass m and for $z \in \mathbb{C}^n$, there is a constant $M > 0$ such that $\lambda_j \geqslant Mmj^{1/2n}$. Hence we conclude that there is a minimum rate of decay for the jth eigenvalue of T_N^N, namely

$$e^{-\lambda_{N,j}} \leqslant e^{-\lambda_j} \leqslant e^{Mmj^{1/2n}}. \tag{V.19}$$

As a consequence, given $\varepsilon > 0$, there exists $J = J(\varepsilon) < \infty$, independent of N, such that for every N,

$$\sum_{j=J}^{\infty} e^{-\lambda_{N,j}} < \tfrac{1}{3}\varepsilon. \tag{V.20}$$

Observe that $U(\theta)^* T_N = T_N U(\theta)^*$, so $U(\theta)^*$ can be diagonalized simultaneously with each T_N^N. Let $e^{i\delta_{N,j} - \lambda_{N,j}}$ denote the spectrum of $U(\theta)\, T_N^N$ in the corresponding orthonormal basis $|N, j\rangle$. We compute the trace of $U(\theta)^*\, T_N^N$ in this basis, and the trace of $U(\theta)^* e^{-\beta H}$ in its basis $|j\rangle$ corresponding to eigenvalues $e^{i\delta_j - \lambda_j}$. Thus using (V.20), we have

$$\begin{aligned} &|\text{Tr}_{\mathfrak{H}}(U(\theta)^* e^{-\beta H}) - \text{Tr}_{\mathfrak{H}}(U(\theta)^* (e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N)| \\ &\quad = |\text{Tr}_{\mathfrak{H}}(U(\theta)^*\, T_N^N) - \text{Tr}_{\mathfrak{H}}(U(\theta)^* e^{-\beta H})| \\ &\quad \leqslant \tfrac{2}{3}\varepsilon + \left| \sum_{j=0}^{J} (e^{i\delta_{N,j} - \lambda_{N,j}} - e^{i\delta_j - \lambda_j}) \right|. \end{aligned} \tag{V.21}$$

Since T_N^N converges strongly to $e^{-\beta H}$ as $N \to \infty$, we can use standard contour integral methods to establish that each eigenvector $|N, j\rangle$ converges strongly to the eigenvector $|j\rangle$ for the limiting operator. It follows that each eigenvalue $e^{i\delta_{N,j} - \lambda_{N,j}}$ converges to $e^{i\delta_j - \lambda_j}$. Hence the finite number of eigenvalues in the sum on the right side of (V.21) converge as $N \to \infty$, uniformly for $0 \leqslant j \leqslant J$. We therefore may choose $N_0(\varepsilon)$ so that for $N > N_0(\varepsilon)$,

$$\left| \sum_{j=0}^{J} \left(e^{i\delta_{N,j} - \lambda_{N,j}} - e^{i\delta_j - \lambda_j} \right) \right| \leqslant \tfrac{1}{3}\, \varepsilon. \tag{V.22}$$

We conclude that we have shown that given $\varepsilon > 0$, there exists N_0 such that for $N > N_0(\varepsilon)$,

$$|\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H}) - \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N)| < \varepsilon. \tag{V.23}$$

This completes the proof of the proposition.

Note that in the notation (II.15) of products that are time-ordered with respect to the action of the semigroup $e^{-\beta H_0}$, we could also write

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, T_N^N) \\ &\quad = \mathrm{Tr}_{\mathfrak{H}} \left(U(\theta)^* \left(\prod_{j=1}^{N} e^{-(\beta/N)\, V(z((j-1/2)\, \beta/N),\, \bar{z}((j-1/2)\, \beta/N))} \right)_+ e^{-\beta H_0} \right). \end{aligned} \tag{V.24}$$

Thus we can interpret the convergence (V.13) as a convergence of expectations of time-ordered products. We need a more general form of convergence of time-ordered products.

Proposition V.3. *Let H be an allowed Hamiltonian. Then*

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (\bar{z}(t_1) \cdots \bar{z}(t_r)\, z(s_1) \cdots z(s_{r'}))_+\, e^{-\beta H}) \\ &\quad = \lim_{N \to \infty} \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (\bar{z}(t_1) \cdots \bar{z}(t_r)\, z(s_1) \cdots z(s_{r'})\, T_N^N)_+) \\ &\quad = \lim_{N \to \infty} \mathrm{Tr}_{\mathfrak{H}} \Bigg(U(\theta)^* \Bigg(\bar{z}(t_1) \cdots \bar{z}(t_r)\, z(s_1) \cdots z(s_{r'}) \\ &\qquad \times \left(\prod_{j=1}^{N} e^{-(\beta/N)\, V(z((j-1/2)\, \beta/N),\, \bar{z}((j-1/2)\, \beta/N))} \right) \Bigg)_+ e^{-\beta H_0} \Bigg), \end{aligned} \tag{V.25}$$

where time ordering in the first term of (V.25) *is defined by $e^{-\beta H}$, while in the other two terms it is defined by $e^{-\beta H_0}$.*

The proof of this proposition is a variation on the proof of Proposition V.2, so we omit the details.

V.3. *Non-Gaussian Feynman–Kac Representations*

With V equal to one of our allowed potentials, consider $H = H_0 + V$. In general the corresponding twisted Gibbs functional (I.2) is non-Gaussian. Recall our definitions of the twisted partition function and the twisted relative partition function,

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H_0}) \qquad \text{and} \qquad \mathfrak{Z}_\gamma^V = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H_0})}. \tag{V.26}$$

From Proposition II.1, generalized to the case $z \in \mathbb{C}^n$,

$$\mathfrak{Z}_\gamma = \prod_{j=1}^{n} |1 - \gamma_j|^{-2}. \tag{V.27}$$

PROPOSITION V.4. *Let $H = H_0 + V$. Then*

(a) *The twisted trace of $e^{-\beta H}$ has the representation*

$$\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H}) = \mathfrak{Z}_\gamma^V \mathfrak{Z}_\gamma, \tag{V.28}$$

where the twisted relative partition function has the representation

$$\mathfrak{Z}_\gamma^V = \int e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta(s)})\, ds}\, d\mu_\gamma > 0.$$

(b) *The twisted Gibbs functional $\langle \cdot \rangle_\gamma^V$ defined by*

$$\langle \cdot \rangle_\gamma^V = \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \cdot e^{-\beta H})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H})} \tag{V.30}$$

has a Feynman–Kac representation given by the measure

$$d\mu_\gamma^V = \frac{e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta(s)})\, ds}\, d\mu_\gamma}{\int e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta(s)})\, ds}\, d\mu_\gamma}. \tag{V.31}$$

Thus for the time-ordered product of coordinates T defined by

$$T = (\bar{z}_{j_1}(t_1) \cdots \bar{z}_{j_r}(t_r)\, z_{j'_1}(s_1) \cdots z_{j'_{r'}}(s_{r'}))_+, \tag{V.32}$$

and for the corresponding function of paths X defined by

$$X(\omega_\theta, \overline{\omega_\theta}) = \overline{\omega_\theta(t_1)\, \omega_\theta(t_2) \cdots \omega_\theta(t_r)}\, \omega_\theta(s_1)\, \omega_\theta(s_2) \cdots \omega_\theta(s_{r'}), \tag{V.33}$$

the expectations satisfy

$$\langle T \rangle_\gamma^V = \int X(\omega_\theta, \overline{\omega_\theta})\, d\mu_\gamma^V. \tag{V.34}$$

Proof. Use Propositions V.2 and V.3 to write the twisted Gibbs functional in the form

$$\frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, T\, e^{-\beta H})}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H})} = \lim_{N\to\infty} \frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, T(e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N)}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N)}. \tag{V.35}$$

We can now rewrite the expressions in the numerator and the denominator using Proposition III.3 (in the n-dimensional case). In order to apply the Gaussian Feynman–Kac formula, we not only need expectations of time-ordered products of coordinates provided by the proposition as such, but we also need the Gaussian representation of bounded exponential functions $e^{-V(z(t))}$ by the corresponding function of paths, $e^{-V(\omega_\theta(t))}$. We obtain this extension from the general functional analysis of measures.

Let us consider first the denominator of (V.35). Using Proposition III.3 we obtain, for the normalized twisted expectation,

$$\frac{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N)}{\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H_0})}$$

$$\int e^{-(\beta/N)\sum_{j=1}^{N} V(\omega_\theta((j+1/2)\,\beta/N),\, \overline{\omega_\theta((j+1/2)\,\beta/N)})}\, d\mu_\gamma. \tag{V.36}$$

The factor $\mathrm{Tr}_{\mathfrak{H}}(U(\theta)\, e^{-\beta H_0})$ is the partition function $\mathfrak{Z}_\gamma$ and normalizes the expectation (V.36). The same factor arises in the numerator of (V.35), and the two partition functions exactly cancel. Using Proposition V.2, and also using cyclicity of the trace to replace $(e^{-\beta H_0/2N} e^{-\beta V/N} e^{-\beta H_0/2N})^N$ by $(e^{\beta H_0/N} e^{-\beta V/N})^N$, we have shown that

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, e^{-\beta H}) &= \lim_{N\to\infty} \mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\, (e^{-\beta H_0/N} e^{-\beta V/N})^N) \\ &= \mathfrak{Z}_\gamma \lim_{N\to\infty} \int e^{-(\beta/N)\sum_{j=1}^{N} V(\omega_\theta(j\beta/N),\, \overline{\omega_\theta(j\beta/N)})}\, d\mu_\gamma \\ &= \mathfrak{Z}_\gamma \int e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta(s)})\, ds}\, d\mu_{m,\beta,\theta} > 0.\end{aligned} \tag{V.37}$$

In the last equality we use the fact that V is bounded from below, so the exponential on the right side of (V.37) is bounded uniformly in N. It also converges pointwise as $N\to\infty$. Thus the integral converges by the dominated convergence theorem.

Note that twist positivity for $\{H, U(\theta), \mathfrak{H}\}$ is a consequence of this Feynman–Kac representation, along with twist positivity for $\{H_0, U(\theta), \mathfrak{H}\}$. We have now proved part (a) of the proposition. In order to establish part (b) of the proposition, we apply Proposition V.3 to expanding the numerator of (V.35). By an argument similar to that above, we obtain

$$\begin{aligned}
&\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \, Te^{-\beta H}) \\
&\quad = \mathfrak{Z}_\gamma \lim_{N\to\infty} \int X(\omega_\theta, \overline{\omega_\theta})\, e^{-(\beta/N)\sum_{j=1}^{N} V(\omega_\theta((j+1/2)\beta/N),\, \overline{\omega_\theta((j+1/2)\beta/N)})}\, d\mu_\gamma \\
&\quad = \mathfrak{Z}_\gamma \int X(\omega_\theta, \overline{\omega_\theta})\, e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta(s)})\, ds}\, d\mu_{m,\beta,\theta}. \qquad \text{(V.38)}
\end{aligned}$$

Taking the ratio of (V.38) and (V.37), we complete the proof of Proposition V.4.

V.4. *The Zero-Mass Limit, Revisited*

It is here that we make distinctions between the general allowed potentials in Definition V.1 and those that are also elliptic or infrared regular. Naturally, we obtain stronger results with more assumptions on V. Recall the definition of Υ_{sing} in (I.16).

PROPOSITION V.5. *Let V be an allowed potential. Consider the limit as $m \to 0$ with β and θ fixed.*

(a) *If $\theta \notin \Upsilon_{\text{sing}}$, then $d\mu_\gamma^V$ converges weakly.*

(b) *If $\theta \in \Upsilon_{\text{sing}}$, there exists a constant $M = M(\beta, V) < \infty$ such that*

$$\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* \, e^{-\beta(H_0+V)}) = \mathfrak{Z}_\gamma^V \mathfrak{Z}_\gamma \leqslant \frac{M}{m^{2n}}. \qquad \text{(V.39)}$$

Proof. A general potential V is bounded from below, so $e^{-\int_0^\beta V\, ds}$ is bounded from above and is independent of m. In Proposition III.6 we showed that in the case $n=1$, the measure $d\mu_\gamma$ converges as $m \to 0$ with fixed β and $\theta \notin \Upsilon_{\text{sing}}$. For vector-valued paths that we consider here, a similar argument shows that the product measure $d\mu_\gamma$ defined in (IV.21) converges. Thus $d\mu_\gamma^V$ converges by the dominated convergence theorem. Furthermore for any θ, we have the elementary bound

$$e^{-\int_0^\beta V(\omega_\theta(s),\, \overline{\omega_\theta}(s))\, ds}\, d\mu_\gamma \leqslant (\sup_z \{e^{-\beta V(z,\bar z)}\})\, d\mu_{m,\beta,\theta}, \qquad \text{(V.40)}$$

so using Proposition II.1, we have

$$\mathfrak{Z}_\gamma^V \leqslant (\sup_z e^{-\beta V(z,\bar z)})\, \mathfrak{Z}_\gamma = (\sup_z e^{-\beta V(z,\bar z)}) \prod_{j=1}^{n} |1-\gamma_j|^{-2}. \qquad \text{(V.41)}$$

As $m \to 0$ with β and θ fixed, $|1-\gamma_j| > \text{const}\, m\beta$. The stated bound (V.39) then follows.

PROPOSITION V.6. *Let V be an allowed, elliptic potential.*

(a) *Let $0<\lambda\leqslant 1$ and $H(\lambda)=H_0+\lambda^2 V$. Then there exists a constant $M=M(V)<\infty$, independent of m, λ, β, θ, such that*

$$\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^* e^{-\beta H(\lambda)}) = \mathfrak{Z}_\gamma^{\lambda^2 V}\mathfrak{Z}_\gamma \leqslant \left(\frac{M}{\beta(m+\lambda)}\right)^{2n}. \tag{V.42}$$

(b) *Suppose in addition that the polynomial V is infrared regular. Consider the limit $m\to 0$ with β, θ (and $\lambda=1$) fixed. Then the heat kernel $e^{-\beta H}$ converges in trace norm, and consequently $\mathfrak{Z}_\gamma^V\mathfrak{Z}_\gamma$ converges as $m\to 0$. Furthermore the measure $d\mu_\gamma^V$ converges weakly as $m\to 0$.*

Remark. Let $\|\!|\cdot|\!\|$ denote the Schatten p-norm,

$$\|\!|T|\!\|_p = \mathrm{Tr}_{\mathfrak{H}}((T^*T)^{p/2})^{1/p}, \tag{V.43}$$

with $p=1$ the trace norm, $p=2$ the Hilbert–Schmidt norm, and $p=\infty$ the operator norm.

Proof. From Proposition V.1(a) we infer the identity of $\mathrm{Tr}(U(\theta)^* e^{-\beta H(\lambda)})$ with $\mathfrak{Z}_\gamma^{\lambda^2 V}\mathfrak{Z}_\gamma$. Let us establish the bound (V.42). For an elliptic potential,

$$\sum_{j=1}^{n} |\omega_{j,\theta}(s)|^2 \leqslant M_1(V(\omega_\theta(s), \overline{\omega_\theta(s)})\, ds + M_2). \tag{V.44}$$

Thus writing $|\omega_\theta(s)|^2=\sum_{j=1}^n |\omega_{j,\theta}(s)|^2$, we infer

$$\begin{aligned} \mathfrak{Z}_\gamma^{\lambda^2 V}\mathfrak{Z}_\gamma &= \int_{\mathscr{S}'} e^{-\int_0^\beta \lambda^2 V(\omega_\theta(s), \overline{\omega_\theta(s)})\, ds}\, d\mu_\gamma \mathfrak{Z}_\gamma \\ &\leqslant \int_{\mathscr{S}'} e^{\lambda^2 M_2\beta} e^{\lambda^2 M_1^{-1}\int_0^\beta |\omega_\theta(s)|^2\, ds}\, d\mu_\gamma\mathfrak{Z}_\gamma. \end{aligned} \tag{V.45}$$

Use the mass renormalization identity of Proposition III.5, with $\varepsilon^2=\lambda^2 M_1^{-1}$, and the identity (V.27) to obtain

$$\begin{aligned} \mathfrak{Z}_\gamma^{\lambda^2 V}\mathfrak{Z}_\gamma &\leqslant (e^{\lambda^2 M_2\beta}\mathfrak{Z}(\gamma, \lambda M_1^{-1/2}))\,\mathfrak{Z}_\gamma \\ &= e^{\lambda^2 M_2\beta} e^{n\beta(m-m')}\left(\prod_{j=1}^n \frac{|1-\gamma_j|^2}{|1-\gamma_j'|^2}\right)\mathfrak{Z}_\gamma \\ &= e^{\lambda^2 M_2\beta} e^{n\beta(m-m')}\left(\prod_{j=1}^n \frac{1}{|1-\gamma_j'|^2}\right), \end{aligned} \tag{V.46}$$

where $m'=(m^2+\lambda^2/M_1)^{1/2}$ and $\gamma_j'=e^{-m'\beta+i\Omega_j\theta}$. Then $|1-\gamma'|>\text{const}\, m'\beta>\text{const}\,\beta(M\lambda)$. Hence we have established the desired bound (V.42).

In order to deal with convergence as $m\to 0$, we use the Feynman–Kac representation to work with Hamiltonian estimates. We establish the *second-order estimate* that follows from our assumed infrared regularity bound on the Laplacian of V.

LEMMA V.7. *Let V be an allowed, elliptic, and infrared regular potential. Let $0 \leqslant M_3$ be sufficiently large that $1 \leqslant V + M_3$, and $M_3 > M_2$, where M_2, M_1 are the constants in* (V.3). *Then with $M_4 = 2nm + M_1 + M_3$, and for any f in the domain of H,*

$$\|\Delta f\|^2 + \|(V + M_3) f\|^2 + m^4 \||z|^2 f\|^2 \leqslant 2 \|(H_0 + V + M_4) f\|^2, \qquad \text{(V.47)}$$

where $\Delta = \sum_{j=1}^n \partial_j \bar{\partial}_j$.

Remark. We work on the domain $\mathscr{D} \times \mathscr{D}$ and expand. We use the *double commutator method*, to write cross terms as a sum of positive terms and lower order terms; see [4] or [7]. The identity

$$D^* D X + X D D^* = D^* X D + D X D^* + [D^*, [D, X]] \qquad \text{(V.48)}$$

gives the following double commutator identity for the Laplacian,

$$-\Delta X - X\Delta = \sum_{j=1}^n (\partial_j^* X \partial_j + \partial_j X \partial_j^*) - \sum_{j=1}^n [\bar{\partial}_j [\partial_j, X]]. \qquad \text{(V.49)}$$

Proof. Computing the square of $H + M_3 + nm$ we have

$$\begin{aligned}(H + M_3 + nm)^2 &= (-\Delta + m^2 |z|^2 + V + M_3)^2 \\ &= (\Delta)^2 + (V + M_3 + m^2 |z|^2)^2 \\ &\quad - \Delta(V + M_3 + m^2 |z|^2) - (V + M_3 + m^2 |z|^2) \Delta \\ &= (\Delta)^2 + (V + M_3)^2 + (m^2 |z|^2)^2 + 2(V + M_3)(m^2 |z|^2) \\ &\quad + \sum_{j=1}^n (\partial_j^* (V + M_3 + m^2 |z|^2) \partial_j + \partial_j (V + M_3 + m^2 |z|^2) \partial_j^*) \\ &\quad + \sum_{j=1}^n [\partial_j^*, [\partial_j, V + M_3 + m^2 |z|^2]] \\ &\geqslant (\Delta)^2 + (V + M_3)^2 + m^4 |z|^4 - nm^2 - \sum_{j=1}^n [\bar{\partial}_j, [\partial_j, V]]. \qquad \text{(V.50)}\end{aligned}$$

The double commutator term is just the Laplacian of V, which obeys the infrared regularity bound (V.3),

$$\begin{aligned}(H + M_3 + nm)^2 &\geqslant (\Delta)^2 + (V + M_3)^2 + m^4 |z|^4 - nm^2 - M_1(V + M_2) \\ &\geqslant (\Delta)^2 + \tfrac{1}{2}(V + M_3)^2 + m^4 |z|^4 - nm^2 - \tfrac{1}{2} M_1^2. \qquad \text{(V.51)}\end{aligned}$$

Thus with $M_4 = 2nm + M_1 + M_3$, we have

$$2(H + M_4)^2 \geqslant (\Delta)^2 + (V + M_3)^2 + m^4 |z|^4. \qquad \text{(V.52)}$$

Since H is essentially self-adjoint on $\mathscr{D}$, the inequality (V.47) follows.

Proof of Proposition V.6(b). Denote the m-dependence of H by $H^{(m)}$, and add a constant to both Hamiltonians so that they are positive. Write

$$
\begin{aligned}
e^{-\beta H^{(m)}} &- e^{-\beta H^{(m')}} \\
&= \int_0^\beta e^{-(\beta-s)\,H^{(m)}}(H^{(m')}-H^{(m)})\,e^{-sH^{(m')}}\,ds \\
&= \int_0^\beta e^{-(\beta-s)\,H^{(m)}}((m'-m)(m'+m)\,|z|^2+n(m-m'))\,e^{-sH^{(m')}}\,ds.
\end{aligned}
\tag{V.53}
$$

Then using Hölder's inequality for Schatten norms, we obtain

$$
\begin{aligned}
&|||e^{-(\beta-s)\,H^{(m)}}(H^{(m')}-H^{(m)})\,e^{-sH^{(m')}}|||_1 \\
&\quad\leqslant |||e^{-(\beta-s)/2H^{(m)}}|||_2\;|||e^{-(\beta-s)/2H^{(m)}}(H^{(m)}+I)^{1/2}|||_\infty \\
&\quad\quad\times |||(H^{(m)}+I)^{-1/2}\,((m'-m)(m'+m)\,|z|^2 \\
&\quad\quad+n(m-m'))(H^{(m')}+I)^{-1/2}|||_\infty \\
&\quad\quad\times |||(H^{(m')}+I)^{1/2}\,e^{-s/2H^{(m')}}|||_\infty\;|||e^{-s/2H^{(m')}}|||_2.
\end{aligned}
\tag{V.54}
$$

We bound this using Lemma V.7 in the form $\|(V+M_3)^{1/2}\,(H^{(m)}+I)^{-1/2}\|\leqslant 2^{1/4}$, along with the elliptic bound on $|z|(M_1(V+M_2))^{-1/2}$, resulting in $\|\,|z|\,(H^{(m)}+I)^{-1/2}\|\leqslant$ const, and similarly with m' replacing m. Thus assuming that m, m' remain bounded, there is a constant independent of the parameters such that for $0<s<\beta$,

$$
\begin{aligned}
&|||e^{-(\beta-s)\,H^{(m)}}(H^{(m')}-H^{(m)})\,e^{-sH^{(m')}}|||_1 \\
&\quad\leqslant \text{const.}(\beta-s)^{-1/2}\,s^{-1/2}\,|m-m'|.
\end{aligned}
\tag{V.55}
$$

Integrating this bound over s, we obtain the desired trace-norm convergence of the heat kernel as $m\to 0$, namely the estimate

$$
|||e^{-\beta H^{(m)}}-e^{-\beta H^{(m')}}|||_1 \leqslant \text{const.}\,|m-m'|.
\tag{V.56}
$$

Using the Feynman–Kac representation of Proposition V.4(a),

$$
\mathrm{Tr}_{\mathfrak{H}}(U(\theta)^*\,e^{-\beta H^{(m)}}) = \mathfrak{Z}_\gamma^{(m)\;V}\mathfrak{Z}_\gamma^{(m)},
\tag{V.57}
$$

we infer the convergence of the product $\mathfrak{Z}_\gamma^{(m)\;V}\mathfrak{Z}_\gamma^{(m)}$ as $m\to 0$. This is the denominator in the representation (V.30) of the twisted Gibbs functional, and it is the normalizing factor for the measure

$$
\mathfrak{Z}_\gamma^{(m)}\,e^{-\int_0^\beta V(\omega_\theta(s),\,\overline{\omega_\theta}(s))\,ds}\,d\mu_\gamma.
\tag{V.58}
$$

After normalization this measure is $d\mu_{\gamma}^{(m)\,V}$. The proof of convergence of the integral of the measure (V.58) on products X of coordinates (V.33) proceeds similarly, and establishes the weak convergence of the perturbed measures $d\mu^{(m)\,V}$ as $m \to 0$. We omit the details.

VI. QUANTUM FIELDS AND RANDOM FIELDS

In this section we generalize the construction in Sections II–V to the case of an n-component, complex quantum field $\varphi(x) = \{\varphi_j(x)\}$. This time-zero field has a spatial coordinate x lying in an s-torus $\mathbb{T}^s$, so our space-time is $\mathbb{T}^s \times \mathbb{R}$. Assume that the periods of the torus are ℓ_i, where $1 \leqslant i \leqslant s$. The spatial volume is $\mathrm{Vol} = \prod_{i=1}^{s} \ell_l$. Let us denote the lattice of momenta k dual to $\mathbb{T}^s$ as

$$\hat{\mathbb{T}} = \left\{ k\colon k = \{k_1, k_2, ..., k_s\}, \text{ where each } k_i = \frac{2\pi}{\ell_i}\,\mathbb{Z} \right\}, \tag{VI.1}$$

and let $kx = \sum_{i=1}^{s} k_i x_i$.

VI.1. *Complex Free Fields*

The quantum-mechanical Hilbert space on which the field acts is on the Fock Hilbert space $\mathfrak{H}$ over the torus $\mathbb{T}^s$. This Hilbert space is the tensor product of the Hilbert spaces for individual, non-interacting oscillators. Each oscillator has a frequency $\mu(k) = (k^2 + m^2)^{1/2}$. Thus for $k \neq 0$ the frequency does not vanish even in the limit $m \to 0$. Each component of the field has a Fourier expansion

$$\varphi_j(x) = \frac{1}{\sqrt{\mathrm{Vol}}} \left(z_j + \sum_{0 \neq k \in \hat{\mathbb{T}}^s} \hat{\varphi}_j(k)\, e^{-ikx} \right). \tag{VI.2}$$

The constant Fourier modes z_j are the coordinates considered in Sections II–V. The non-constant Fourier modes have the form

$$\hat{\varphi}_j(k) = \frac{1}{\sqrt{2\mu(k)}} \left(a_{+,\,j}(k)^* + a_{-,\,j}(-k) \right), \tag{VI.3}$$

where we express them in terms of canonical annihilation and creation operators $a_{\pm}^{\#}$ that satisfy

$$[a_{\pm,\,j}(k), a_{\pm,\,j'}(k')^*] = \delta_{j,\,j'} \delta_{k,\,k'} I, \tag{VI.4}$$

and where all other commutators between pairs of $a_{\pm,\,j}^{\#}$'s vanish.

Define the mutually commuting number operators $N_{\pm,\,j}(k)$ by

$$N_{\pm,\,j}(k) = a_{\pm,\,j}(k)^*\, a_{\pm,\,j}(k). \tag{VI.5}$$

Then express three fundamental operators in terms of the time zero Fourier components:

$$H_0 = \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}^s} \mu(k)(N_{+,j}(k) + N_{-,j}(k)),$$

$$J = \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}^s} \Omega_j(N_{+,j}(k) - N_{-,j}(k)), \tag{VI.6}$$

$$P_i = \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}^s} k_i(N_{+,j}(k) + N_{-,j}(k)).$$

These operators are the free-field Hamiltonian, the twist generator, and the components of the momentum operator, respectively. With $x \in \mathbb{T}^s$, and $xP = \sum_{i=1}^{s} x_i P_i$, let

$$\varphi(x+y) = e^{-ixP} \varphi(y)\, e^{ixP}. \tag{VI.7}$$

As we observed in the Introduction, in the field case we introduce the $(s+1)$-parameter symmetry group $U(\tau, \theta) = e^{i\tau P + i\theta J}$. This group acts on the components of the field as

$$U(\tau, \theta)\, \varphi_j(x)\, U(\tau, \theta)^* = e^{i\Omega_j \theta} \varphi_j(x - \tau), \tag{VI.8}$$

and it implements a twist on Fourier components,

$$U(\tau, \theta)\, \hat{\varphi}_j(k)\, U(\tau, \theta)^* = e^{ik\tau + i\Omega_j \theta} \hat{\varphi}_j(k). \tag{VI.9}$$

In keeping with the notation in Sections II–IV, we define $\overline{\varphi_j(x)} = \varphi_j(x)^*$, with the adjoint taken in the sense of a densely defined sesquilinear form on $\mathfrak{H}$. Also for $t > 0$, we define the imaginary time field $\varphi_j(x, t)$ as an operator with domain equal to the range of e^{-sH}, where $s > t$, namely

$$\varphi_j(x, t) = e^{-tH} \varphi_j(x)\, e^{tH}. \tag{VI.10}$$

Let

$$\overline{\varphi_j}(x, t) = e^{-tH} \overline{\varphi_j}(x)\, e^{tH} = e^{-tH} \varphi_j(x)^*\, e^{tH}. \tag{VI.11}$$

In order to simplify our notation, we replace the parameter γ used in previous sections by a family of parameters $\gamma = \{\gamma_j(k)\}$ that contain information on the dependence of γ on j, k, β, τ, and θ. Set

$$\gamma = \{\gamma_j(k)\}, \qquad \text{where} \quad \gamma_j(k) = e^{-\mu(k)\beta + ik\tau + i\Omega_j \theta}, \tag{VI.12}$$

with $k \in \hat{\mathbb{T}}^s$ and $1 \leqslant j \leqslant n$. Then we designate partition functions and expectations as before by $\mathfrak{Z}_\gamma$ or $\langle \cdot \rangle_\gamma$, and thereby we designate the dependence on all relevant variables. For example, we write the free-field twisted partition function as

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(U(\tau, \theta)^* e^{-\beta H_0})$$
$$= \prod_{j=1}^{n} \prod_{k} \mathrm{Tr}_{\mathfrak{H}_{\{k,j,+\}}}(\bar{\gamma}_j(k)^{N_{+,j}(k)}) \, \mathrm{Tr}_{\mathfrak{H}_{\{k,j,-\}}}(\gamma_j(-k)^{N_{-,j}(k)}). \qquad \text{(VI.13)}$$

Here $\mathfrak{H}_{\{k,j,\pm\}}$ is the Hilbert space for the $\{k, j, \pm\}$ degrees of freedom, and $\mathfrak{H} = \bigotimes_{\{k,j,\pm\}} \mathfrak{H}_{\{k,j,\pm\}}$. In the case of paths we retain the notation $\Phi_{\tau,\theta}$, and we also use $C_{\tau,\theta}$ for the pair correlation operator, in order to emphasize the dependence on these variables.

PROPOSITION VI.1. *With the above assumptions, the free-field partition function is twist positive,*

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(e^{-i\theta J - i\tau P - \beta H}) = \prod_{j=1}^{n} \prod_{k \in \hat{\mathbb{T}}^s} \frac{1}{|1-\gamma_j(k)|^2} > 0. \qquad \text{(VI.14)}$$

Proof. The trace factorizes, as indicated in (VI.13). For a particular term in the product, perform the sum as in the proof of Proposition II.1. We obtain for given k and j the contribution to the product equal to

$$(1-\bar{\gamma}_j(k))^{-1} (1-\gamma_j(-k))^{-1}. \qquad \text{(VI.15)}$$

Thus

$$\mathfrak{Z}_\gamma = \prod_{k,j} (1-\bar{\gamma}_j(k))^{-1} (1-\gamma_j(-k))^{-1}. \qquad \text{(VI.16)}$$

This product converges as $|\gamma_j(k)| < e^{-\beta |k|}$. Combining the result for modes k and $-k$ yields the product (VI.14), and the proof is complete.

VI.2. *Twisted Expectations and the Pair Correlation Function*

Introduce the twisted expectation

$$\langle \cdot \rangle_\gamma = \frac{\mathrm{Tr}_{\mathfrak{H}}(\cdot \, e^{-i\theta J - i\tau P - \beta H})}{\mathrm{Tr}_{\mathfrak{H}}(e^{-i\theta J - i\tau P - \beta H})}. \qquad \text{(VI.17)}$$

Note that the expectation of one field vanishes,

$$\langle \varphi_i(x,t) \rangle_\gamma = \langle \overline{\varphi_i}(x,t) \rangle_\gamma = 0. \qquad \text{(VI.18)}$$

Furthermore,

$$\langle \varphi_i(x,t) \, \varphi_j(y,s) \rangle_\gamma = \langle \overline{\varphi_i}(x,t) \, \overline{\varphi_j}(y,s) \rangle_\gamma = 0. \qquad \text{(VI.19)}$$

Define the pair correlation function $C_{\tau,\theta}(x-y, t-s)$ as the expectation of the time-ordered product of a field and a conjugate field. In particular, let

$$C_{\tau,\theta}(x-y, t-s)_{ij} = \langle (\overline{\varphi_i}(x,t) \, \varphi_j(y,s))_+ \rangle_\gamma. \qquad \text{(VI.20)}$$

This function vanishes unless $i = j$. Furthermore the fields $\varphi_j(x)$ are periodic with $x \in \mathbb{T}^s$, so the pair correlation function has a partial Fourier representation

$$C_{\tau,\theta}(x-y, t-s)_{ij} = \frac{1}{\mathrm{Vol}} \sum_{k \in \hat{\mathbb{T}}^s} \hat{C}_{\tau,\theta}(k; t-s)_{ij}\, e^{ik(x-y)}, \tag{VI.21}$$

where k ranges over the lattice (VI.1).

PROPOSITION VI.2. *The twisted Gibbs functional* (VI.17) *for the complex, massive, free scalar field on the space-time* $\mathbb{T}^s \times [0, \beta]$ *is a Gaussian functional. The pair correlation has the Fourier representation* (VI.21), *where*

$$\hat{C}_{\tau,\theta}(k; t-s)_{ij} = \frac{1}{2\mu(k)} \left(\left(\frac{\gamma_j(k)}{1-\gamma_j(k)} \right) e^{-\mu(k)(t-s)} + \left(\frac{\overline{\gamma_j(k)}}{1-\overline{\gamma_j(k)}} \right) e^{\mu(k)(t-s)} + e^{-\mu(k)\,|t-s|} \right) \delta_{ij}. \tag{VI.22}$$

Proof. In order to establish (VI.22), carry out the argument for each degree of freedom in exactly the same fashion as for the analysis in Section II for the oscillator. This result generalizes (II.26). In the mode labelled by k we replace m by $\mu(k)$. Second, we need to perform the twist arising from the $e^{-i\tau P}$ group. But for each k, this simply modifies the angle $\Omega_j\theta$, replacing it by the angle $\Omega_j\theta + k\tau$. With our definition (VI.12) of γ to include the proper parameter dependence, we obtain the Fourier coefficients (VI.22) in a straightforward way. The effect of complex conjugating $\gamma_j(k)$ in one term gives the correct relation between the signs of the terms $\Omega_j\theta$ and $k\tau$. Once we have obtained the formula for the pair correlation function, the proof of the Gaussian character of the functional (I.24) follows the proof of Proposition II.3. We omit the details.

PROPOSITION VI.3. (a) *The pair correlation function and its Fourier coefficients satisfy twist relations. If* $0 \leqslant t \leqslant s \leqslant \beta$ *then*

$$C_{\tau,\theta}(x-y, t-s+\beta)_{jj} = e^{-i\Omega_j\theta} C_{\tau,\theta}(x-y-\tau, t-s)_{jj}, \tag{VI.23}$$

while if $0 \leqslant s \leqslant t \leqslant \beta$, *then*

$$C_{\tau,\theta}(x-y, t-s-\beta)_{jj} = e^{i\Omega_j\theta} C_{\tau,\theta}(x-y+\tau, t-s)_{jj}. \tag{VI.24}$$

(b) *The kernel* $C_{\tau,\theta}(x-y, t-s)_{ij}$ *is hermitian in the sense that*

$$C_{\tau,\theta}(x-y, t-s)_{ij} = \overline{C_{\tau,\theta}(y-x, s-t)_{ji}}. \tag{VI.25}$$

(c) *For* $0 \leqslant t \leqslant s \leqslant \beta$, *the coefficients in the Fourier representation of the pair correlation function satisfy*

$$\hat{C}_{\tau,\theta}(k; t-s+\beta)_{jj} = e^{-i\Omega_j\theta - ik\tau} c_{\tau,\theta}(k; t-s)_{jj}. \tag{VI.26}$$

On the other hand, for $0 \leqslant s \leqslant t \leqslant \beta$,

$$\hat{C}_{\tau,\theta}(k; t-s-\beta)_{jj} = e^{-i\Omega_j\theta - ik\tau} C_{\tau,\theta}(k; t-s)_{jj}. \tag{VI.27}$$

(d) *As a consequence, each* $\hat{C}_{\tau,\mathbb{T}}(k; t-s)_{jj}$ *has the representation*

$$\hat{C}_{\tau,\theta}(k; t-s)_{jj} = \sum_{E \in K_{\tau,\theta,j}} \frac{1}{E^2 + k^2 + m^2} e^{iE(t-s)}, \tag{VI.28}$$

where

$$K_{\tau,\theta,j} = \frac{1}{\beta} \{2\pi\mathbb{Z} + \Omega_j\theta + k\tau\}. \tag{VI.29}$$

Remark. The original definition of the pair correlation function $C_{\tau,\theta}(x-y, t-s)$ is restricted to the domain $x-y \in \mathbb{T}^s$ and $t-s \in [-\beta, \beta]$. The twist relation (I.25) provides a natural extension of $C_{\tau,\theta}(x-y, t-s)$ to $\mathbb{T}^s \times \mathbb{R}$, with (VI.23), (VI.24), and (VI.25) holding throughout.

Proof. The boundary condition for the pair correlation function can be established by using the above definitions and cyclicity of the trace. Consider the case $0 \leqslant t \leqslant s \leqslant \beta$. Then

$$\begin{aligned} C_{\tau,\theta}(x-y, t-s+\beta)_{jj} &= \langle (\overline{\varphi_j}(x, \beta-s+t)\, \varphi_j(y, 0))_+ \rangle_\gamma \\ &= \langle \varphi_j(y, 0)\, \overline{\varphi_j}(x, \beta-s+t) \rangle_\gamma \\ &= e^{-i\Omega_j\theta} \langle \overline{\varphi_j}(x, \beta-s+t)\, \varphi_j(y+\tau, \beta) \rangle_\gamma \\ &= e^{-i\Omega_j\theta} C_{\tau,\theta}(x-y-\tau, t-s)_{jj}. \end{aligned} \tag{VI.30}$$

For the case $0 \leqslant s \leqslant t \leqslant \beta$, write

$$\begin{aligned} C_{\tau,\theta}(x-y, t-s-\beta)_{jj} &= \langle (\overline{\varphi_j}(x, t-s)\, \varphi_j(y, \beta))_+ \rangle_\gamma \\ &= \langle \overline{\varphi_j}(x, t-s)\, \varphi_j(y, \beta) \rangle_\gamma \\ &= e^{i\Omega_j\theta} \langle \varphi_j(y-\tau, 0)\, \overline{\varphi_j}(x, t-s) \rangle_\gamma \\ &= e^{i\Omega_j\theta} \langle (\overline{\varphi_j}(x, t-s)\, \varphi_j(y-\tau, 0))_+ \rangle_\gamma \\ &= e^{i\Omega_j\theta} C_{\tau,\theta}(x-y+\tau, t-s)_{jj}. \end{aligned} \tag{VI.31}$$

Thus we have established (VI.23) and (VI.24).

The hermiticity condition (VI.25) follows directly from inspecting the representation (VI.21)–(VI.22). In fact, (VI.22) shows that $C_{\tau,\theta}(x-y, t-s)_{ij}$ is diagonal in the matrix index *ij*, so we write the representation (VI.21) as

$$\overline{C_{\tau,\theta}(y-x,s-t)_{jj}}=\frac{1}{\mathrm{Vol}}\sum_{k\in\hat{\mathbb{T}}^s}\overline{\hat{C}_{\tau,\theta}(k;s-t)_{jj}}\,e^{ik(x-y)}$$
$$=\frac{1}{\mathrm{Vol}}\sum_{k\in\hat{\mathbb{T}}^s}\hat{C}_{\tau,\theta}(k;t-s)_{jj}\,e^{ik(x-y)}=C_{\tau,\theta}(x-y,t-s)_{jj}. \tag{VI.32}$$

We translate this boundary condition into a condition on the Fourier coefficients $\hat{C}_{\tau,\theta}(k;t-s)_{jj}$ by taking the Fourier series in the variable $x-y$. This gives (VI.26) and (VI.27). Thus

$$\hat{C}_{\tau,\theta}(k;t-s)_{jj}=\sum_{E\in K_{\tau,\theta,j}} f(k;E)\,e^{iE(t-s)}. \tag{VI.33}$$

Finally, as in Section II.6, we calculate the coefficients in the Fourier representation similarly to the proof of Proposition II.4. The spectrum of $C_{\tau,\theta}$ then follows as claimed, namely (VI.47) is a union of the contributions from the different components of the covariance operator on the matrix diagonal, and the individual spectra resulting from (VI.39).

VI.3. *The Free–Field Pair Correlation Operator*

The pair correlation operator $C_{\tau,\theta}$ is the integral operator with the pair correlation function as its integral kernel.

$$(C_{\tau,\theta}f)_i(x,t)=\sum_{j=1}^{n}\int_{\mathbb{T}^s\times[0,\beta]} C_{\tau,\theta}(x-y,t-t')_{ij}\,f_j(y,t')\,dy\,dt'. \tag{VI.34}$$

We identify the pair correlation operator as the resolvent of a twisted Laplace operator $\Delta_{\tau,\theta}$. Introduce the Hilbert space $\mathscr{K}=\mathscr{L}^2(\mathscr{C})\otimes\mathbb{C}^n$ on which the pair correlation operator acts. Here $\mathscr{C}$ denotes the compactified space-time $\mathbb{T}^s\times[0,\beta]$. Let $\mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ denote functions on $\mathscr{C}$ that have Fourier representations of the form

$$f(x,t)=\sum_{\substack{E\in(2\pi/\beta)\,\mathbb{Z}\\ k\in\mathbb{T}^s}}\hat{f}(k,E)\,e^{ikx+iEt}e^{-i(\Omega_j\theta+k\tau)\,t/\beta}. \tag{VI.35}$$

Here the set $\hat{\mathbb{T}}^s$ denotes the lattice $\prod_{j=1}^{s}(2\pi/\ell_j)\,\mathbb{Z}$ dual to the torus $\mathbb{T}^s$. We assume that the coefficients $\hat{f}(k,E)$ decrease faster than the inverse of any polynomial function of k^2+E^2. The functions (VI.35) satisfy the boundary condition

$$f(x,\beta)=e^{-i\Omega_j\theta}f(x-\tau,0), \tag{VI.36}$$

relating the two ends of the cylinder. The space of C^∞ functions $\mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ is dense in $L^2(\mathscr{C})$. Endowed with the topology given by the countable set of norms

$$\|f\|_n=\sup_{k,E}(1+|k|^2+E^2)^n\,|\hat{f}(k,E)|, \qquad \text{for} \quad n\in\mathbb{Z}_+, \tag{VI.37}$$

the space $\mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ is a Schwartz space of C^∞ functions. Furthermore, the representation (VI.35) shows that functions $f \in \mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ extend from $\mathscr{C} = \mathbb{T}^s \times [0, \beta]$ to C^∞ functions on $\mathbb{T}^s \times \mathbb{R}$, which satisfy the *twist relation*

$$f(x, t+\beta) = e^{i\Omega_j\theta} f(x+\tau, t). \tag{VI.38}$$

The functions $e^{ikx+iEt}e^{-i(k\tau)t/\beta}e^{-i\Omega_j\theta t/\beta} \in \mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ are all simultaneous eigenfunctions of $-i(\partial/\partial t)$ and of $-i(\partial/\partial x_i)$. The joint spectrum of these two operators on these eigenvectors is the set of

$$K_{\tau,\theta,j} \times \hat{\mathbb{T}}^s, \tag{VI.39}$$

where $K_{\tau,\theta,j}$ is defined in (VI.29). As a consequence, the domain $\mathscr{S}^{(j)}_{\tau,\theta}(\mathscr{C})$ is a domain of essential self-adjointness for the twisted Laplace operator

$$\Delta^{(j)}_{\tau,\theta} = \frac{\partial^2}{\partial t^2} + \sum_{1 \leqslant i \leqslant s} \frac{\partial^2}{\partial x_i^2}. \tag{VI.40}$$

The dense domain of definition

$$\mathscr{S}_{\tau,\theta} = \mathscr{S}^{(1)}_{\tau,\theta}(\mathscr{C}) \oplus \mathscr{S}^{(2)}_{\tau,\theta}(\mathscr{C}) \oplus \cdots \oplus \mathscr{S}^{(n)}_{\tau,\theta}(\mathscr{C}) \subset \mathscr{K} \tag{VI.41}$$

provides a dense domain for the Laplace operator $\Delta_{\tau,\theta}$. This is an $n \times n$ matrix with each entry an operator on $L^2(\mathscr{C})$. The entries are

$$\{(\Delta_{\tau,\theta})_{jj'}\} = \{\Delta^{(j)}_{\tau,\theta}\delta_{jj'}\}. \tag{VI.42}$$

Denote the corresponding resolvent by

$$(-\Delta_{\tau,\theta} + m^2)^{-1}. \tag{VI.43}$$

It has matrix elements

$$\{(-\Delta^{(j)}_{\tau,\theta} + m^2)^{-1}\,\delta_{jj'}\}, \tag{VI.44}$$

so that for $f_j \in L^2(\mathscr{C})$ and $f = \{f_j\} \in \mathscr{K}$,

$$((-\Delta_{\tau,\theta} + m^2)^{-1}\, f)_j\,(x, t) = \sum_{j'=1}^{n} \int_{\mathscr{C}} (-\Delta_{\tau,\theta} + m^2)^{-1}\,(x-y, t-t')_{jj'}\, f_{j'}(y, t')\, dy\, dt'. \tag{VI.45}$$

We say for short that $(-\Delta_{\tau,\theta} + m^2)^{-1}\,(x-y, t-t')$ denotes the integral kernel for $(-\Delta_{\tau,\theta} + m^2)^{-1}$.

Theorem VI.4. *Consider the resolvent of the twisted Laplace operator $\Delta_{\tau,\theta}$ defined as the closure of the Laplace operator with the domain $\mathscr{S}_{\tau,\theta}(\mathscr{C})$ and acting as*

(VI.45). *This equals the pair correlation operator* $C_{\tau,\theta}$ *by the integral kernel* (VI.22). *Namely*

$$C_{\tau,\theta} = (-\Delta_{\tau,\theta} + m^2)^{-1}. \tag{VI.46}$$

Furthermore, the spectrum of $C_{\tau,\theta}$ *is the set*

$$(E^2 + k^2 + m^2)^{-1}, \qquad \textit{where} \quad E \in \bigcup_{1 \leqslant j \leqslant n} K_{\tau,\theta,j} \quad \textit{and} \quad k \in \hat{\mathbb{T}}^s, \tag{VI.47}$$

with $K_{\tau,\theta,j}$ *defined in* (VI.29).

Proof. The proof of the theorem follows the proof of Theorem III.1. The spectrum is given in Proposition VI.3(c).

Define the singular set $\Upsilon_{\text{sing}} = \Upsilon_{\text{sing}}(\Omega_j, \ell_j)$ appropriate for complex quantum fields by

$$\Upsilon_{\text{sing}} = \left\{ \tau, \theta \colon 0 \in \bigcup_{1 \leqslant j \leqslant n} \; \bigcup_{k \in \hat{\mathbb{T}}^s} \{2\pi\mathbb{Z} - \Omega_j \theta - k\tau\} \right\}. \tag{VI.48}$$

For $\tau, \theta \notin \Upsilon_{\text{sing}}$, let

$$M = \sup_{1 \leqslant j \leqslant n} \; \sup_{E \in K_{\tau,\theta,j},\, k \in \hat{\mathbb{T}}^s} \frac{1}{E^2 + k^2}. \tag{VI.49}$$

Corollary VI.5. *Let* $m > 0$. *The operator* $C_{\tau,\theta}$ *on* $\mathscr{K}$ *with integral kernel* (VI.22) *is positive, and compact. Assume also that* $M < \infty$ *and* $\tau, \theta \notin \Upsilon_{\text{sing}}$. *Then the operator* $C_{\tau,\theta}$ *is norm convergent as* $m \to 0$.

VI.4. *Random Fields*

In this section we discuss the appropriate path space for random fields satisfying a twist relation (I.25) leading to a Feynman–Kac formula for quantum fields with the twist operator $U(\tau, \theta)$. These random fields are just elements of $\mathscr{S}'_{\tau,\theta}$, the dual space to $\mathscr{S}_{\tau,\theta}$. We let $\Phi_{\tau,\theta}(x, t)$ denote an element of $\mathscr{S}'_{\tau,\theta}$. The space $\mathscr{S}'_{\tau,\theta}$ has a dense subspace of C^∞ functions,

$$\mathscr{S}_{\tau,-\theta} \subset \mathscr{S}'_{\tau,\theta}. \tag{VI.50}$$

We infer from Proposition VI.2 that the pair correlation operator $C_{\tau,\theta}$ maps an $\mathscr{S}_{\tau,\theta}$ into itself, and in fact this map is continuous. Define the adjoint operator

$$C^+_{\tau,\theta} \colon \mathscr{S}'_{\tau,\theta} \to \mathscr{S}'_{\tau,\theta} \qquad \text{by the relation} \quad (C^+_{\tau,\theta}\Phi_{\tau,\theta})(f) = \Phi_{\tau,\theta}(C_{\tau,\theta} f). \tag{VI.51}$$

It then follows that the kernel of $C^+_{\tau,\theta}$ satisfies the boundary condition

$$C^+_{\tau,\theta}(x - y, t - s + \beta)_{jj} = e^{i\Omega_j\theta} C^+_{\tau,\theta}(x - y - \tau, t - s)_{jj}. \tag{VI.52}$$

We write for the corresponding paths the relation (I.25).

VI.5. *The Gaussian Feynman–Kac Measure*

The twisted Gibbs state of a free field $\varphi(x) = \{\varphi_j(x)\}$ has a Feynman–Kac representation with a Gaussian measure on $\mathcal{S}'_{\tau,\theta}$. Define $d\mu_\gamma(\Phi_{\tau,\theta})$ as the Gaussian probability measure on $\mathcal{S}'_{\tau,\theta}(\mathcal{C})$ with mean zero and covariance $C_{\tau,\theta}$. Since we have established that the twisted, free-field Gibbs functional is Gaussian, we have as a consequence of the properties established in Proposition VI.4,

PROPOSITION VI.6. (a) *The twisted Gibbs functional for a free field $\varphi(x)$ with mass $m>0$ has a Feynman–Kac representation*

$$\langle(\overline{\varphi_{j_1}}(x_1,t_1)\,\overline{\varphi_{j_2}}(x_2,t_2)\cdots\overline{\varphi_{j_r}}(x_r,t_r)\,\varphi_{j'_1}(x'_1,t'_1)\,\varphi_{j'_2}(x'_2,t'_2)\cdots\varphi_{j'_{r'}}(x'_{r'},t'_{r'}))_+\rangle_\gamma$$

$$=\int_{\mathcal{S}'_{\tau,\theta}(\mathcal{C})}\overline{\Phi_{\tau,\theta,j_1}(x_1,t_1)\,\Phi_{\tau,\theta,j_2}(x_2,t_2)}\cdots\overline{\Phi_{\tau,\theta,j_r}(x_r,t_r)}$$

$$\times\Phi_{\tau,\theta,j'_1}(x'_1,t'_1)\,\Phi_{\tau,\theta,j'_2}(x'_2,t'_2)\cdots\Phi_{\tau,\theta,j'_{r'}}(x'_{r'},t'_{r'})\,d\mu_\gamma(\Phi_{\tau,\theta}). \tag{VI.53}$$

(b) *Let $\tau,\theta\notin\Upsilon_{\text{sing}}$ and let $M<\infty$. As $m\to 0$, with β,τ,θ fixed, the measures $d\mu_\gamma(\Phi_\tau,\theta)$ converge weakly as measure on $\mathcal{S}'_{\tau,\theta}(\mathcal{C})$, and the expectations*

$$\langle(\overline{\varphi_{j_1}}(x_1,t_1)\,\overline{\varphi_{j_2}}(x_2,t_2)\cdots\overline{\varphi_{j_r}}(x_r,t_r)\,\varphi_{j'_1}(x'_1,t'_1)\,\varphi_{j'_2}(x'_2,t'_2)\cdots\varphi_{j'_{r'}}(x'_{r'},t'_{r'}))_+\rangle_\gamma \tag{VI.54}$$

converge as distributions in $\bigotimes^{r+r'}\mathcal{S}'_{\tau,\theta}(\mathcal{C})$.

VI.6. *Non-linear Quantum Fields and Non-Gaussian Measures*

Finally, we remark on the construction of non-Gaussian, twist-invariant Gibbs functionals for fields. These measures give Feynman–Kac representations for twisted Gibbs functionals constructed from certain non-linear quantum fields. In this paper we do not wish to consider ultraviolet questions, so we use a regularized interaction potential V. We follow the (standard) procedure introduced in Section V to construct Feynman–Kac representations for non-linear oscillators. First, we introduce a regularized field $\varphi_{j,\chi}(x)$ from which we construct the non-linear interaction. Let

$$\varphi_{j,\chi}(x)=\frac{1}{\sqrt{\text{Vol}}}\left(z_j+\sum_{k\neq 0}\hat{\varphi}_j(k)\,\chi(k)\,e^{-ikx}\right), \tag{VI.55}$$

where $0\leqslant\chi(k)\leqslant 1$, $\chi(0)=1$, and $\chi(k)$ is rapidly decreasing as $|k|\to\infty$. Likewise, let $\Phi_{\tau,\theta,\chi}(x,t)$ denote the random field $\Phi_{\tau,\theta}(x,t)$ after convolution in the spatial variable x with the function whose Fourier coefficients are $\chi(k)$. Given a holomorphic, quasihomogeneous potential function W as in Section V, we form the perturbed Hamiltonian

$$H=H_0+\int_{\mathbb{T}^s}V(\{\varphi_{j,\chi}(x)\},\{\overline{\varphi_{j,\chi}(x)}\})\,dx. \tag{VI.56}$$

This Hamiltonian then defines a twisted Gibbs expectation. It has a Feynman–Kac representation given by the measure

$$d\mu_\gamma^V(\Phi_{\tau,\theta}(\cdot)) = \frac{e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\theta,\chi}(y,s),\, \overline{\Phi_{\tau,\theta,\chi}(y,s)})\, ds\, dy}\, d\mu_\gamma}{\int e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\theta,\chi}(y,s),\, \overline{\Phi_{\tau,\theta,\chi}(y,s)})\, ds\, dy}\, d\mu_\gamma}. \tag{VI.57}$$

Since this measure is regularized, we can construct it by methods similar to those in Section V. We do not give the details, as we will return elsewhere to analyze such problems as well as to establish their dependence on the regularization. As previously, we obtain

PROPOSITION VI.7. (a) *The $U(\tau,\theta)$-twisted Gibbs functional for a field Hamiltonian $H = H_0 + V$ has a Feynman–Kac representation given by the measure* (VI.57). *Let $\tau, \theta \notin \Upsilon_{\text{sing}}$ and let β, τ, θ be fixed. As $m \to 0$, the measures $d\mu_\gamma^V(\Phi_{\tau,\theta})$ converge weakly in $\mathscr{S}'_{\tau,\theta}(\mathscr{C})$, as do the expectations of products of fields.*

VI.7. *Real Fields*

In order to treat real scalar fields $\varphi(x)$ it is sufficient to make minor modifications of our preceding analysis. In particular, there is only one set of creation and annihilation operators for each $\{k, j\}$. Thus in place of (VI.3) we have

$$\hat{\varphi}_j(k) = \frac{1}{\sqrt{2\mu(k)}}\,(a_j(k)^* + a_j(-k)). \tag{VI.58}$$

As a consequence the translation group acts as a symmetry of H_0 as before, but there is no θ-twist symmetry. We define the parameters

$$\gamma = \{\gamma(k)\}, \qquad \text{where} \quad \gamma(k) = e^{-\mu(k)\beta + ik\tau}. \tag{VI.59}$$

Then the modified results for the real case are

PROPOSITION VI.8. (a) *The twisted partition function for the real, free scalar field is*

$$\mathfrak{Z}_\gamma = \mathrm{Tr}_{\mathfrak{H}}(e^{i\tau P - \beta H}) = \prod_{k \in \hat{\mathbb{T}}^s} \frac{1}{|1-\gamma(k)|^n}. \tag{VI.60}$$

(b) *The real free-field pair correlation function is a multiple of the identity in the matrix index,*

$$C_\gamma(x-y, t-s)_{ij} = \langle (\varphi_i(x,t)\,\varphi_j(y,s))_+ \rangle_\gamma = C_\gamma(x-y, t-s)\,\delta_{ij}, \tag{VI.61}$$

and satisfies the twist relation

$$C_\gamma(x-y+\tau, t-s+\beta) = C_\gamma(x-y, t-s). \tag{VI.62}$$

(c) *The diagonal elements of the real, free-field pair correlation function have Fourier representation of the form* (VI.21) *with coefficients*

$$\hat{C}_\tau(k; t-s) = \frac{1}{2\mu(k)}\left(\left(\frac{\gamma(k)}{1-\gamma(k)}\right) e^{-\mu(k)(t-s)} + \left(\frac{\overline{\gamma(k)}}{1-\overline{\gamma(k)}}\right) e^{\mu(k)(t-s)} + e^{-\mu(k)\,|t-s|}\right). \tag{VI.63}$$

(d) *The real, free-field pair correlation operator is the resolvent of the twisted Laplace operator Δ_r defined on the real functions in $\mathcal{S}_{\tau,0}(\mathscr{C})$,*

$$C_\tau = (-\Delta_\tau + m^2)^{-1}. \tag{VI.64}$$

(e) *The real random paths $\Phi_\tau(x, t)$ satisfy*

$$\Phi_\tau(x+\tau, t+\beta) = \Phi_\tau(x, t). \tag{VI.65}$$

As a consequence, there exists a class of interacting Hamiltonians

$$H = H_0 + V \tag{VI.66}$$

that are twist positive and have a Feynman–Kac representation with a twist $U(\tau)$. These interactions arise from real, translation-invariant, cutoff interactions with a potential V that is bounded from below. The corresponding Feynman–Kac measure has the form

$$d\mu_\gamma^V(\Phi_{\tau,\theta}(\cdot)) = \frac{e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\chi}(y,s))\, ds\, dy}\, d\mu_\gamma}{\int e^{-\int_{\mathscr{C}} V(\Phi_{\tau,\chi}(y,s))\, ds\, dy}\, d\mu_\gamma}. \tag{VI.67}$$

We intend to return to these properties in another work in more detail.

REFERENCES

1. I. M. Gelfand and N. Ya. Vilenkin, "Generalized Functions," Vol. 4, "Applications of Harmonic Analysis," Academic Press, San Diego, 1964.
2. J. Glimm and A. Jaffe, "Quantum Physics," Springer-Verlag, Berlin/New York, 1987.
3. R. Høgh-Krohn, *Comm. Math. Phys.* **38** (1974), 195.
4. A. Jaffe, "Dynamics of a Cut-off $\lambda\Phi^4$ Field Theory," Ph.D. thesis Princeton University, 1965; *in* "Mathematical Theory of Elementary Particles" (R. Goodman and E. Segal, Eds.), MIT Press, Cambridge, MA, 1966.
5. A. Jaffe, *J. Math. Phys.* **6** (1965), 1172.
6. A. Jaffe, Quantum harmonic analysis and geometric invariants, *Adv. in Math.* **143** (1999), 1.
7. A. Jaffe, The holonomy expansion, index theory, and approximate supersymmetry, *Ann. Phys.*, to appear.
8. A. Wightman, *in* "High Energy Electromagnetic Interactions and Field Theory, 1964 Cargèse Lectures" (M. Lévy, Ed.), Gordon Breach, New York, 1967.
9. E. Witten, *Int. J. Mod. Phys. A* **9** (1994), 4738.