Replica Condensation and Tree Decay

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Abstract

We give an intuitive method—using local, cyclic replica symmetry—to isolate exponential tree decay in truncated (connected) correlations. We give an expansion and use the symmetry to show that all terms vanish, except those displaying *replica condensation*. The condensation property ensures exponential tree decay.

We illustrate our method in a low-temperature Ising system, but expect that one can use a similar method in other random field and quantum field problems. While considering the illustration, we prove an elementary upper bound on the entropy of random lattice surfaces.

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I Introduction

Symmetry is used widely in physics to unify laws or simplify results. Global symmetries often arise and are characterized by Lie groups or their representation acting on a manifold. Some symmetries, such as gauge symmetry, are local; they are characterized by the action of a group on a bundle over a manifold. Global replica symmetry has been introduced as a symmetry of the Hamiltonian of certain interacting systems such as Ising models, random fields, and quantum fields, leading to valuable insights.

In §III we study *local* replica symmetry. This is *not* a symmetry of the Hamiltonian in general, but it *is* a symmetry within certain spin configurations. This enables us to simplify our expansion of certain expectations in the low-temperature Ising system in order to exhibit a desired property: exponential tree decay of truncated correlations. This low-temperature expansion only serves to illustrate our method. We plan to investigate the use of our method in other high-temperature and low-temperature situations for random and quantum fields.

Consider the truncated expectations $\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle^{\mathrm{T}}$, defined in §IV.1. The Ising spins σ_i are maps from the unit lattice $\mathbb{Z}^d$ in $d \geq 2$ dimensions to ± 1. The Hamiltonian is $H = \frac{1}{2}\|\nabla\sigma\|^2$, and the Gibbs factor is $e^{-\beta H}$, where β denotes the inverse temperature. We show in §VIII that there are constants a, b such that for $\delta_n = \beta - b\ln n \geq 1$,

$$\left|\,\langle\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}\rangle^{\mathrm{T}}\right| \leq a\,n^n\,e^{-\delta_n\tau(i_1,\ldots,i_n)}\;, \tag{I.1}$$

where $\tau(i_1,\ldots,i_n)$ is the length of the minimal tree connecting the n points $i_1,\ldots,i_n$. Note the condition $\delta_n \geq 1$ requires that $\beta \geq \beta_n$, where β_n grows at least as fast as $O(\ln n)$. It would be of interest to eliminate the n-dependence from the minimum value of β.

Our method uses replica variables, comprising n identical, independent copies of the original system; one considers expectations in the replicated system that are product expectations for the individual systems. Replica symmetry is the symmetry of these expectations under a permutation of the copies. For a system in a finite volume Λ, with $i_1,\ldots,i_n \in \Lambda$, the same estimate holds uniformly in Λ. Our method requires unbroken replica symmetry, so one must impose the same boundary conditions in each replica copy.

We develop a low-temperature expansion, based on the intuitive idea that individual terms with less than the desired exponential tree-graph decay sum to zero (vanish) due to symmetry under the local cyclic replica group. In §VIII we define and establish convergence of this expansion. The terms in the expansion are parameterized by *replica continents*. These replica continents are bounded by *random surfaces*. The convergence of our expansion relies on an interplay between energy and entropy estimates; in particular we give entropy estimates bounding the number of random surfaces that occur in our expansion, as well as energy estimates showing that large islands are suppressed at a desired rate.

Key to our method is the use of local cyclic replica symmetry, to show that all non-zero terms in our expansion display *replica condensation*, defined in §V. By this we mean that all the lattice sites $i_1,\ldots,i_n$ must live on a single continent. The size of the boundary of the continent must therefore be larger than $\tau(i_1,\ldots,i_n)$; this is the source of the exponential tree decay.

I.1 The Ising Model as Illustration

The Ising system is the simplest example of a statistical mechanics interaction. We present our method for such a model on a unit cubic lattice $\mathbb{Z}^d$, with $d \geq 2$, although our methods clearly apply in more generality. The Ising Hamiltonian in volume $\Lambda \subset \mathbb{Z}^d$ is

$$H_\Lambda = H_\Lambda(\sigma) = \frac{1}{2}\|\nabla\sigma\|^2_{\ell^2(\Lambda)} = \sum_{nn\in\Lambda} (1 - \sigma_i\sigma_j) \ , \tag{I.2}$$

where nn denotes the sum over nearest-neighbor pairs of sites in the lattice, namely sites with $|i - j| = 1$. The partition function

$$\mathcal{Z}_{\Lambda,\beta} = \sum_{\substack{\sigma_i \\ i\in\Lambda}} e^{-\beta H_\Lambda(\sigma)} \tag{I.3}$$

normalizes statistical averages $\langle f \rangle_{\Lambda,\beta}$ of a function f, namely

$$\langle f \rangle_{\Lambda,\beta} = \frac{1}{\mathcal{Z}_{\Lambda,\beta}} \sum_{\substack{\sigma_i \\ i\in\Lambda}} f(\sigma)\, e^{-\beta H_\Lambda(\sigma)} \ . \tag{I.4}$$

Often f is a monomial in spins, $f = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}$. The expectation $\langle \ \cdot \ \rangle_{\Lambda,\beta}$ is linear, so one can express the expectation of a general f as a limit of finite linear combinations of expectations of the form $\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n} \rangle_{\Lambda,\beta}$.

II The Correspondence $\mathbb{Z}^d \leftrightarrow \mathbb{R}^d$

Each subset $X \subset \mathbb{Z}^d$ of sites in the lattice $\mathbb{Z}^d$ can be identified with a subset $X \subset \mathbb{R}^d$. Define the latter as the union of closed, unit d-cubes $\square_i$ centered at the lattice sites $i \in X$, as we illustrate in the upper part of Figure 1.

Connectedness: We say that $X \subset \mathbb{Z}^d$ is *connected* if any two sites in X can be connected by a continuous path through nearest-neighbor lattice sites in the set X. This agrees with the notion that the interior of the set $X \subset \mathbb{R}^d$ is connected in the ordinary sense. Two cubes are connected if they share a $(d-1)$-dimensional face, but they are disconnected if they only touch on a corner of dimension $\leq (d-2)$.

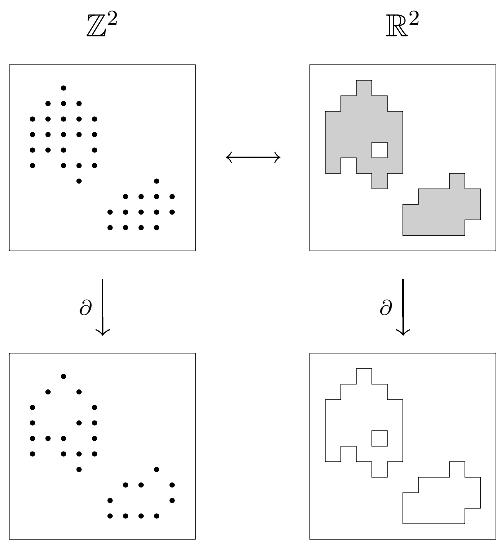


Figure 1: An example for the correspondence between subsets of $\mathbb{Z}^d$ and $\mathbb{R}^d$, and their boundaries.

Boundary: The boundary $\partial X \subset \mathbb{R}^d$ allows us to define the set $\partial X \subset \mathbb{Z}^d$ of boundary lattice sites. These boundary sites $\partial X \in \mathbb{Z}^d$ are those lattice sites in X lying in cubes that share a $(d-1)$-dimensional face with the boundary $\partial X \subset \mathbb{R}^d$.

By $|\partial X|$ we always refer to the area of the $(d-1)$-dimensional surface in $\mathbb{R}^d$ and not the number of points in $\mathbb{Z}^d$. (A single cube $\square_i$, for example, contains exactly 1 boundary lattice site, while $|\square_i| = 2d$.) In most instances we will call this area the "length" of the boundary, but in some cases we will also call it the number of faces of the boundary surface. We illustrate the correspondence between the boundary lattice sites and the boundary of regions in $\mathbb{R}^d$ in the lower part of Figure 1.

Surface: More generally let a *face* in $\mathbb{R}^d$ denote a $(d-1)$-cube; such a cube lies in the boundary of two d-cubes in $\mathbb{R}^d$. A *surface* Y is a union of $(d-1)$-faces, and its area $|Y|$ is the number of $(d-1)$-faces in Y. Lattice sites in Y may lie on either side of the surface Y, but could be limited by selecting an orientation to appropriate sets of faces in Y.

Connected Surface: Define two faces to be adjacent, if they share a $(d-2)$-cube. Likewise, define Y to be *connected* if any two faces in Y can be reached by a continuous path through a sequence of adjacent faces in Y.

III Replica Variables and Symmetry

Choose $n \in \mathbb{Z}_+$ and consider n independent copies of a statistical-mechanical or quantum-field system; these are called n replicas. One can study the properties of expectations under the group of permutations of the replica variables (the *replica group*). The n-element subgroup of cyclic permutation of all the copies is abelian, and it provides useful one-dimensional representations of replica symmetry.

III.1 Replica Variables

We assume that the different replicas are identical and independent. They are defined on the same lattice, they have the same form of interaction, they are given identical boundary conditions, etc. We label the spin variable at the lattice site i by $\sigma_i^{(\alpha)}$, where $\alpha = 1, 2, \ldots, n$ denotes the index of the copy. We also consider the replica spins at site i as a vector $\vec{\sigma}_i$ with the vector components $\sigma_i^{(\alpha)}$.

III.2 The Global Replica Group

The *global replica group* is the symmetric group S_n comprising elements $\pi \in S_n$ with action,

$$\pi : (1, \ldots, n) \mapsto (\pi_1, \ldots, \pi_n) \ . \tag{III.1}$$

The element $\pi \in S_n$ acts on the spins, giving a unitary representation,

$$\boxed{\sigma_i^{(\alpha)} \mapsto (\pi\sigma_i)^{(\alpha)} = \sigma_i^{(\pi^{-1}\alpha)}} \ , \qquad \text{for } \alpha = 1, \ldots, n \ , \text{ and for all } \ i \ . \tag{III.2}$$

The *global cyclic replica group* S_n^{c} is the subgroup of cyclic permutations of n objects, and is generated by the permutation π^0,

$$\pi^0 : (1, \ldots, n) \mapsto (2, \ldots, n, 1) \ . \tag{III.3}$$

Treating the indices α modulo n, substitute $\alpha = n$ for $\alpha = 0$ and write

$$\boxed{\sigma_i^{(\alpha)} \mapsto \left(\pi^0\sigma_i\right)^{(\alpha)} = \sigma_i^{(\alpha-1)}} \ , \qquad \text{for } \alpha = 1, \ldots, n \ , \text{ and for all } \ i \ . \tag{III.4}$$

The matrix representation of (III.4) is $\vec{\sigma}_i \mapsto \pi^0 \vec{\sigma}_i$, where

$$\left(\pi^0 \vec{\sigma}_i\right)^{(\alpha)} = \sum_{\alpha'=1}^{n} \left(\pi^0\right)_{\alpha\,\alpha'} \sigma_i^{(\alpha')}\ , \qquad \text{and} \quad \left(\pi^0\right)_{\alpha\alpha'} = \delta_{\alpha-1\ \alpha'}\ . \tag{III.5}$$

III.3 The Local Cyclic Replica Group

Let $\mathcal{K}$ denote a subset of the lattice $\mathbb{Z}^d$. The *local cyclic replica group* $S_n^c(\mathcal{K})$ is a bundle over S_n^c defined as the action of S_n^c on the spins in $\mathcal{K}$ and the identity on the complement. This group is generated by $\pi^0_{\mathcal{K}}$ which has the representation on spins,

$$\pi^0_{\mathcal{K}} \vec{\sigma}_i = \begin{cases} \pi^0 \vec{\sigma}_i\ , & \text{when} \quad i \in \mathcal{K} \\ \vec{\sigma}_i\ , & \text{when} \quad i \notin \mathcal{K} \end{cases}\ . \tag{III.6}$$

III.4 Irreducible Representations

The cyclic replica group is abelian, so its irreducible representations are one dimensional. We transform from $\vec{\sigma}_i$ to a set of coordinates $\vec{s}_i = U\vec{\sigma}_i$ to reduce the representation of S_n^c. In particular, let $\omega = e^{2\pi i/n}$ denote the primitive n^{th} root of unity. Define

$$\boxed{s_i^{(\alpha)} = \frac{1}{n^{1/2}} \sum_{\alpha'=1}^{n} \omega^{\alpha(\alpha'-1)} \sigma_i^{(\alpha')}}\ , \qquad \text{for} \ \ \alpha = 1, \ldots, n\ . \tag{III.7}$$

Note that for $n > 2$ the s-variables may be complex, even though the original σ-spins are real. The choice (III.7) defines the entries of the matrix U as $U_{\alpha\alpha'} = n^{-1/2}\omega^{\alpha(\alpha'-1)}$. This is Fourier transform in the replica space.

Proposition III.1. *The matrix U is unitary with eigenvalues ω^α, for $\alpha = 1, \ldots, n$. Let D be the diagonal matrix with $D_{\alpha\alpha'} = \omega^\alpha \delta_{\alpha\alpha'}$. Then*

$$\pi^0 \vec{s}_i = D\vec{s}_i\ . \tag{III.8}$$

Proof. For ν an integer (modulo n),

$$\sum_{\alpha=1}^{n} \omega^{-\nu\alpha} = n\,\delta_{\nu 0}\ . \tag{III.9}$$

Thus

$$(UU^*)_{\alpha\alpha'} = \sum_{\beta=1}^{n} U_{\alpha\beta}\overline{U_{\alpha'\beta}} = \frac{1}{n}\sum_{\beta=1}^{n} \omega^{(\alpha-\alpha')(\beta-1)} = \delta_{\alpha\alpha'} \ . \tag{III.10}$$

Since π^0 acts on the $\vec{\sigma}_i$ components according to (III.4), this means that

$$\left(\pi^0 \vec{s}_i\right)^{(\alpha)} = \omega^{\alpha} \left(\vec{s}_i\right)^{(\alpha)} = \sum_{\alpha'=1}^{n} D_{\alpha\alpha'} \left(\vec{s}_i\right)^{(\alpha)} \ , \tag{III.11}$$

which is (III.8). □

The inverse change of coordinates is

$$\boxed{\sigma_i^{(\gamma)} = \frac{1}{n^{1/2}} \sum_{\alpha=1}^{n} \omega^{-(\gamma-1)\alpha} s_i^{(\alpha)}}\ , \qquad \text{for } \ \gamma = 1, \ldots, n \ . \tag{III.12}$$

A further corollary of the unitarity of U is the fact that for any i, j

$$\sum_{\alpha=1}^{n} \sigma_i^{(\alpha)} \sigma_j^{(\alpha)} = \langle \vec{\sigma}_i, \vec{\sigma}_j \rangle_{\ell^2} = \langle U\vec{\sigma}_i, U\vec{\sigma}_j \rangle_{\ell^2} = \langle \vec{s}_i, \vec{s}_j \rangle_{\ell^2} = \sum_{\alpha=1}^{n} \overline{s_i^{(\alpha)}} s_j^{(\alpha)} \ . \tag{III.13}$$

In particular, the expression on the right side of this identity is always real. Furthermore, each individual term on the right is invariant under the elements of the local, cyclic replica group $S_n^c(\mathcal{K})$ as long as both $i, j \in \mathcal{K}$ or both $i, j \notin \mathcal{K}$.

III.5 Replica Boundary Conditions

We consider finite volume Hamiltonians that, along with their boundary conditions, have the global replica group as a symmetry. If one wished to investigate the breaking of the replica group in the infinite volume limit, then one might explicitly break replica symmetry in a finite volume by imposing different boundary conditions for different replica copies of the system.

Since our system is originally given in terms of the variables σ_i, one describes the boundary conditions in the volume Λ in terms of the variables σ_i for $i \in \partial\Lambda$, with $\partial\Lambda$ defined in §II.

It is natural to ensure symmetry under the replica group by specifying the same boundary condition on each component of the vector spin

$$\sigma_i^{(\alpha)} = \sigma_i \;, \qquad \text{for all } \; i \in \partial\Lambda \;, \; \text{ and all } \; \alpha = 1, \ldots, n \;. \tag{III.14}$$

In order to simplify the discussion, we impose $+1$ boundary conditions in each replica copy: set

$$\boxed{\vec{\sigma}_i = (+1, \ldots, +1)} \;, \;\; \text{when } \; i \in \partial\Lambda \;. \tag{III.15}$$

The resulting boundary conditons for $\vec{s}$ are

$$\vec{s}_i = \left(0, 0, \ldots, 0, n^{1/2}\right) \;, \;\; \text{when } \; i \in \partial\Lambda \;. \tag{III.16}$$

III.6 Replica Symmetry is Global, not Local

Define the total replica Hamiltonian H_{replica} as the sum of the Hamiltonians for the replica copies of the Hamiltonian in volume Λ,

$$\boxed{H_{\text{replica}} = H_{\text{replica}}(\vec{\sigma}) = \frac{1}{2}\|\nabla\vec{\sigma}\|_{\ell^2(\Lambda)} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\text{nn}\in\Lambda}\left(\sigma_i^{(\alpha)} - \sigma_j^{(\alpha)}\right)^2} \;. \tag{III.17}$$

Proposition III.2. *Consider the replica Hamiltonian* (III.17)*.*

i. As a function of the variables $\vec{s}$, one has

$$\boxed{H_{\text{replica}} = \frac{1}{2}\|\nabla\vec{\sigma}\|_{\ell^2(\Lambda)} = \frac{1}{2}\|\nabla\vec{s}\|_{\ell^2(\Lambda)} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\text{nn}\in\Lambda}\left|s_i^{(\alpha)} - s_j^{(\alpha)}\right|^2} \;. \tag{III.18}$$

ii. The replica Hamiltonian (III.18) *is invariant under a global replica permutation $\pi \in S_n$ defined in* (III.2)*, namely*

$$\boxed{H_{\text{replica}}(\pi\vec{s}) = H_{\text{replica}}(\vec{s})} \;. \tag{III.19}$$

iii. In general, the replica Hamiltonian is not *invariant under the local cyclic replica group $S_n^{\mathrm{c}}(\mathcal{K})$ defined in* (III.6)*.*

Proof. The relation (III.13) shows that H_{replica} has the form (III.18). The invariance under the global replica group follows by considering the effect on H_{replica} expressed in the $\vec{\sigma}$ variables, where the transformation permutes the various terms $H_\Lambda(\sigma^{(\alpha)})$ in the first expression for H_{replica} in (III.17).

In order to see that $H_{\text{replica}}(\vec{\sigma})$ is not invariant under the local cyclic replica group, we give a configuration $\vec{\sigma}$ and set $\mathcal{K}$ that provides a counterexample in the case $n = 2$. It is easiest to visualize this configuration by illustrating it; see the left side of Figure 2. We choose $\mathcal{K}$ to be the centermost square in the configuration (with $\sigma^{(1)} = +1$ and $\sigma^{(2)} = -1$), and choose $\pi_{\mathcal{K}} \in S_n^c(\mathcal{K})$ to flip the spins in $\mathcal{K}$. The action of $\pi_{\mathcal{K}}$ produces the configuration on the right side of the figure, and it lowers the energy by $4|\partial\mathcal{K}|$. In other words, $H_{\text{replica}}(\vec{\sigma}) - H_{\text{replica}}(\pi_{\mathcal{K}}\vec{\sigma}) = 4|\partial\mathcal{K}|$, showing that H_{replica} is not invariant under the action of $S_n^c(\mathcal{K})$.

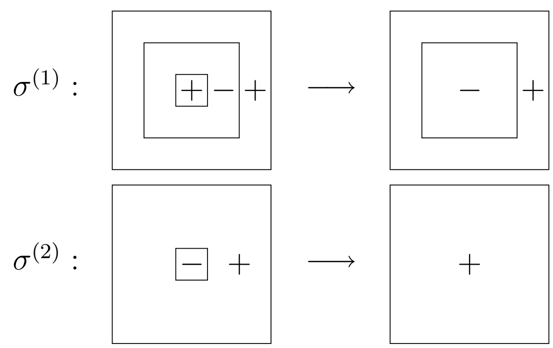


Figure 2: A counter-example to local cyclic replica symmetry.

□

IV Expectations

Define the expectation $\ll \cdot \gg_{\Lambda,\beta}$ for the replicated system as follows: for a function $F(\vec{\sigma})$, let

$$\ll F \gg_{\Lambda,\beta} = \frac{1}{\mathfrak{z}} \sum_{\substack{\vec{\sigma}_i \\ i\in\Lambda}} F(\vec{\sigma}) e^{-\beta H_{\text{replica}}(\vec{\sigma})} , \tag{IV.1}$$

where $\mathfrak{z} = \mathcal{Z}^n$, with $\mathcal{Z}$ is given in (I.3). In case that $F(\vec{\sigma}) = f(\sigma^{(\alpha)})$ only depends on one component $\sigma^{(\alpha)}$, the expectation $\ll \cdot \gg_{\Lambda,\beta}$ reduces to the expectation $\langle \,\cdot\, \rangle_{\Lambda,\beta}$. In this case

$$\ll f(\sigma^{(\alpha)}) \gg_{\Lambda,\beta} = \langle f(\sigma) \rangle_{\Lambda,\beta} \ , \qquad \text{for } \ \alpha = 1, \ldots, n \ . \tag{IV.2}$$

We now introduce the generating function $S(\mu)$ for expectations of products of spins. Let μ be a function from Λ to $\mathbb{C}$ and let

$$\sigma(\mu) = \sum_{i\in\Lambda} \mu_i \, \sigma_i \ , \qquad \text{and correspondingly } \ \sigma^{(\alpha)}(\mu) = \sum_{i\in\Lambda} \mu_i \, \sigma_i^{(\alpha)} \ . \tag{IV.3}$$

Then define

$$S(\mu) = \left\langle e^{\sigma(\mu)} \right\rangle_{\Lambda,\beta} = \ll e^{\sigma^{(\alpha)}(\mu)} \gg_{\Lambda,\beta} \ . \tag{IV.4}$$

The expectations of n spins are derivatives of the generating function,

$$\langle \sigma_{i_1} \, \sigma_{i_2} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta} = \left. \frac{\partial^n}{\partial\mu_{i_1}\partial\mu_{i_2}\cdots\partial\mu_{i_n}} S(\mu) \right|_{\mu_i=0} = \ll \sigma_{i_1}^{(1)} \, \sigma_{i_2}^{(1)} \cdots \sigma_{i_n}^{(1)} \gg_{\Lambda,\beta} \ . \tag{IV.5}$$

The expectations (IV.5) are n-multi-linear, symmetric, functions of the spins,

$$\langle \sigma(\mu)^n) \rangle_{\Lambda,\beta} = \sum_{i_1,\ldots,i_n=1}^{n} \mu_{i_1} \cdots \mu_{i_n} \, \langle \sigma_{i_1} \, \sigma_{i_2} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta} \ . \tag{IV.6}$$

One can recover the expectation $\langle \sigma_{i_1} \, \sigma_{i_2} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta}$ from the expectations of powers of $\sigma(\mu)$ by polarization,

$$\langle \sigma_{i_1} \, \sigma_{i_2} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta} = \frac{1}{2^n n!} \sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \epsilon_1 \cdots \epsilon_n \, \left\langle (\epsilon_1 \sigma_{i_1} + \cdots + \epsilon_n \sigma_{i_n})^n \right\rangle_{\Lambda,\beta} \ . \tag{IV.7}$$

IV.1 Truncated Expectations

The truncated expectation of a product of n spins is a generalization of the correlation of two spins. The truncated expectation vanishes asymptotically as one translates any subset of the spin locations a large distance away from the others.

The generating function of the connected expectations is

$$G(\mu) = \ln S(\mu) = \ln \left\langle e^{\sigma(\mu)} \right\rangle_{\Lambda,\beta} \ . \tag{IV.8}$$

One defines the truncated (connected) expectations as

$$\langle \sigma_{i_1}\, \sigma_{i_2}\, \sigma_{i_3} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta} \quad = \quad \frac{\partial^n}{\partial \mu_{i_1} \partial \mu_{i_2} \cdots \partial \mu_{i_n}} G(\mu) \Big|_{\mu_i = 0} \, . \tag{IV.9}$$

A standard representation of $\langle \sigma_{i_1}\, \sigma_{i_2}\, \sigma_{i_3} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta}$ in terms of sums of products of expectations can be formulated in terms of the set $\mathcal{P}$ of partitions of $\{i_1, i_2, \ldots, i_n\}$. Suppose that a set $P \in \mathcal{P}$ has cardinality $|P|$. Then

$$\langle \sigma_{i_1}\, \sigma_{i_2}\, \sigma_{i_3} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta} = \sum_{\mathcal{P}} \prod_{P \in \mathcal{P}} \langle \sigma^P \rangle^{\mathrm{T}}_{\Lambda,\beta} \, . \tag{IV.10}$$

Like the expectations (IV.5), the n-truncated expectations satisfy the n-multi-linear relation (IV.6)–(IV.7). Thus

$$\langle \sigma(\mu)^n \rangle^{\mathrm{T}}_{\Lambda,\beta} = \sum_{i_1,\ldots,i_n=1}^{n} \mu_{i_1} \cdots \mu_{i_n} \langle \sigma_{i_1}\, \sigma_{i_2}\, \sigma_{i_3} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta} \, , \tag{IV.11}$$

and

$$\langle \sigma_{i_1}\, \sigma_{i_2} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta} = \frac{1}{2^n n!} \sum_{\epsilon_1,\ldots,\epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n \langle (\epsilon_1 \sigma_{i_1} + \cdots + \epsilon_n \sigma_{i_n})^n \rangle^{\mathrm{T}}_{\Lambda,\beta} \, . \tag{IV.12}$$

IV.2 Truncated Functions as Replica Expectations

The form of the replica variables $\vec{s}$ leads to an elementary representation of the truncated (connected) expectations of products of spins. Ultimately we show that this yields exponential decay at low temperatures with a rate governed by the length of the shorted tree-graph connecting all the spins. (A similar argument presumably works at high temperature.)

Our expansion method uses replica symmetry to arrange that each term in the expansion either exhibits the desired decay rate, or else it is canceled by other terms as a consequence of local cyclic replica symmetry. We begin by establishing a known representation of the connected correlation of n spins as an expectation of n replica variables introduced above. This representation was discovered by P. Cartier (unpublished); our presentation is based on Sylvester's treatment [2] using $s^{(1)}$. Let g.c.d. denote the greatest common divisor.

Proposition IV.1. *Let $\vec{s}$ be defined in* (III.7) *with n replica copies, and let $\gamma \in (1, \ldots, n)$ satisfy* g.c.d.$(n, \gamma) = 1$. *Then*

$$\boxed{\langle \sigma_{i_1}\, \sigma_{i_2} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta} = n^{(n-2)/2} \ll s^{(\gamma)}_{i_1}\, s^{(\gamma)}_{i_2} \cdots s^{(\gamma)}_{i_n} \gg_{\Lambda,\beta}}\,. \tag{IV.13}$$

Lemma IV.2. *For all $\gamma = 1, \ldots, n$,*

$$\ll s^{(\gamma)}_{i_1}\, s^{(\gamma)}_{i_2} \cdots s^{(\gamma)}_{i_n} \gg^{\mathrm{T}}_{\Lambda,\beta} = n^{-(n-2)/2}\ \langle \sigma_{i_1}\, \sigma_{i_2} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta}\ . \tag{IV.14}$$

Proof. Using the multi-linearity (IV.11), and its analog for the expectations $\langle\ \cdot\ \rangle_{\Lambda,\beta}$ and $\ll\ \cdot\ \gg_{\Lambda,\beta}$ of the truncated functions, we infer that

$$\begin{aligned}
&\ll s^{(\gamma)}_{i_1}\, s^{(\gamma)}_{i_2} \cdots s^{(\gamma)}_{i_n} \gg^{\mathrm{T}}_{\Lambda,\beta} \\
&\qquad = n^{-n/2} \ll \sum_{\alpha_1,\ldots,\alpha_n=1}^{n} \omega^{\gamma\alpha_1+\cdots+\gamma\alpha_n-\gamma n}\, \sigma^{(\alpha_1)}_{i_1}\, \sigma^{(\alpha_2)}_{i_2} \cdots \sigma^{(\alpha_n)}_{i_n} \gg^{\mathrm{T}}_{\Lambda,\beta} \\
&\qquad = n^{-n/2} \sum_{\alpha_1,\ldots,\alpha_n=1}^{n} \omega^{\gamma\alpha_1+\cdots+\gamma\alpha_n-\gamma n} \ll \sigma^{(\alpha_1)}_{i_1}\, \sigma^{(\alpha_2)}_{i_2} \cdots \sigma^{(\alpha_n)}_{i_n} \gg^{\mathrm{T}}_{\Lambda,\beta}\ .
\end{aligned} \tag{IV.15}$$

Since the different components of $\vec{\sigma}_i$ are independent, the expectations on the right vanishes unless $\alpha_1 = \cdots = \alpha_n$. In this case the truncated expectation of each copy equals the truncated expectation of the original spins, and the sum yields n such terms. Therefore (IV.14) holds as claimed.

□

Lemma IV.3. *Let $k\gamma \neq 0$* (modulo n)*. Then*

$$\boxed{\ll s^{(\gamma)}_{i_1}\, s^{(\gamma)}_{i_2} \cdots s^{(\gamma)}_{i_k} \gg_{\Lambda,\beta} = 0}\,. \tag{IV.16}$$

Proof. Expand the expectation

$$\begin{aligned}
&\ll s^{(\gamma)}_{i_1}\, s^{(\gamma)}_{i_2} \cdots s^{(\gamma)}_{i_k} \gg_{\Lambda,\beta} \\
&\qquad = \frac{1}{n^{n/2}} \sum_{\alpha_1,\ldots,\alpha_n=1}^{n} \omega^{\gamma\alpha_1+\cdots+\gamma\alpha_k-\gamma k} \ll \sigma^{(\alpha_1)}_{i_1}\, \sigma^{(\alpha_2)}_{i_2} \cdots \sigma^{(\alpha_n)}_{i_k} \gg_{\Lambda,\beta} \\
&\qquad = \frac{1}{n^{n/2}} \sum_{\alpha_1,\ldots,\alpha_n=1}^{n} \omega^{\gamma\alpha_1+\cdots+\gamma\alpha_k-\gamma k} \ll \sigma^{(\alpha_1-1)}_{i_1}\, \sigma^{(\alpha_2-1)}_{i_2} \cdots \sigma^{(\alpha_n-1)}_{i_k} \gg_{\Lambda,\beta}\ .
\end{aligned} \tag{IV.17}$$

In the second equality, we use the symmetry of the expectation $\ll \cdot \gg_{\Lambda,\beta}$ under the global cyclic replica group $S_n^c \ni \pi^0$. Therefore

$$\ll s_{i_1}^{(\gamma)} s_{i_2}^{(\gamma)} \cdots s_{i_k}^{(\gamma)} \gg_{\Lambda,\beta} = \omega^{\gamma k} \ll s_{i_1}^{(\gamma)} s_{i_2}^{(\gamma)} \cdots s_{i_k}^{(\gamma)} \gg_{\Lambda,\beta} . \tag{IV.18}$$

As long as $\gamma k \neq 0$ (modulo n), it is the case that $\omega^{\gamma k} \neq 1$. Therefore the expectation must vanish. □

Proof of the Proposition. The relation (IV.10) also holds for the replica expectations,

$$\ll s_{i_1}^{(\gamma)} s_{i_2}^{(\gamma)} \cdots s_{i_n}^{(\gamma)} \gg_{\Lambda,\beta} = \sum_{\mathcal{P}} \prod_{P \in \mathcal{P}} \ll s^{(\gamma)\,P} \gg_{\Lambda,\beta}^{\mathrm{T}} . \tag{IV.19}$$

Because g.c.d.$(n,\gamma) = 1$, it is the case that $k\gamma \neq 0$ (modulo n) for all $k = 1, \ldots, n-1$. Thus we can apply Lemma IV.3 to each such k, and only the partition P with all n elements in one set survives in (IV.19). We infer

$$\ll s_{i_1}^{(\gamma)} s_{i_2}^{(\gamma)} s_{i_3}^{(\gamma)} \cdots s_{i_n}^{(\gamma)} \gg_{\Lambda,\beta} = \ll s_{i_1}^{(\gamma)} s_{i_2}^{(\gamma)} s_{i_3}^{(\gamma)} \cdots s_{i_n}^{(\gamma)} \gg_{\Lambda,\beta}^{\mathrm{T}} . \tag{IV.20}$$

Using Lemma IV.2 then completes the proof.

□

V Replica Condensation

In this section we investigate certain classes of configurations $\vec{\sigma}$ of the replica spins. We see that for each class of configurations, there is a local cyclic replica group (see §III.3) under which the Hamiltonian H_{replica} of (III.17) is invariant. This leads to the phenomenon of *replica condensation* in which all the spin localizations $i_1, \ldots, i_n$ must be localized within a given region $\mathcal{K} \subset \Lambda$ that we call a *continent.*

V.1 Continents

Each configuration of spins $\vec{\sigma}$ in the volume Λ defines a *sea* $\mathcal{S}(\vec{\sigma})$, surrounding a set of *continents* $\mathcal{K}(\vec{\sigma})$. The sea starts at the boundary boundary $\partial\Lambda$ of the region Λ. The boundary of a continent appears if any one of the components of $\vec{\sigma}$ changes its value. Continents have a substructure arising from the different configurations of the individual components $\sigma^{(\alpha)}$ within the continent. We say more about this substructure when defining *replica continent contours* in §VI.2. In the following we utilize the notion of "connectedness" introduced in §II.

Definition V.1. *Consider a configuration $\vec{\sigma}$. The* replica sea $\mathcal{S}(\vec{\sigma})$ *is the connected component of the set $\{i \mid \vec{\sigma}_i = (+1, \ldots, +1)\}$ that meets the boundary $\partial\Lambda$ of Λ. The* continents $\mathcal{K}_j$ *are the connected components of the complementary set, $\mathcal{S}^c(\vec{\sigma}) = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$. The set of continents $\mathcal{K}(\vec{\sigma})$ is*

$$\mathcal{K}(\vec{\sigma}) = \{\mathcal{K}_1, \ldots, \mathcal{K}_r\} \; . \tag{V.1}$$

We illustrate this definition in Figure 3.

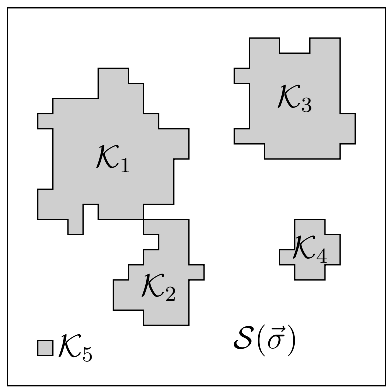


Figure 3: The set of continents $\mathcal{K}(\vec{\sigma}) = \{\mathcal{K}_1, \ldots, \mathcal{K}_5\}$ in the sea $\mathcal{S}(\vec{\sigma})$.

V.2 Local Cyclic Replica Symmetry

In §III.6 we saw that a global replica symmetry transformation leaves $H_{\text{replica}}(\vec{\sigma})$ invariant, and that a local replica symmetry transformation does not necessarily do so. We now recover local cyclic replica symmetry by choosing the localization $\mathcal{K}$ in $S_n^c(\mathcal{K})$ to be a continent.

Proposition V.2. *Let $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$. Then the local cyclic replica group $S_n^c(\mathcal{K})$ defined in* (III.6) *preserves the continent $\mathcal{K}$ and the Hamiltonian $H_{\text{replica}}(\vec{\sigma})$. For $\pi_{\mathcal{K}} \in S_n^c(\mathcal{K})$,*

$$H_{\text{replica}}(\vec{\sigma}) = H_{\text{replica}}(\pi_{\mathcal{K}}(\vec{\sigma})) \; . \tag{V.2}$$

Proof. The action of $S_n^c(\mathcal{K})$ on $\vec{\sigma}$ leaves invariant spins $\vec{\sigma}_i = (+1, \ldots, +1)$, so it changes neither the sea $\mathcal{S}(\vec{\sigma})$ nor the definition of continents. Hence it also does not change the contribution of nearest neighbor spins to the energy either inside or outside the continent. The local permutation also does not alter the energy across the island boundary, because all the components outside the island have value $+1$ and are invariant under the permutation. $\square$

V.3 Symmetry Ensures Condensation

We now establish the property of condensation. We use the representation (IV.13) for the truncated correlation function of n spins. We may choose any γ with g.c.d.$(n, \gamma) = 1$, so for simplicity we consider the case $\gamma = 1$.

Proposition V.3 (Condensation). *In the expectation* $\ll s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} \gg_{\Lambda,\beta}$, *any configuration* $\vec{\sigma}$ *giving a nonzero contribution has all the sites* $i_1, \ldots, i_n \in \mathcal{K}$ *lying in a single continent* $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$.

Lemma V.4. *Consider a given configuration* $\vec{\sigma}$ *and a continent* $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$ *containing at least one but not all the sites* $i_1, \ldots, i_n$. *Let* $\pi_{\mathcal{K}}^k$ *denote* $\pi_{\mathcal{K}}$ *applied* k *times. Then*

$$\begin{aligned}
\sum_{k=0}^{n-1} & \left(\pi_{\mathcal{K}}^k s_{i_1}^{(1)}\right) \cdots \left(\pi_{\mathcal{K}}^k s_{i_n}^{(1)}\right) \; e^{-\beta H_{\mathrm{replica}}\left(\pi_{\mathcal{K}}^k(\vec{\sigma})\right)} \\
& = \sum_{k=0}^{n-1} s_{i_1}^{(1)} \left(\pi_{\mathcal{K}}^k(\vec{\sigma})\right) \cdots s_{i_n}^{(1)} \left(\pi_{\mathcal{K}}^k(\vec{\sigma})\right) \; e^{-\beta H_{\mathrm{replica}}\left(\pi_{\mathcal{K}}^k(\vec{\sigma})\right)} \\
& = 0 \; .
\end{aligned} \tag{V.3}$$

Proof. From Proposition V.2 we infer that the energy in the permuted configuration is unchanged by the permutation,

$$H_{\mathrm{replica}}\left(\pi_{\mathcal{K}}^k(\vec{\sigma})\right) = H_{\mathrm{replica}}(\vec{\sigma}) \; . \tag{V.4}$$

Therefore, we only need consider the changes to the spins $s_{i_k}^{(1)}$. Let $l = |\{k | i_k \in \mathcal{K}\}|$ denote the number of sites $i_1, \ldots, i_k$ that lie in $\mathcal{K}$; clearly $1 \leq l < n$. According to Proposition III.1, the application of $\pi_{\mathcal{K}}$ to $s_i^{(1)}$ gives a phase ω for $i \in \mathcal{K}$. The sum equals

$$\sum_{k=0}^{n-1} \omega^{kl} s_{i_1}^{(1)} s_{i_2}^{(1)} \cdots s_{i_n}^{(1)} e^{-\beta H_{\text{replica}}(\vec{\sigma})} = s_{i_1}^{(1)} s_{i_2}^{(1)} \cdots s_{i_n}^{(1)} e^{-\beta H_{\text{replica}}(\vec{\sigma})} \sum_{k=0}^{n-1} \omega^{kl} = 0 \ . \tag{V.5}$$

□

Proof of Proposition V.3. The expectation is

$$\ll s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} \gg_{\Lambda,\beta} = \sum_{\vec{\sigma}} s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} e^{-\beta H_{\text{replica}}(\vec{\sigma})} / \mathfrak{Z} \ . \tag{V.6}$$

If $\vec{\sigma}$ is a configuration where some site i_k lies in the sea $i_k \in \mathcal{S}(\vec{\sigma})$ then the spin has the value of the boundary, $s_{i_k}^{(1)} = 0$. We also have $s_{i_k}^{(1)} = 0$, if $i_k \in \mathcal{K}$ and all the $\sigma^{(\alpha)}$ take the same values on $\mathcal{K}$.

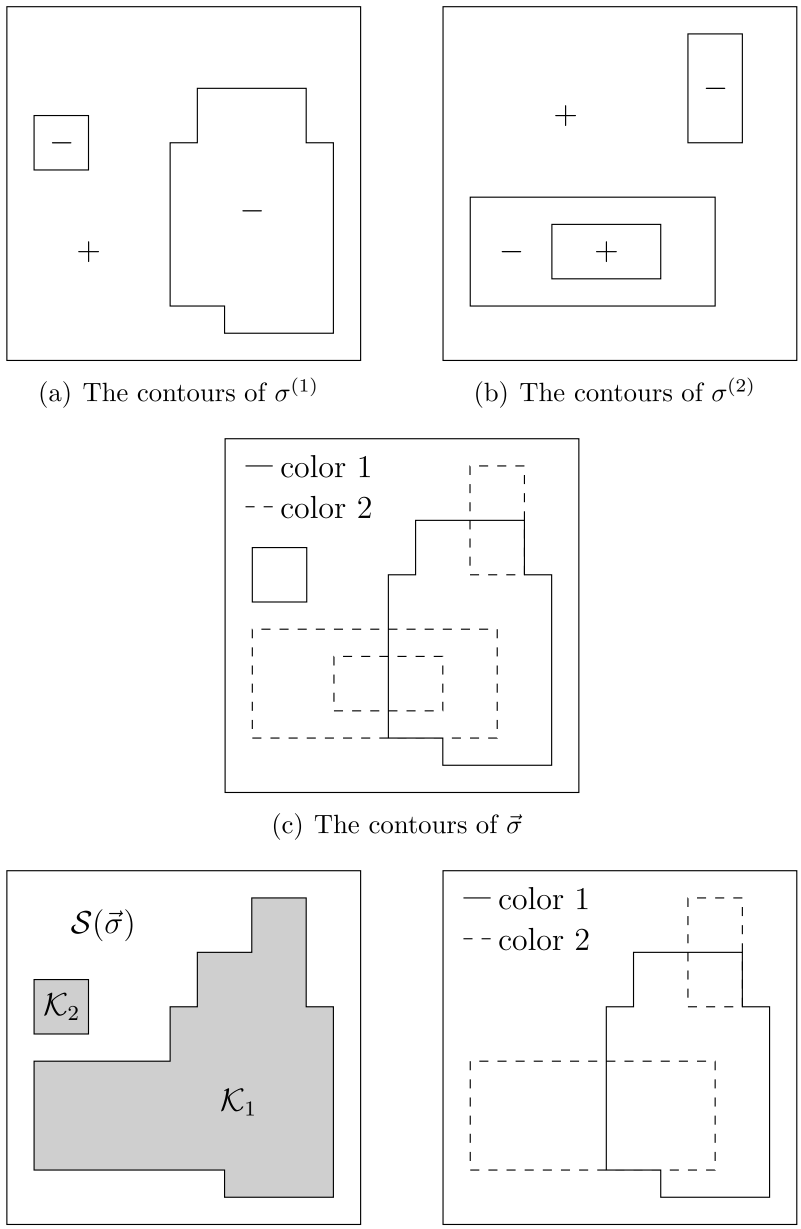
Therefore, the only contributing configurations have all the sites i_k lying in continents where $\pi_{\mathcal{K}}$ actually yields new configurations. In this case, the sum in Lemma V.4 is a sub-sum of (V.6). According to the lemma the sum is only nonzero if all or none of the i_k lie in the contintent $\mathcal{K}$. □

VI Contours and the Energy

VI.1 Contours for Vector Spins $\vec{\sigma}$

For each component $\sigma^{(\alpha)}$ of the vector spin, we can define contours in the usual statistical mechanics sense. These contours are the boundaries between islands with different values of $\sigma^{(\alpha)}$, as defined in §II. They are subsets of the lattice dual to $\mathbb{Z}^d$, consisting of $(d-1)$-faces of d-cubes.

The $\vec{\sigma}$ contours are the direct sum of contours in the individual components. In order to picture the boundaries of $\vec{\sigma}$, we assign colors to the different components, corresponding to the label α used above. We illustrate these contours for a particular configuration in the case $n = 2$ in Figure 4(a)–Figure 4(c).



(a) The contours of $\sigma^{(1)}$

(b) The contours of $\sigma^{(2)}$

(c) The contours of $\vec{\sigma}$

(d) The set of continents $\mathcal{K}(\vec{\sigma}) = \{\mathcal{K}_1, \mathcal{K}_2\}$

(e) The continent contours $\vec{C}(\mathcal{K}_1, \vec{\sigma})$

Figure 4: An illustration of contours and continents in the case $n = 2$.

VI.2 Replica Continent Contours

Here we define appropriate *replica continent contours* $\vec{C}(\mathcal{K},\vec{\sigma})$ in order to analyze the probability $\Pr(r)$ of the occurrence of configurations containing a continent with a contour of length r. We do not define $\vec{C}$ as the boundary $\partial\mathcal{K}$. The problem is: while this boundary is a contour for $\vec{\sigma}$, it is not necessarily a contour for a component $\sigma^{(\alpha)}$. Usually $\partial\mathcal{K}$ consists of segments of contours of the components.

To estimate $\Pr(r)$ we use the relation between the configuration $\vec{\sigma}$ with the replica continent contour $\vec{C}(\mathcal{K},\vec{\sigma})$ and a configuration $\vec{\sigma}^*$ with the contour removed. This transformation removes all the contours of the component spins $\sigma^{(\alpha)}$ that contribute to $\partial\mathcal{K}$. With this motivation, we now give the appropriate construction.

Definition VI.1. *For $\mathcal{K}\in\mathcal{K}(\vec{\sigma})$ define the* replica continent contour *of $\mathcal{K}$ in the configuration $\vec{\sigma}$ as the vector $\vec{C}(\mathcal{K},\vec{\sigma})$ with components*

$$C^{(\alpha)}(\mathcal{K},\vec{\sigma}) = \textit{union of contours } C \textit{ for } \sigma^{(\alpha)} \textit{ with } |C\cap\partial\mathcal{K}|\neq 0\ , \qquad \text{(VI.1)}$$

where $|\cdot|$ is the measure of $(d-1)$-surfaces. This is the subset of contours for $\vec{\sigma}$ meeting the boundary of the continent $\partial\mathcal{K}$.

See the example in Figure 4(e). In a generic configuration, these contours touch the boundary and penetrate arbitrarily into the interior of the continent.

Several different configurations of the spin $\vec{\sigma}$ may have different contours, but a common continent $\mathcal{K}$. Define the set of possible contours for the continent $\mathcal{K}$ as

$$C(\mathcal{K}) \ = \ \Big\{\vec{C}(\mathcal{K},\vec{\sigma})\,|\quad \text{where } \ \mathcal{K}\in\mathcal{K}(\vec{\sigma})\Big\}\,. \qquad \text{(VI.2)}$$

Finally, the length of any contour $\vec{C}\in C(\mathcal{K})$ is just the sum over the length of the constituent contours,

$$\Big|\vec{C}\Big| = \sum_{\alpha=1}^{n} \big|\, C^{(\alpha)} \big|\ . \qquad \text{(VI.3)}$$

With these definitions it is obvious that removing $\vec{C}(\mathcal{K},\vec{\sigma})$ in the configuration $\vec{\sigma}$ is well-defined. We just remove the respective contours $C^{(\alpha)}(\mathcal{K},\vec{\sigma})$ for the components $\sigma^{(\alpha)}$, by flipping the sign of all the spins inside these contours.

Definition VI.2. *For a configuration $\vec{\sigma}$ and a continent $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$, write $\vec{\sigma}^*$ for the configuration where the contour $\vec{C}(\mathcal{K}, \vec{\sigma})$ for the continent has been removed as described above.*

As a consequence of the removal of the replica continent contour the energy H_{replica} is decreased by two times the length of the removed contours. This the generalization of the fact that for each component spin, the energy is given by two times the total length of the contours,

$$H_{\text{replica}}(\vec{\sigma}^*) = H_{\text{replica}}(\vec{\sigma}) - 2\left|\vec{C}(\mathcal{K}, \vec{\sigma})\right| . \tag{VI.4}$$

VII Counting Random Surfaces in $\mathbb{R}^d$

In order to prove the tree decay we need an exponential bound on the number of possible connected contours. These are surfaces in $\mathbb{R}^d$ composed of r faces, each a unit $(d-1)$-cube. We call these *random surfaces* and prove a bound that holds for general connected unions of faces, as defined in §II. We also use the term *adjacent faces* as in that section, to indicate that two faces share a $(d-2)$-dimensional cube.

Definition VII.1. *Let $N(r)$ denote the number of connected, random surfaces of dimension $(d-1)$, which contain exactly r faces, including a given face S_0.*

Proposition VII.2. *There is a constant a (independent of dimension) such that for $k_d = a2^d$,*

$$N(r) \le k_d^r . \tag{VII.1}$$

Proof. The idea of the proof is to map each connected surface onto a rooted tree-graph, whose edges connect the centers of adjacent faces of the surface, and which touches each face. We say that the graph *covers* the surface. One then counts the number of possible surfaces that can correspond to one graph. The product of the number of possible tree graphs, times the number of surfaces per graph, gives our bound.

The tree graph will have length r and $r-1$ edges; the root of the tree is the center p_1 of S_0, see Figure 5. The first branch of the tree connects the root p_1 to the center p_2 of a face adjacent to S_0. From there we draw another edge connecting to the center of a new adjacent face (but we do not return

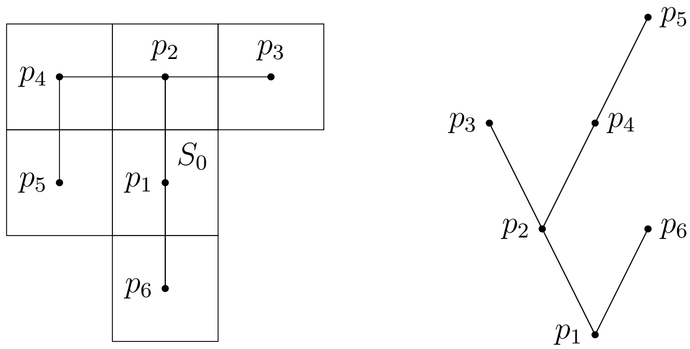


Figure 5: A surface covered by a tree graph and the corresponding tree graph rooted at the center of S_0. To simplify the illustration, all the angles between the faces are set to 180°, while in general they may equal 90°, 180°, or 270°.

to p_1). If all the adjacent surface elements are already in the tree graph, we cannot continue this branch. At this point we move in the reverse direction along the branch, face by face, until we reach a face having an adjacent face that is not yet covered by the tree. Starting at this place we start a new branch. We continue in this manner until we cover the entire surface.

In this manner we assign at least one tree diagram to every connected surface. This also means that every possible connected surface with r faces can be constructed by choosing a tree graph and attaching new faces in the order given by the tree structure. The number of planar tree graphs with $r-1$ edges is the Catalan number C_{r-1}, see example 6.19.e of Stanley [1]. Hence

$$C_{r-1} = \frac{1}{r}\binom{2(r-1)}{r-1} \leq (2e)^r \ , \tag{VII.2}$$

where the bound follows from the elementary inequality

$$\binom{v}{w} \leq \left(\frac{ev}{w}\right)^w \ . \tag{VII.3}$$

An upper bound for the number of ways to add a single face as one builds up the surface along the tree-graph is $2^{d-1}\cdot 3$. A face has 2^{d-1} sides to attach an adjacent face, and every attachment can be done with one of the angles 90°, 180°, or 270°. Therefore we infer the bound (VII.1) with $a = 3e$, namely

$$\begin{aligned} N(r) &\leq (2^{d-1}\cdot 3)^r C_{r-1} \\ &\leq (2^{d-1}\cdot 3\cdot 2e)^r = k_d^r . \end{aligned} \tag{VII.4}$$

□

VIII Tree Decay

In this section we prove the decay bound for the truncated correlation functions. We base the proof on condensation. Starting from the representation (IV.13), namely

$$\ll s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} \gg_{\Lambda,\beta} = n^{-(n-2)/2} \langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta}^{\mathrm{T}} , \qquad \text{(VIII.1)}$$

we use the fact established in Proposition V.3 that every non-vanishing contribution contains a continent $\mathcal{K}$ with all the points $i_1, \ldots, i_n$.

Proposition VIII.1. *There are constants a, b depending on d, but independent of Λ, such that if $1 \leq \delta = \beta - b \ln n$ (hence requiring $\beta \geq \beta_n = O(\ln n)$), then the truncated correlation functions satisfy*

$$\left| \langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta}^{\mathrm{T}} \right| \leq a\, n^n\, e^{-\delta \tau(i_1,\ldots,i_n)} . \qquad \text{(VIII.2)}$$

Here $\tau(i_1, \ldots, i_n)$ is the length of the shortest tree connecting $i_1, \ldots, i_n$.

VIII.1 Outline of the Proof

We have shown in Proposition V.3 that each non-vanishing contribution to the expectation (VIII.1) contains a condensate continent $\mathcal{K}$ containing all the points $i_1, \ldots, i_n$. As a consequence, every possible replica contour $\vec{C} \in C(\mathcal{K})$ has minimal length $\tau(i_1, \ldots, i_n)$.

We formulate the sum over configurations

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle_{\Lambda,\beta}^{\mathrm{T}} = n^{(n-2)/2} \frac{1}{\mathfrak{Z}} \sum_{\vec{\sigma}} s_{i_1} \cdots s_{i_n} e^{-\beta H_{\mathrm{replica}}(\vec{\sigma})} , \qquad \text{(VIII.3)}$$

as a sum over configurations with contours $\vec{C}$ of length r and a sum over r. We claim that the probability $\Pr(r)$ that a replica contour $\vec{C}$ occurs with $\left|\vec{C}\right| = r$ satisfies the bound

$$\Pr(r) \leq e^{-\beta \left|\vec{C}\right|} = e^{-\beta r} . \qquad \text{(VIII.4)}$$

To complete the proof we use the entropy bound Proposition VII.2, along with an estimate on the number of configurations that contain a given contour $\vec{C}$. These estimates, together with the fact that $\left|s_i^{(1)}\right| \leq n^{1/2}$, yield the desired bound. We now break the proof into a sequence of elementary steps.

VIII.2 Details of the Proof

Rewrite the Sum: Consider the sum (VIII.3), with the restriction of Proposition V.3. Recall that the replica continent borders $\vec{C} = \vec{C}(\mathcal{K}, \vec{\sigma})$, and the set of configurations containing such a replica continent $C(\mathcal{K}) \ni \vec{C}(\mathcal{K}, \vec{\sigma})$ is given in Definition VI.1. One can rewrite the sum as an iterated sum,

$$\sum_{\vec{\sigma}} = \sum_{r=\tau(i_1,\ldots,i_n)}^{\infty} \sum_{\mathcal{K},\vec{C}}{}^{\prime} \sum_{\vec{\sigma}}{}^{\prime\prime} . \tag{VIII.5}$$

For fixed $\mathcal{K}$ and $\vec{C}$, the sum $\sum''$ denotes the sum over configurations containing the continent $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$ with the continent border $\vec{C} = \vec{C}(\mathcal{K}, \vec{\sigma})$,

$$\sum_{\vec{\sigma}}{}^{\prime\prime} = \sum_{\vec{\sigma} \text{ with } \mathcal{K} \in \mathcal{K}(\vec{\sigma})\ ,\ \vec{C} = \vec{C}(\mathcal{K}, \vec{\sigma})} . \tag{VIII.6}$$

The sum $\sum'$ ranges over the possible continents $\mathcal{K}$ containing the n sites $i_1, \ldots, i_n$, and their possible borders $\vec{C}$ of length $\left|\vec{C}\right| = r$. Thus

$$\sum_{\mathcal{K},\vec{C}}{}^{\prime} = \sum_{\mathcal{K} \supset \{i_1,\ldots,i_n\}} \sum_{\substack{\vec{C} \in C(\mathcal{K}) \\ \text{with } \left|\vec{C}\right| = r}} . \tag{VIII.7}$$

Finally we sum over r, which is bounded from below by the minimal size $\tau(i_1, \ldots, i_n)$.

One interprets the sum $\sum''$ as the *energy* contribution to the sum, namely the probability

$$\Pr(r) = \frac{1}{3} \sum{}^{\prime\prime} e^{-\beta H(\vec{\sigma})} , \tag{VIII.8}$$

for the states $\vec{\sigma}$ with $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$. Likewise one interprets the sum $\sum'$ as the *entropy* contribution to the sum. Define the *entropy factor* $\mathcal{N}(r)$ by

$$\mathcal{N}(r) = \sum_{\mathcal{K},\vec{C}}{}' 1 \ . \tag{VIII.9}$$

The entropy counts the number of different shapes for $\vec{C}$.

Using $|\sigma_i| = 1$, one has $\left|s_i^{(1)}\right| \leq n^{1/2}$. Thus we obtain the bound

$$\begin{aligned} \left|\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle^{\mathrm{T}}_{\Lambda,\beta}\right| &= n^{(n-2)/2} \left| \ll s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} \gg_{\Lambda,\beta} \right| \\ &\leq n^{(n-2)/2} \frac{1}{3} \sum_{r=\tau}^{\infty} \sum_{\mathcal{K},\vec{C}}{}' \sum_{\vec{\sigma}}{}'' \left| s_{i_1}^{(1)} \cdots s_{i_n}^{(1)} \right| e^{-\beta H(\vec{\sigma})} \\ &\leq n^{(n-2)/2} \sum_{r=\tau}^{\infty} n^{n/2} \mathcal{N}(r) \ \Pr(r) \\ &= n^{n-1} \sum_{r=\tau}^{\infty} \mathcal{N}(r) \Pr(r) \ . \end{aligned} \tag{VIII.10}$$

In the following we prove bounds on $\Pr(r)$ and on $\mathcal{N}(r)$ that depend only on r, on β, and on the dimension d.

Bound the Entropy: We show that there are constants A, B depending only on d such that $\mathcal{N}(r)$ satisfies the exponential bound,

$$\mathcal{N}(r) \leq AB^r n^r \ . \tag{VIII.11}$$

We obtain this result by constructing the border contour $\partial\mathcal{K}$ and attaching l colored sub-contours. In this way one constructs any possible $\vec{C}$ satisfying the conditions above. The geometry of the contour (which must surround i_1) requires that the starting face we choose in constructing $\vec{C}$ must lie in a cube of side-length $(r-1)$, centered at i_1. Such a cube contains at most dr^d possible starting faces.

Using Proposition VII.2, the number of possible border contours is less than $dr^d N(r) \leq dr^d k_d^r$. We now build up the full contour $\vec{C}$ by attaching at least one and at most r subcontours to $\partial\mathcal{K}$ to obtain the total number of faces r. This can be done in a number of ways. For l sub-contours, the number of ways is bounded by the product of combinatorial factors:

r^l	for the starting faces on $\partial\mathcal{K}$,
k_d^r	for the shapes,
n^l	for the colors,
$\binom{r-1}{l-1}$	for the lengths,
$1/l!$	as the ordering of the subcontours is irrelevant.

Therefore

$$\mathcal{N}(r) \le d r^d k_d^r \sum_{l=1}^{r} r^l k_d^r n^l \binom{r-1}{l-1} \frac{1}{l!} \,. \tag{VIII.12}$$

We use the elementary inequalities

$$r^d \le d! e^r \,, \qquad \text{and } \binom{r-1}{l-1} \le 2^r \,. \tag{VIII.13}$$

Then

$$\mathcal{N}(r) \le d r^d k_d^{2r} n^r \sum_{l=1}^{r} \frac{r^l}{l!} \binom{r-1}{l-1} \le d d! \left(2e^2 k_d^2\right)^r n^r \,. \tag{VIII.14}$$

This bound has the form (VIII.11) with $A = dd!$ and $B = 2e^2 k_d^2$.

Bound the Energy Factor: The energy bound has the form

$$\Pr(r) \le e^{-\beta r}, \tag{VIII.15}$$

where $\mathcal{K}$ (implicitely contained in $\sum'$) is any fixed connected set with $\{i_1, \ldots, i_n\} \subset \mathcal{K}$ and $\vec{C} \in C(\mathcal{K})$ is any fixed extended border with $\left|\vec{C}\right| = r$. The idea is to compare every summand in the numerator to a summand in the denominator. For any given $\vec{\sigma}$ with $\mathcal{K} \in \mathcal{K}(\vec{\sigma})$ and $\vec{C}(\mathcal{K}, \vec{\sigma}) = \vec{C}$, we can take away the contours in $\vec{C}$ obtaining the unique $\vec{\sigma}^*$ as described in Definition VI.2. Because of the difference in energy this gives an additional factor $e^{-\beta r}$ for the term in the numerator. As the procedure works for all the summands, we infer

$$\Pr(r) = \frac{\sum''_{\vec{\sigma}} e^{-\beta H(\vec{\sigma})}}{\sum_{\vec{\sigma}} e^{-\beta H(\vec{\sigma})}} \le \frac{\sum''_{\vec{\sigma}} e^{-\beta H(\vec{\sigma}^*)} e^{-\beta r}}{\sum''_{\vec{\sigma}} e^{-\beta H(\vec{\sigma}^*)}} = e^{-\beta r}. \tag{VIII.16}$$

Tree Decay: The bound (VIII.2) now follows. Using (VIII.10), one has

$$\begin{aligned}\left|\langle\sigma_{i_1}\cdots\sigma_{i_n}\rangle^{\mathrm{T}}_{\Lambda,\beta}\right| &\leq n^{n-1}\sum_{r=\tau}^{\infty}\mathcal{N}(r)\,\mathrm{Pr}(r) \leq n^{n-1}\sum_{r=\tau}^{\infty}AB^r n^r e^{-\beta r}\\ &= n^{n-1}A\sum_{r=\tau}^{\infty}e^{-(\beta-b\ln n)r}\ , \end{aligned}\tag{VIII.17}$$

where $b = \ln B$ and where $\tau = \tau(i_1, \ldots, i_n)$. The last sum converges for $\beta > b\ln n$. With $1 \leq \delta_n = \beta - b\ln n$, this gives

$$\left|\langle\sigma_{i_1}\cdots\sigma_{i_n}\rangle^{\mathrm{T}}_{\Lambda,\beta}\right| \leq n^{n-1}A(1-e^{-\delta_n})^{-1}\ e^{-\delta_n\tau}\ . \tag{VIII.18}$$

As $1 \leq \delta_n$, one can take

$$a = A\,(1-e^{-\delta_n})^{-1} \leq Ae(e-1)^{-1}\ . \tag{VIII.19}$$

This completes the proof of Proposition VIII.1. □

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[1] Richard Stanley. *Enumerative Combinatorics, Volume 2*. Cambridge University Press. Cambridge Studies in Advanced Mathematics 62, Cambridge, New York 1999.

[2] Garrett Sylvester. Representations and inequalities for Ising model Ursell functions. *Comm. Math. Phys.*, 42:209–220, 1975.