Mathematical Truth & Beauty in Physics

Friedrich Hirzebruch Lecture

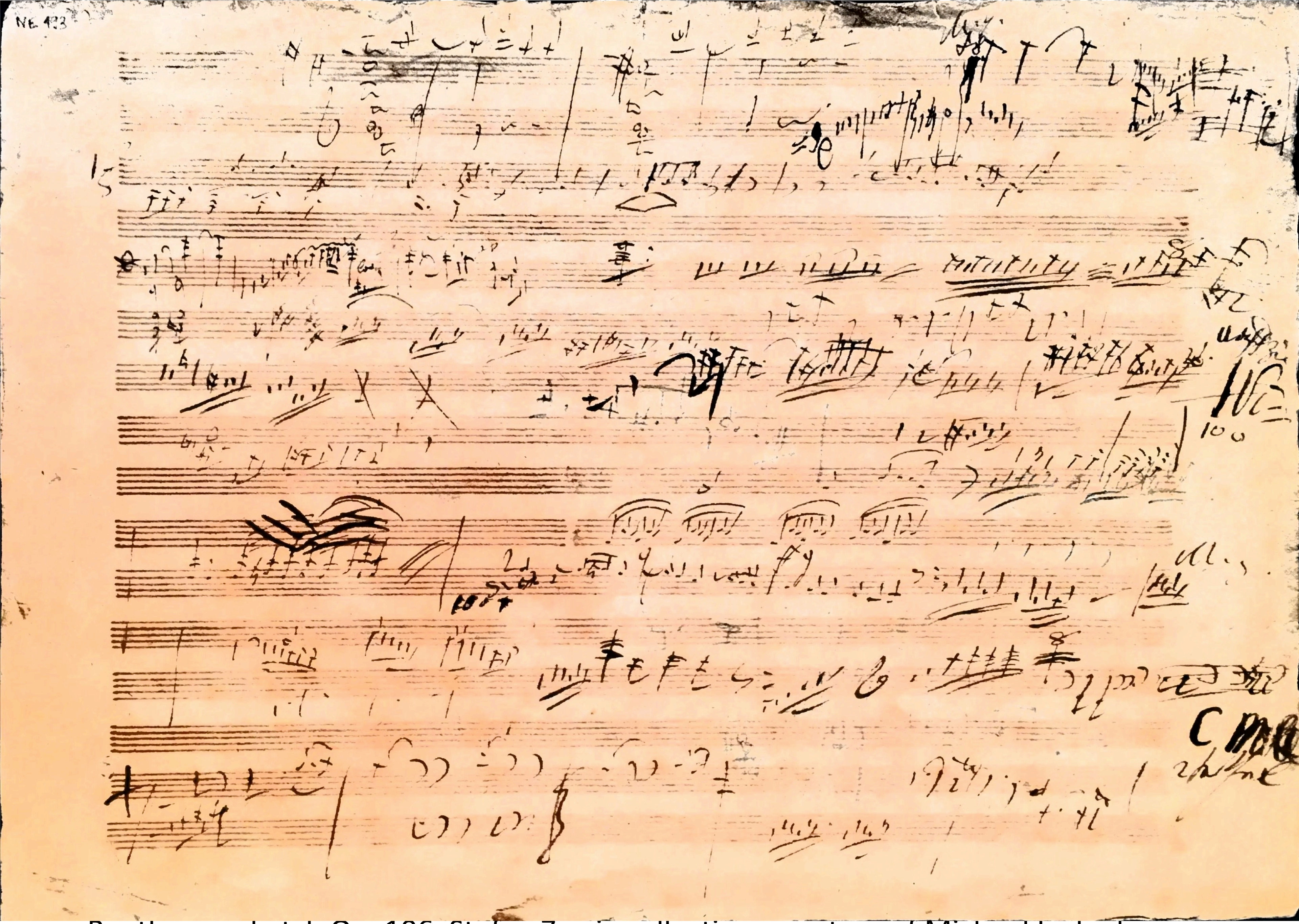
13 October, 2015

18^{th} Century

- Mozart's *Die Entführung aus dem Serail* or "The Abduction from the Seraglio" K. 384 (1782)

- M's counting total number of bars of each act and performing various calculations. Leads to arithmetic series: 2, 3, 5, 6, 28

Courtesy of Robert D. Levin, musician and Mozart expert, Harvard University



Beethoven sketch Op. 106, Stefan Zweig collection, courtesy of Michael Ladenburger, Beethoven Haus Bonn

19th Century

- "Beauty is truth, truth beauty, -that is all ye know on earth, and all ye need to know."
- Ode to a Grecian Urn: John Keats

20th Century

(Dirac, Seminar IAS, Princeton 1961)

- Oppenheimer: "Professor Dirac. At what point did you realize that your relativistic equation for the electron was correct?"

- Dirac: "I was working at home and had to go to the university library to check the spectrum of hydrogen.
- "But as I was getting my bicycle out of the garage, I realized that this equation was so beautiful, it had to be right!"

21st Century

"I hope that you will agree that the only answer to the Question: *Does the world embody beautiful ideas?* … is a resounding Yes!"

Frank Wilczek, "A Beautiful Question: Finding Nature's Deep Design," p321

Optimist

Truth = Beauty!

But the perception of either requires enormous skill!

And enormous time!

= Hard Work

Skeptic

Truth and Beauty generally go together:

But not always!

Mathematical Beauty

Lecture by Fritz Hirzebruch

Occam's razor

Elegance of presentation

- Perspective
- Unifying different ideas

Subjective beauty is difficult to make universal

Scientific Beauty

Occam's razor: simplicity
Elegance, symmetry, unity
Subjective beauty is difficult to make universal

Mathematical Truth

- It is *proof* that is our device for establishing the absolute and irrevocable truth of statements in our subject. This is the reason that we can depend on mathematics that was done by Euclid 2300 years ago as readily as we believe in the mathematics that is done today. No other discipline can make such an assertion. Steven Krantz 2007

Physical Truth

Truth in Physics: Experiment

Objective: Qualitative and Quantitative

- Qualitative: Simple, Elegant Laws
  - Equations of motion
  - Symmetries of nature
  - Observed particles, charges,…
- Quantitative: Spectral Frequencies
  - Constants of Nature
  - Magnetic Moments
  - Particle Masses
  - Cross sections

Truth

		Verification
Mathematics	CONJECTURE TRIAL AND ERRO COMPUTER SIMULATIONS	
Physics	Conjecture Trial and Error Theoretical Physics	Experimental Physics Observation

Truth

		Verification
Mathematics	CONJECTURE TRIAL AND ERRO COMPUTER SIMULATIONS	Proof
Physics	Conjecture Trial and Error Theoretical Physics	Experimental Physics Observation

Truth

		Verification
Mathematics	CONJECTURE TRIAL AND ERRO COMPUTER SIMULATIONS	Experimental Mathematics Proof
Physics	Conjecture Trial and Error Theoretical Physics	Experimental Physics Observation

Truth

	Conjecture	Verification
Mathematics	TRIAL AND ERROR CALCULATION COMPUTER SIMULATIONS	Experimental Mathematics Proof
Physics	Theoretical Physics Theoretical Physics	Experimental Physics Observation

Truth

	Conjecture	Verification
Mathematics	CONJECTURE TRIAL AND COMPUTER SIMULATIONS	Experimental Mathematics Proof
Physics	Theoretical Physics Computer Simulation Trial and Error	Experimental Physics Observation

Truth

	Conjecture	Verification
Mathematics	Theoretical Mathematics Computer Simulation MPUTER SIMULATIONS	Experimental Mathematics Proof
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Example: Fermat's Last Theorem

- Theorized in 1637

$$x^n + y^n = z^n$$

no integer solutions if n > 2

- Experimental Verification 1994, Taylor and Wiles

Riemann Hypothesis posed 1859

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Zeros on negative real axis and real z = ½ (the critical line)

“Theoretical” Evidence

The 10^{22}-nd zero of the Riemann zeta function

A. M. Odlyzko

Abstract. Recent and ongoing computations of zeros of the Riemann zeta function are described. They include the computation of 10 billion zeros near zero number 10^{22}. These computations verify the Riemann Hypothesis for those zeros, and provide evidence for additional conjectures that relate these zeros to eigenvalues of random matrices.

Still no experimental verification! (proof)

Physics: Traditional Understanding

- Newton: planetary motion 17-18th century
- Maxwell: electricity and magnetism 19th
- Boltzmann, Gibbs: statistical physics 19th
- Einstein: relativity, gravitation, Browian 20th motion,...
- Bohr, Schrödinger, Heisenberg, Dirac: 20th

  particle based quantum theory
- ALL PARTS OF MATHEMATICS

Summary Lesson

- In physics one tries to isolate a part of the whole and describe it by mathematics.

Consider:

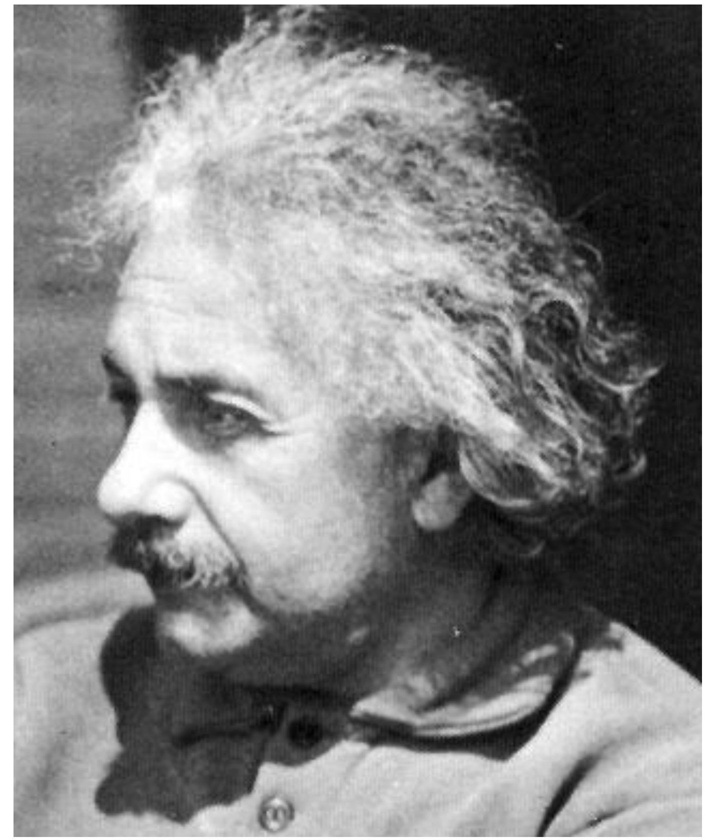
The TWO cornerstones of 20^{th} century physics.

Quantum Theory

$$i\hbar \frac{d\psi}{dt} = H\psi$$

1926

Relativity



$$E = mc^2$$

1905

Relativity Predicts that Particles can Decay and New Ones Created

- Non-relativisitic quantum theory inadequate.
- Proposal: Describe particles by a field of waves

  Forces arise from the field

  All observable phenomena predicted by field

Field and Waves

- Field $\varphi = \varphi(x)$

- Field satisfies a wave equation. Example:

$$(\Box + m^2)\varphi(x) + \varphi^3(x) = 0$$

- Matter field: a field of waves
- Equations given names:
Klein-Gordon, Maxwell, Dirac, Weyl, Yang-Mills,
Quantum Electrodynamics,
Quantum Chromodynamics, Standard Model, ...

Field and Particles

- Particles are "Quanta" of the Field φ
- Ground State (empty) Ω
- One Particle State $\varphi\,\Omega$
- Three Particle State: disentangle $\varphi\,\varphi\,\varphi\,\Omega$

WAVES & PARTICLES

- Waves and particles together in one. This is a way to resolve the "duality" of waves and particles.

Ultimate Quantitative Test

Magnetic moment μ of an electron. Most accurate measurement ever made.

1913 1928

$$\mu = \kappa \frac{e\hbar}{mc} , \quad \kappa_{\mathrm{Bohr}} = \frac{1}{2} , \quad \kappa_{\mathrm{Dirac}} = 1 .$$

$$\kappa = 1,001 \text{ Kusch } (1947)$$

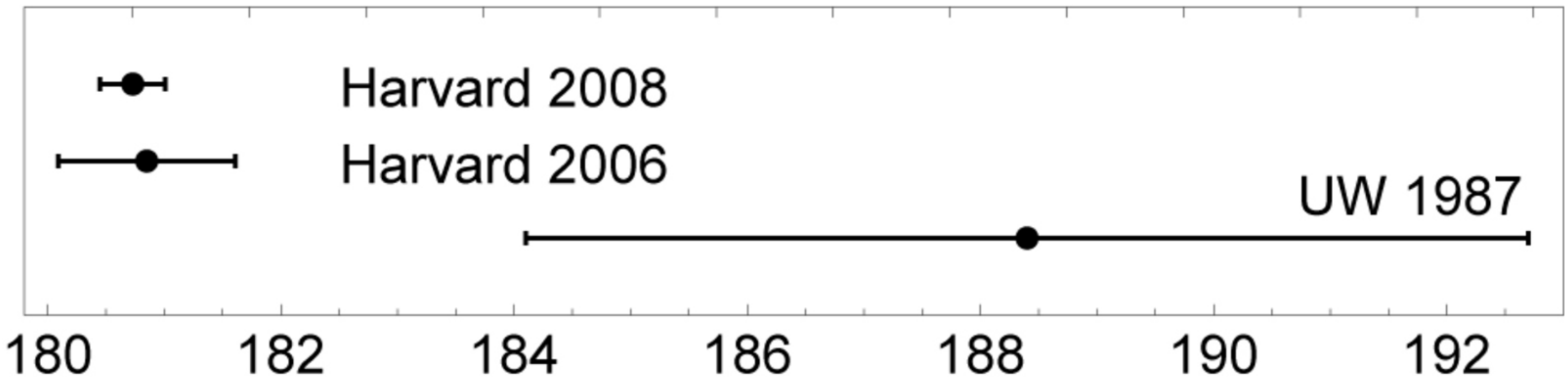
$$\kappa = 1,001\ 159\ 652\ 180\ 73(\pm 28)$$

Deviation from 1 is effect of fields: interaction between the Dirac field of electron and the Maxwell field of electromagnetism.

Represents triumph of fields as basic paradigm; particles are consequences of fields.

Measure: Parts in 10^{12}
Gabrielse (2006, 2008)

$$(\kappa - 1.001\ 159\ 652\ 000) * 10^{12} = 180.73 \pm .28$$



A.I.P. "Physics Achievement of the Year: 2006"

Experiments refined for 65+ Years
Calculation refined for 65+ Years

Most accurate measurement ever performed.

Complete agreement between experiment and rules for calculation!

Something must be "right"!!

23 Nobel Prize winners connected with this spectacular achievement.

YET: Are THE two cornerstones of 20th century physics mathematically compatible?

This *is a* central "big" question!

Short answer: We only know a partial yes!!

Slightly Longer Answer

Exact "non-perturbative" theory:

Constructive Quantum Field Theory

d=2 ✔	life on a line + time
d=3 ✔	flatland + time
d=4 ?	our space-time
d=5 ✗	
d=10,11 ?????	world of strings

Historical Departure

Classical mechanics, classical gravitation, classical Maxwell theory, fluid mechanics, non-relativistic quantum theory, statistical mechanics,..., each has a logical foundation. Each is a branch of *mathematics*.

But: perturbative renormalization analysis leads physicists to believe that rules for quantum electrodynamics are logically inconsistent.

The most accurately known electrodynamics' effect may have no logical explanation using only electrodynamics!

Must Study Yang-Mills

- Yang-Mills theory "asymptotically free"
  - Goes beyond dimension counting (Sobolev)
- This avoids difficulty as does QCD (YM version of electrodynamics)
- However "standard model" has "Higgs field"

A. Weil on Riemann Hypothesis

- 1. When I was young I wanted to prove the RH.
- 2. As I got older I wanted to see the proof.
- 3. Now I would only like to know that it has been proved!

Closed Form of Solution

Solve algebraically

- Appears intractable

First Approach

1. Canonical Approach on Hilbert Space
   - Follow Heisenberg:

$$[q,p] = i\hbar$$

$$[q(x), p(x')] = i\hbar\delta(x - x')$$

   - Hamiltonian

$$H = H(q,p) = H^* \geqslant 0$$

Second Approach

2. Path Integral Approach: Classical

$$\langle A \rangle = \int A(\Phi)\, d\mu(\Phi) \qquad A \in \mathcal{E}$$

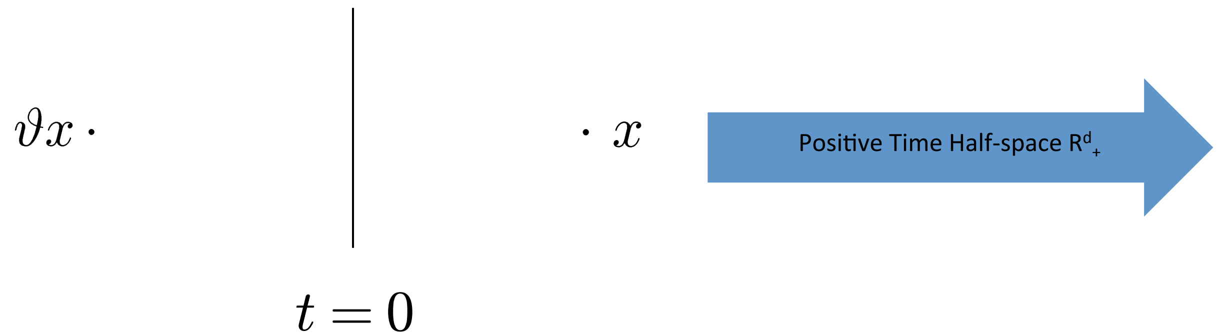
$$\langle A \rangle \text{``}=\text{''} \frac{1}{Z} \int A(\Phi)\, e^{-\mathfrak{A}}\, d\mu_C(\Phi)$$

Both Approaches

- Analysis in infinite dimensions
  - (each Fourier mode)
- Symmetry Key
- Localization (Divide and Conquer)
  - Renormalization Group Ideas
  - Localization (exponential decay)

Unify the Two Approaches

Reflection Positivity:
Konrad Osterwalder and Robert Schrader



Reflection:

$\vartheta(A) =$   $= A$

Osterwalder-Schrader Quantization

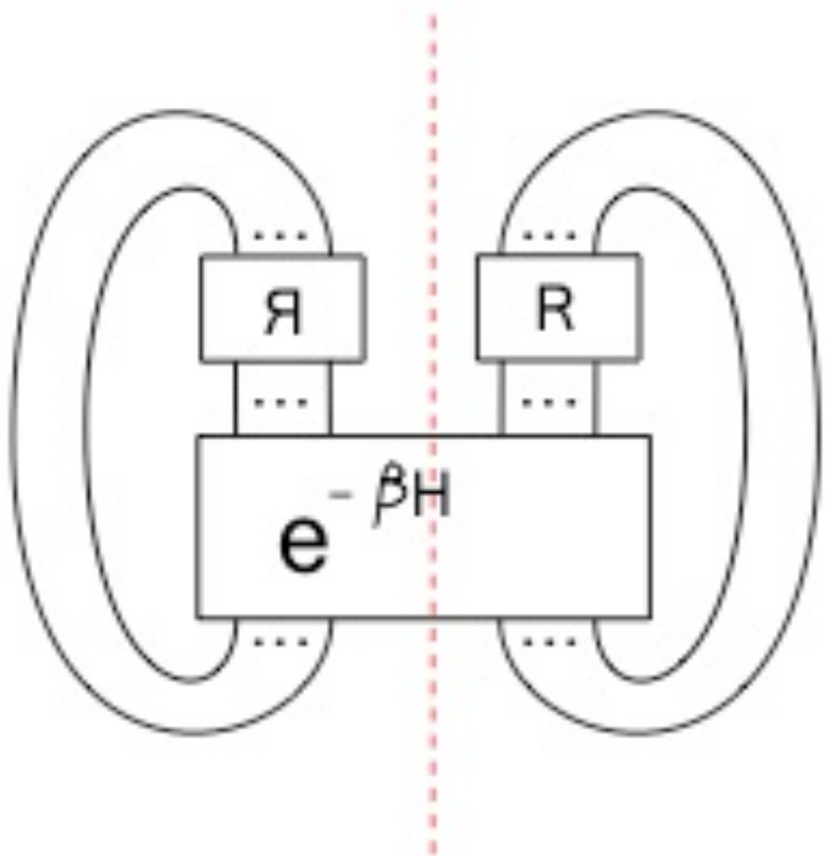
$$\underbrace{\langle \vartheta(A) B \rangle}_{\text{classical}} = \underbrace{\langle A,\, B \rangle_{\mathcal{H}}}_{\text{quantum}}$$

$$B(t) \text{ translate by } t > 0$$

$$\underbrace{\langle \vartheta(A) B(t) \rangle_{\mathcal{E}}}_{\text{classical}} = \underbrace{\langle A, e^{-tH} B \rangle_{\mathcal{H}}}_{\text{quantum}}$$

Pictorial Interpretation
in context of planar algebras

$$\vartheta(A) =$$

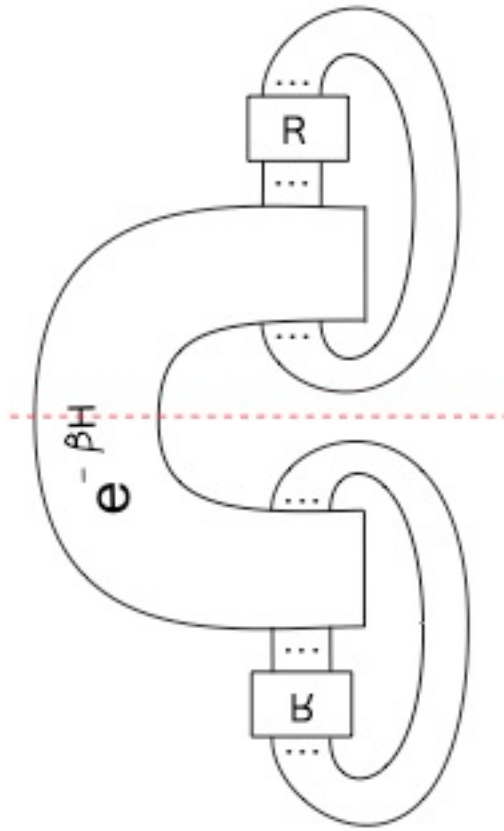


$$= A$$

Multiplication and Reflection

- Multiplication: bottom to top
- Horizontal Reflection:
  - Reflection in time
- Vertical Reflection:
  - Hilbert Space Adjoint
- Assume Invariance of Picture Under Isotopy

180° Rotn. × Reflection = *



$$\langle A^* A \rangle \geqslant 0$$

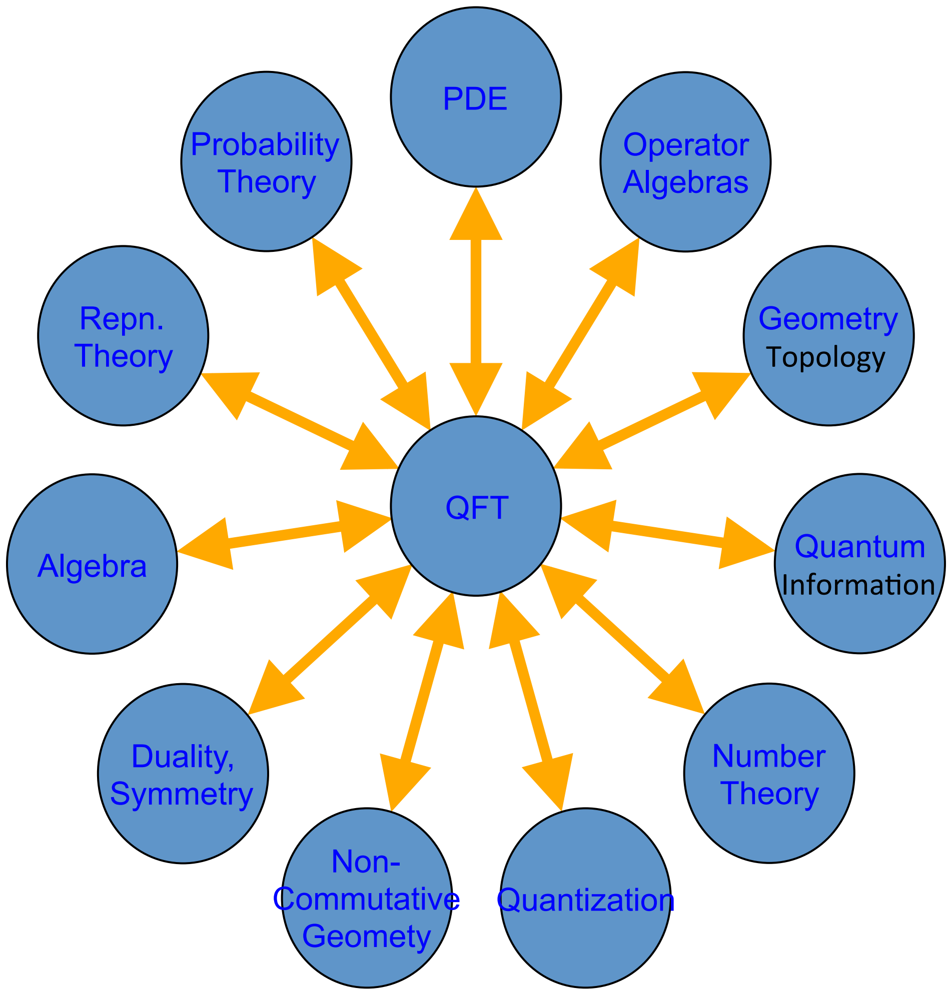
RP Permeates Physics, Mathematics

- Quantum Field Theory
- Condensed Matter Physics of Phase Transitions
- Topological Field Theory
- Quantum Information Theory
- String theory (?)
- Oscillating functionals+
- Planar Algebras*
- Tomita-Takesaki Theory
- Non-commutative Geometry
- Super Symmetry
- Para Symmetry*

+recent work with Christian Jäkel

*new work jointly with Zhengwei Liu & Bas Janssens

Whatever the outcome: QFT Encodes Many Landscapes



Mathematical Beauty and Physical Truth will enjoy a *long* and *happy* marriage!

To enable this requires the attention of young, genius mathematicians!