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STOCHASTIC PDE, REFLECTION POSITIVITY AND QUANTUM FIELDS

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We outline some known relations between classical random fields and quantum fields. In the scalar case, the existence of a quantum field is equivalent to the existence of a Euclidean-invariant, reflection-positive (RP) measure on the Schwartz space tempered distributions. Martin Hairer recently investigated random fields in a series of interesting papers, by studying non-linear stochastic partial differential equations, with a white noise driving term. To understand such stochastic quantization, we consider a linear example. We ask: does the measure on the solution induced by the stochastic driving term yield a quantum field? The RP property yields a general method to implement quantization. We show that the RP property fails for finite stochastic parameter λ, although RP does hold in the limiting case $\lambda = \infty$.

1. Introduction. We outline the relation between classical random fields and quantum fields. In the scalar case, the existence of a quantum field is equivalent to the existence of a Euclidean-invariant, reflection-positive measure (RP) on tempered distributions on space-time. We review some results in constructive quantum field theory, and their relation to the development of renormalization group methods.

Martin Hairer recently investigated random fields in a series of interesting papers, by studying non-linear stochastic partial differential equations with a white noise driving term. To understand properties of such stochastic quantization, we consider an elementary example—the massive free (linear) Euclidean field—and the corresponding family of Gaussian measure $d\mu_\lambda$. Here λ denotes the stochastic parameter.

We ask: does the measure $d\mu_\lambda$ on random fields yield a corresponding quantum field? The RP property enables the standard method to obtain a quantization. In §3 we demonstrate that the RP property fails for $\lambda < \infty$, although it holds in the limiting case $\lambda = \infty$. We raise the question in §4 of how one would approximate $d\mu_\lambda$ in the non-linear case in order to obtain an approximation that preserves RP.

1.1. *The Free Euclidean Field.* The free relativistic quantum field $\varphi(x)$ is a Wightman field on a Fock-Hilbert space $\mathcal{H}$. It arises from the Osterwalder-

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Schrader quantization of the Gaussian measure $d\mu_C(\Phi)$ with characteristic function

$$S_C(f) = e^{-\frac{1}{2}\langle \bar{f}, Cf\rangle_{L^2}} , \quad \text{where} \quad C = (-\Delta + m^2)^{-1} . \tag{1.1}$$

Here space-time is d-dimensional, and one requires $m^2 > 0$ if $d = 1, 2$. This field was introduced by Kurt Symanzik [27] as a random field and studied extensively in the free-field case by Edward Nelson [20], and later by many others. It is well-understood that such a random field is equivalent to a classical field acting on a Euclidean Fock space $\mathcal{E}$ with no-particle state Ω^E, see for example [12]. In terms of annihilation and creation operators satisfying $[A(k)\,,\, A(k')^*] = \delta(k-k')$, one has

$$\Phi(x) = \frac{1}{(2\pi)^{d/2}} \int (A(k)^* + A(-k))\, \frac{1}{(k^2+m^2)^{1/2}}\, e^{ikx} dk . \tag{1.2}$$

In this framework the field has a Gaussian characteristic functional

$$S_C(f) = \left\langle \Omega^E, e^{i\Phi(f)}\Omega^E \right\rangle_{\mathcal{E}} = \int_{\mathcal{S}'} e^{i\Phi(f)} d\mu_C(\Phi) . \tag{1.3}$$

Nelson's Markov field construction is not sufficiently robust to work for non-Gaussian examples, for the global Markov property required in Nelson's construction has never been established for the known interacting field theories. Furthermore random Markov fields are tied to classical probability theory, and so they do not accommodate a theory of fermions.

Konrad Osterwalder and Robert Schrader solved this problem in 1972 when they discovered the fundamental *reflection positivity* property, [24, 25]. This construction is so simple and beautiful, it should be a part of every book on quantum theory. Unfortunately that must wait for a number of new books to be written!

1.2. *Reflection Positivity and Osterwalder-Schrader Quantization.* The connection of $d\mu_C(\Phi)$ to quantum field theory is given through its property of reflection-positivity. There is a similar property for fermion fields and for gauge fields, as well as for fields of higher spin. So reflection positivity can be formulated to connect all known quantum theories with corresponding classical ones.

One identifies a time direction t for quantization, and writes $x = (t, \vec{x})$. Let $\vartheta : (t, \vec{x}) \mapsto (-t, \vec{x})$ denote time reflection, and Θ its push forward to $\mathcal{S}'(\mathbb{R}^d)$. Then RP requires that for $A(\Phi)$ an element of the polynomial algebra $\mathcal{E}_+$ generated by random fields $\Phi(f)$ with $f \in \mathcal{S}(\mathbb{R}^d_+)$, one has

$$0 \leqslant \langle A, A\rangle_{\mathcal{H}} = \langle A, \Theta A\rangle_{\mathcal{E}} . \tag{1.4}$$

Let $\mathcal{N}$ denote the null space of this positive form and $\mathcal{E}_+/\mathcal{N}$ the space of equivalence classes differing by a null vector. The Hilbert space of quantum theory $\mathcal{H}$ is the completion of the pre-Hilbert space $\mathcal{E}_+/\mathcal{N}$, in this inner product. The vectors in $\mathcal{H}$ are called the OS quantization of vectors in $\mathcal{E}_+$. Operators T acting on $\mathcal{E}_+$ and preserving $\mathcal{N}$, also have a quantization $\widehat{T}$ as operators on $\mathcal{H}$, defined by $\widehat{T}\widehat{A} = \widehat{TA}$. This is summarized in the commuting exact diagram of Figure 1.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{N} \arrow[r, "T"] \arrow[d, "\mathrm{Id}"'] & \mathcal{N} \arrow[d, "\mathrm{Id}"] \\
\mathcal{E}_+ \arrow[r, "T"] \arrow[d, "\wedge"'] & \mathcal{E}_+ \arrow[d, "\wedge"] \\
\mathcal{E}_+/\mathcal{N} \arrow[r, "\widehat{T}"] \arrow[d] & \mathcal{E}_+/\mathcal{N} \arrow[d] \\
0 & 0
\end{tikzcd}$$

FIG 1. *OS Quantization of Vectors* $[A] \in \mathcal{E}_+/\mathcal{N} \mapsto \widehat{A}$, *and of Operators* $T \mapsto \widehat{T}$.

1.3. *Non-Gaussian Examples.* Many families of non-Gaussian measures $d\mu(\Phi)$ on $\mathcal{S}'(\mathbb{R}^d)$ that are Euclidean-invariant and reflection-positive are known. The first examples were shown to exist in space-time of two dimensions, $d = 2$, by Glimm and Jaffe [6, 7, 8, 9], and Glimm, Jaffe, and Spencer [13, 14, 15, 16]. Additional examples were given by Guerra, Rosen, and Simon [5] and others..

In the more difficult case of $d = 3$ space-time dimensions, the only complete example known is the Φ_3^4 theory. Glimm and Jaffe proved that in a finite volume, a reflection-positive measure exists for all couplings [10]. They showed that one has a convergent sequence of renormalized, approximating action functionals $\mathfrak{A}_n$ whose exponentials $e^{-\mathfrak{A}_n}$ when multiplied by the standard Gaussian measure $d\mu_C(\Phi)$ converge weakly. But the limit is inequivalent to the Gaussian. This limit agrees in perturbation theory with the standard perturbation theory in physics texts for φ_3^4. The physics result established in this paper is that the renormalized φ_3^4 Hamiltonian H in a finite spatial volume is bounded from below. Joel Feldman and Osterwalder combined the stability result of [10] with a modified version of the cluster expansions for Euclidean fields [13], to obtain a Euclidean-invariant, reflection-positive measure on $\mathbb{R}^3$ for small coupling [3].

The original stability bound paper [10] took several years to finish. In that paper we developed a method to show stability in a region of a cell in phase space of size $O(1)$, and to show independence of different phase space cells, with a quantitative estimate of rapid polynomial decay in terms of a dimensionless distance between cells. This analysis allowed us to analyze partial expectation of degrees of freedom associated with the phase cells.

It turned out that the ideas we used overlap a great deal with the "renormalization group" methods developed by Kenneth Wilson [28], which appeared while we were still developing our non-perturbative methods for constructive QFT. One major difference in Wilson's approach, and what makes it so appealing, is that his methods are iterative. Our original methods were inductive, using a somewhat different method on each length scale. Wilson achieved this simplicity by ignoring effects which appeared to be small.

While many persons have attempted to reconcile these two methods, much more work needs to be done. In spite of qualitative advances, the conceptually-simpler renormalization group methods have not yet been used to establish the physical clustering properties, that were proved earlier using the inductive methods. The most detailed studies of Φ_3^4 using the renormalization group methods have been carried out by Brydges, Dimock, and Hurd [1, 2]. My undergraduate student David Moser gave a nice exposition and also some refinements [23].

2. The Stochastic PDE Approach to the Free Field. The idea of quantization through SPDE goes back to Edward Nelson [20, 22] and Parisi-Wu [26]. Recently Martin Hairer reinvestigated these questions and has made substantial progress [17, 18], as well as in his many other recent works on the ArXiv. The most interesting case is his new look at the Φ_3^4 measure.

In this work we consider only the Gaussian case corresponding to the massive free field. Our goal is to understand more about the method, and the relation between SPDE and relativistic quantum field theory. Let $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ denote the Laplacian on $\mathbb{R}^d$, where the Laplacian involves the Euclidean space-time coordinates $\vec{x} = (x_1, \ldots, x_{d-1})$ and a time coordinate $x_d = t$. The SPDE also involves the auxiliary stochastic parameter $\lambda \in \mathbb{R}_+$. The field $\Phi_\lambda(x)$ satisfies the fundamental equation

$$\frac{\partial \Phi_\lambda(x)}{\partial \lambda} = -\frac{1}{2}(-\Delta + m^2)\Phi_\lambda(x) + \xi_\lambda(x) , \tag{2.1}$$

where $\xi_\lambda(x)$ is white noise. In other words, $\xi_\lambda(x)$ has a Gaussian probability distribution with mean zero and with covariance $\delta(\lambda-\lambda')\delta(x-x')$. Let $d\nu_\lambda(\xi)$ denote this Gaussian measure.

The solutions $\Phi_\lambda(x)$ to the fundamental SPDE have a distribution $d\mu_\lambda(\Phi_\lambda)$ corresponding to the measure $d\nu_\lambda(\xi)$. The distribution $d\mu_\lambda(\Phi)$ is Gaussian. It is claimed that

$$d\mu_C(\Phi) = \lim_{\lambda\to\infty} d\mu_\lambda(\Phi)\ . \tag{2.2}$$

In this section we verify the form of $d\mu_\lambda$ and its $\lambda\to\infty$ limit.

2.1. *The Classical Heat Kernel.* Let $\mathfrak{K}_\lambda(x,x')$ denote the heat kernel for the related equation, namely the integral kernel of the linear transformation $e^{-\frac{\lambda}{2}(-\Delta+m^2)}$. In other words the kernel $\mathfrak{K}_\lambda(x,x')$ satisfies the equation

$$\frac{\partial}{\partial\lambda}\mathfrak{K}_\lambda(x,x') = \frac{1}{2}\left(\Delta-m^2\right)\mathfrak{K}_\lambda(x,x')\ , \tag{2.3}$$

with initial data

$$\mathfrak{K}_0(x,x') = \lim_{\lambda\to 0+}\mathfrak{K}_\lambda(x,x') = \delta(x-x')\ . \tag{2.4}$$

Given $f(x)$, define $f_\lambda(x)$ for $0<\lambda$ as

$$f_\lambda(x) = \int_{\mathbb{R}^d}\mathfrak{K}_\lambda(x,x')\,f(x')\,dx' = \left(e^{-\frac{\lambda}{2}(-\Delta+m^2)}\,f\right)(x)\ , \tag{2.5}$$

which is a solution to the homogeneous equation for $0<\lambda$

$$\frac{\partial f_\lambda(x)}{\partial\lambda} = -\frac{1}{2}\left(-\Delta+m^2\right)f_\lambda(x)\ , \tag{2.6}$$

with initial data f. The corresponding inhomogeneous equation arises if one is given a forcing term $\xi_\lambda(x)$ and desires to solve the equation

$$\frac{\partial f_\lambda(x)}{\partial\lambda} = -\frac{1}{2}\left(-\Delta+m^2\right)f_\lambda(x)+\xi_\lambda(x)\ . \tag{2.7}$$

The solution with initial data $f(x)$ at $\lambda=0$ is

$$f_\lambda(x) = \int_{\mathbb{R}^d}dx'\,\mathfrak{K}_\lambda(x,x')\,f(x') + \int_0^\lambda d\alpha\int_{\mathbb{R}^d}dx'\,\mathfrak{K}_{\lambda-\alpha}(x,x')\,\xi_\alpha(x')\ . \tag{2.8}$$

2.2. *The Random Field Solution to the SPDE.* For each given $\xi_\lambda(x)$ the solution to the fundamental equation (2.1) with initial data $\Phi_0(x)$ is given by (2.8) as

$$\Phi_\lambda(x) = \int_{\mathbb{R}^d} \mathfrak{K}_\lambda(x,x')\Phi_0(x')dx' + \int_0^\lambda d\alpha \int_{\mathbb{R}^d} dx'\, \mathfrak{K}_{\lambda-\alpha}(x,x')\xi(x')\ . \tag{2.9}$$

Since this solution is linear in ξ, it is clear that the Gaussian distribution of ξ will yield a Gaussian distribution of Φ_λ.

2.2.1. *First Moment.* As ξ_λ has mean zero, and $d\nu_\lambda(\xi)$ is a probability measure, the first moment of $\Phi_\lambda(x)$ equals

$$\langle \Phi_\lambda(x)\rangle_{\nu_\lambda} = \int \Phi_\lambda(x)\, d\nu_\lambda(\xi) = \int_{\mathbb{R}^d} dx'\, \mathfrak{K}_\lambda(x,x')\Phi_0(x')\ . \tag{2.10}$$

One could also write

$$\langle \Phi_\lambda(x)\rangle_{\nu_\lambda} = \left(e^{-\frac{\lambda}{2}\left(-\Delta+m^2\right)}\,\Phi_0\right)(x)\ . \tag{2.11}$$

It is desirable to have the mean of $d\mu_\lambda(\Phi)$ to be zero. So we assume that $\Phi_0 = 0$. Then $\langle \Phi_\lambda(x)\rangle_{\nu_\lambda} = 0$. Otherwise we subtract the mean to obtain this result. On the other hand, it is clear that in the $\lambda \to \infty$ limit the mean (2.11) will tend to zero. Thus the initial data will be wiped out in the limiting distribution, and whether or not we start with mean zero, we obtain mean zero in the limit.

2.2.2. *Second Moment.* The second moment of $\Phi_\lambda(x)$ at a given λ defines the covariance D_λ of the distribution. We claim that covariance for $\Phi_\lambda(x)$ is

$$D_\lambda = (I - e^{-\lambda C^{-1}})C\ . \tag{2.12}$$

If one considers the action of D_λ as a transformation on L^2, then the self-adjoint, positive transformation $(-\Delta+m^2) \geqslant m^2$, yields $\left\|e^{-\lambda C^{-1}}\right\| \leqslant e^{-\lambda m^2}$. One infers that

$$0 \leqslant D_\lambda\ , \tag{2.13}$$

so D_λ is a bone-fide covariance.

To show that D_λ has the form claimed, calculate

$$
\begin{aligned}
D_\lambda(x,y) &= \langle \Phi_\lambda(x)\,\Phi_\lambda(y)\rangle_{\nu_\lambda} = \int \Phi_\lambda(x)\,\Phi_\lambda(y) d\nu_\lambda \\
&= \int_0^\lambda d\alpha \int_0^\lambda d\beta \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy'\, \mathfrak{K}_{\lambda-\alpha}(x,x')\mathfrak{K}_{\lambda-\beta}(y,y') \left\langle \xi_\alpha(x')\xi_\beta(y')\right\rangle_{\nu_\lambda} \\
&= \int_0^\lambda d\alpha \int_0^\lambda d\beta \int_{\mathbb{R}^d} dx' \int_{\mathbb{R}^d} dy'\, \mathfrak{K}_{\lambda-\alpha}(x,x')\mathfrak{K}_{\lambda-\beta}(y',y)\, \delta(\alpha-\beta)\, \delta(x'-y') \\
&= \int_0^\lambda d\alpha\, \mathfrak{K}_{2(\lambda-\alpha)}(x,y) \\
&= \left(I - e^{-\lambda C^{-1}}\right) C(x,y)\ ,
\end{aligned} \tag{2.14}
$$

as claimed. Here we use the symmetry and multiplication law for the semi-group $e^{-\lambda C^{-1}}$ with integral kernel $\mathfrak{K}_\lambda(x,y)$.

It is clear that $D_\lambda(x,y)$ has the same local singularity on the diagonal as $C(x,y)$, for the difference

$$
C(x,y) - D_\lambda(x,y) = \left(e^{-\lambda(-\Delta+m^2)}C\right)(x,y) \tag{2.15}
$$

is the Fourier transform of a multiple of $e^{-\lambda(k^2+m^2)}\left(k^2+m^2\right)^{-1}$. So it is an element of Schwartz space for every $0<\lambda$.

2.2.3. *General Moments.* It follows from the Gaussian property that the odd moments of $d\mu_\lambda(\Phi)$ vanish, and that the even moments yield

$$
S_\lambda(f) = \left\langle e^{i\Phi_\lambda(f)}\right\rangle_{\nu_\lambda} = e^{-\frac{1}{2}\langle \bar{f}, D_\lambda f\rangle_{L^2}}\ . \tag{2.16}
$$

2.3. *The Limit $\lambda\to\infty$.* The measure $d\mu_\lambda$ is a Gaussian on $\mathcal{S}'(\mathbb{R}^d)$, with mean 0 and covariance (2.12). Under what conditions does this family of Gaussian measures converge to a limiting measure $d\mu$ as $\lambda\to\infty$? This question is answered in [4], and a sufficient condition for convergence is the convergence $D_\lambda f\to Cf$ in the topology of the Schwartz space $\mathcal{S}$ for every $f\in\mathcal{S}$.

It is clear that in the sense of the weak limit of operators on $\mathcal{S}(\mathbb{R}^d)$, the operators $e^{-\lambda C^{-1}}C$ converge in the $\mathcal{S}$ topology to zero as $\lambda\to\infty$. As a consequence $D_\lambda \to_{\mathcal{S}} C$ and as the weak limit of measures

$$
\lim_{\lambda\to\infty} S_\lambda(f) = S_C(f) = e^{-\frac{1}{2}\langle \bar{f}, Cf\rangle_{L^2}}\ , \quad \text{and} \quad \lim_{\lambda\to\infty} d\mu_\lambda(\Phi) = d\mu_C(\Phi)\ . \tag{2.17}
$$

3. The Measure $d\mu_\lambda$ is Not Reflection Positive.

3.1. *RP for the Gaussian with Covariance* D_λ. For a Gaussian measure $d\mu_\lambda(\Phi)$ with covariance D_λ, it is well known that the RP property (1.4) for $d\mu_\lambda$ is equivalent to RP for the covariance D_λ, see for example [12]. The RP property for D_λ on $\mathbb{R}^d_+$, with respect to time reflection ϑ means: for each function $f \in L^2(\mathbb{R}^d_+)$ supported in the positive-time half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$,

$$0 \leqslant \langle f, \vartheta D_\lambda f \rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.1}$$

This form defines the pre-inner product on the one-particle space $L^2(\mathbb{R}^d_+)$ as

$$\langle f, f \rangle_{\mathcal{H}} = \langle f, \vartheta D_\lambda f \rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.2}$$

(Note that RP of a covariance on $\mathbb{R}^d$ ensures RP on various compactified space-times, such as RP on the torus $\mathbb{T}^d$ [19].)

It is also well-known that $C = (-\Delta + m^2)^{-1}$ is reflection positive with respect to ϑ. This can be verified on $\mathbb{R}^d$ by analysis of the Fourier transform. On subdomains of $\mathbb{R}^d$, or in other geometries, it is convenient to note the equivalence between *RP* for C and the monotonicity of covariance operators $C_D \leqslant C_N$ with Dirichlet and with Neumann boundary data on the reflection plane [11].

In general one cannot expect to recover the reflection-positivity property without preserving positivity in all intermediate approximations. So we ask whether the covariance D_λ is reflection positive for $0 < \lambda < \infty$. In order to establish a counterexample to RP for the Gaussian measure, it is only necessary to find one function f, supported at positive time, for which

$$\langle f, \vartheta\, D_\lambda\, f \rangle_{L^2} < 0 \ . \tag{3.3}$$

We now give such a counterexample to RP for $d\mu_\lambda$.

3.2. *RP Fails if* $\boldsymbol{d = 1}$ *and* $\boldsymbol{\frac{1}{2} < \lambda m^2 < \infty}$. To make things as simple as possible, we first inspect the case $d = 1$. Since

$$\|f\|^2_{\mathcal{H}} = \langle f, \vartheta D_\lambda f \rangle_{L^2} \leqslant \langle f, D_\lambda f \rangle_{L^2} \leqslant \langle f, C f \rangle_{L^2} = \langle f, f \rangle_{\mathfrak{H}_{-1}} \ , \tag{3.4}$$

the inner product (3.2) extends by continuity from L^2 to the Sobolev space $f \in \mathfrak{H}_{-1}(\mathbb{R}^d_+)$. For $d = 1$ the Sobolev space contains the Dirac measure δ_t localized at $0 \leqslant t$, so in our example we choose f to be a linear combination of two Dirac delta functions localized at two distinct non-negative times $0 \leqslant s < t$.

Define

$$f(u) = e^{ms}\,\delta_s(u) - e^{mt}\,\delta_t(u)\;, \tag{3.5}$$

so

$$(\vartheta f)(u) = e^{ms}\,\delta_s(-u) - e^{mt}\,\delta_t(-u) = e^{ms}\,\delta_{-s}(u) - e^{mt}\,\delta_{-t}(u)\;. \tag{3.6}$$

We choose f of this form, as it is in the null space of the RP inner product for the covariance C. In fact

$$\begin{aligned}
\langle f, \vartheta C f\rangle_{L^2} &= \langle \vartheta f, C f\rangle_{L^2} = \left\langle (e^{ms}\delta_{-s} - e^{mt}\delta_{-t}), C(e^{ms}\delta_s - e^{mt}\delta_t)\right\rangle_{L^2} \\
&= e^{2ms}\;\langle \delta_{-s}, C\delta_s\rangle + e^{2mt}\;\langle \delta_{-t}, C\delta_t\rangle \\
&\quad -e^{m(t+s)}\;\langle \delta_{-s}, C\delta_t\rangle - e^{m(t+s)}\;\langle \delta_{-t}, C\delta_s\rangle \\
&= \frac{1}{2m}\,(1+1-1-1) = 0\;.
\end{aligned} \tag{3.7}$$

Hence

$$\langle f, \vartheta D_\lambda f\rangle_{L^2} = -e^{-\lambda m^2}\left\langle f, \vartheta e^{\lambda\Delta} C f\right\rangle_{L^2}\;. \tag{3.8}$$

We infer that f gives a counterexample to RP for $\lambda m^2 < \infty$ in case for some s, t,

$$0 < \left\langle \vartheta f, e^{\lambda\Delta} C f\right\rangle_{L^2}\;. \tag{3.9}$$

Define

$$W_{\lambda,m} = (4\pi\lambda)^{1/2}\,2m\,e^{\lambda\Delta}\,C\;. \tag{3.10}$$

Recall that the operator $2mC$ has integral kernel

$$2mC(u,s) = e^{-m|u-s|}\;. \tag{3.11}$$

Furthermore the operator $(4\pi\lambda)^{1/2}\,e^{\lambda\Delta}$ has the integral kernel

$$\left((4\pi\lambda)^{1/2}\,e^{\lambda\Delta}\right)(t,u) = \;e^{-\frac{(t-u)^2}{4\lambda}}\;. \tag{3.12}$$

Thus the integral kernel $W_{\lambda,m}(t,s) = W_{\lambda,m}(t-s)$ of $W_{\lambda,m}$ equals

$$W_{\lambda,m}(t-s) = \int_{-\infty}^{\infty} du\, e^{-\frac{(t-s-u)^2}{4\lambda} - m|u|} = W_{\lambda,m}(s-t)\;. \tag{3.13}$$

The second equality in (3.13) shows that $W_{\lambda,m}$, which is real, is also hermitian. Remark that

$$W_{\lambda,m}(t) = m^{-1}\,W_{\lambda m^2,1}(mt)\;. \tag{3.14}$$

Thus without loss of generality we may study $W(t) = W_{\lambda,1}(t)$.

3.2.1. *An Indicative Bound.* One can rewrite the condition for violation of RP in (3.9) in terms of W. If for some $0 \leqslant s < t$,

$$e^{s+t}\, W(s+t) < \frac{1}{2}\left(e^{2s}\, W(2s) + e^{2t}\, W(2t)\right) , \tag{3.15}$$

then RP would fail. With the Schwarz inequality, and the comparison of geometric and arithmetic means one has the following bound: while close, as one can take $\exp\left((t-s)^2/4\lambda\right) \approx 1$, we need more.

$$\begin{aligned}
&e^{t+s}\, W(s+t) \\
&\quad= e^{t+s} \int_{-\infty}^{\infty} du\, e^{-\frac{(s+t-u)^2}{4\lambda}}\, e^{-|u|} \\
&\quad= e^{t+s}\, e^{-\frac{(s+t)^2}{4\lambda}} \int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{4\lambda}-|u|}\, e^{\frac{su+tu}{2\lambda}} \\
&\quad\leqslant e^{t+s}\, e^{-\frac{(s+t)^2}{4\lambda}} \left(\int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{4\lambda}-|u|} e^{\frac{su}{\lambda}}\right)^{1/2} \left(\int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{4\lambda}-|u|} e^{\frac{tu}{\lambda}}\right)^{1/2} \\
&\quad= e^{t+s}\, e^{\frac{(s-t)^2}{4\lambda}} \left(\int_{-\infty}^{\infty} du\, e^{-\frac{(u-2s)^2}{4\lambda}-|u|}\right)^{1/2} \left(\int_{-\infty}^{\infty} du\, e^{-\frac{(u-2t)^2}{4\lambda}-|u|}\right)^{1/2} \\
&\quad= e^{t+s}\, e^{\frac{(s-t)^2}{4\lambda}}\, W(2s)^{1/2}\, W(2t)^{1/2} \\
&\quad\leqslant \frac{1}{2}\, e^{\frac{(s-t)^2}{4\lambda}} \left(e^{2s}\, W(2s) + e^{2t}\, W(2t)\right)
\end{aligned}$$

.

3.2.2. *A Numerical Check.* We have used Mathematica to make a numerical check of whether f violates RP in case $s = 0$ and $\lambda = 1$. We have plotted the function

$$F(t) = \frac{1}{2}\left(W(0) + e^{2t}\, W(2t)\right) - e^{t}\, W(t)\ , \quad \text{with} \quad W(t) = \int_{-\infty}^{\infty} du\, e^{-\frac{(t-u)^2}{4}-|u|}\ . \tag{3.16}$$

If $F(t) > 0$ for any positive t, then RP fails to hold.

The Mathematica plot of $F(t)$ appears in Figure 2. Clearly there are values of $t \in (0, 1.5)$ for which $F(t)$ is positive.[1] So this indicates for the value of $\lambda = m = 1$ that we tested, RP does not hold for $\Phi_\lambda(t)$ in the measure $d\mu(\Phi_\lambda)$.

[1] I am grateful to Alex Wozniakowski for assisting me to use Mathematica to test whether $F(t)$ changes sign.

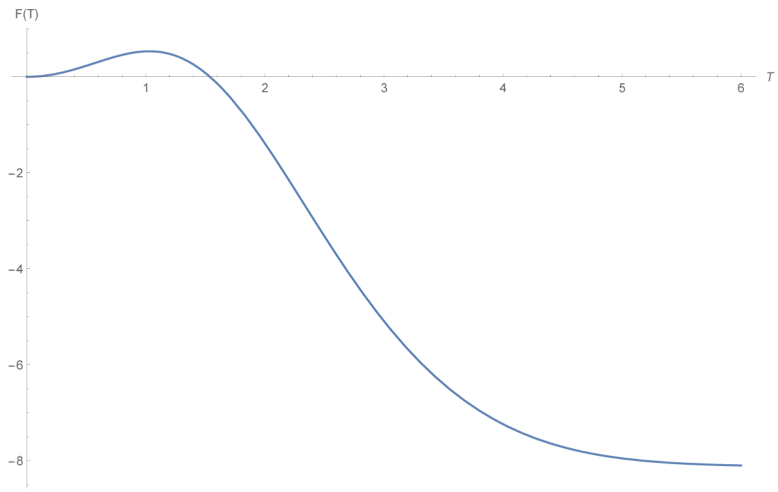

Fig 2. *RP of D_λ requires $F(t) = \frac{1}{2}\,W(0) + \frac{1}{2}\,e^{2t}\,W(2t) - e^t\,W(t) \leqslant 0$ for all $0 \leqslant t$.*

3.2.3. *The Proof.* Since the RP violation occurs for small t, we can give a proof of the existence of the counterexample expanding $F(t)$ as a power series in t. First we use the form (3.13) to extend $W(t)$ (defined here for positive t as an even function to negative t.) Then it is clear from the form of $F(t)$, that $F(t)$ is an analytic function at $t = 0$. Therefore the sign of $F(t)$ for small $0 < t$ is determined by the first non-zero term in the power series at $t = 0$, and this will be the term of second order. For this calculation we introduce the parameter λ. We show that RP is violated for $\frac{1}{2} \leqslant \lambda m^2$.

From (3.13) one has

$$W_\lambda(t) = W_\lambda(0) + \frac{t^2}{2} W''_\lambda(0) + O(t^4)\ , \tag{3.17}$$

and both $0 < W_\lambda(0)$, as well as

$$W''_\lambda(0) = c_\lambda - \frac{1}{2\lambda}\, W_\lambda(0)\ , \ \text{ with }\ 0 < c_\lambda = \frac{1}{4\lambda^2} \int_{-\infty}^{\infty} du\, u^2 e^{-\frac{u^2}{4\lambda} - |u|}\ . \tag{3.18}$$

The corresponding expansion for $F(t)$ is

$$
\begin{aligned}
F(t) &= \frac{1}{2}W_\lambda(0) + \frac{1}{2}\left(1+2t+2t^2\right)\left(W_\lambda(0)+2t^2W_\lambda''(0)\right) \\
&\qquad - \left(1+t+\frac{t^2}{2}\right)\left(W_\lambda(0)+\frac{t^2}{2}W_\lambda''(0)\right) + O(t^3) \\
&= \frac{t^2}{2}\left(W_\lambda(0)+W_\lambda''(0)\right) + O(t^3) \\
&= \frac{t^2}{2}\left(c_\lambda + \left(1-\frac{1}{2\lambda}\right)W_\lambda(0)\right) + O(t^3)\ . \qquad (3.19)
\end{aligned}
$$

As $0 < c_\lambda, W_\lambda(0)$, the leading non-zero coefficient in $F(t)$ is strictly positive for $\frac{1}{2} \leqslant \lambda$. (It is also valid for some $\lambda m^2 < \frac{1}{2}$.) Therefore there is an $\epsilon > 0$ such that for $t \in (0,\epsilon)$,

$$
0 < F(t)\ , \quad \text{for} \quad t \in (0,\epsilon)\ . \qquad (3.20)
$$

Hence RP does not hold for $\frac{1}{2} \leqslant \lambda < \infty$. Reinterpreting this with respect to the scaled function $W_{\lambda,m}(t)$ according to (3.14), we infer: RP fails for $\frac{1}{2} \leqslant \lambda m^2 < \infty$.

3.3. *RP Fails if* $\boldsymbol{1 < d}$ *and* $\boldsymbol{\lambda, m^2 < \infty}$. For $1 < d$ we show that RP for $d\mu_\lambda(\Phi_\lambda)$ fails for all $\lambda m^2 \in (0,\infty)$. Denote the Sobolev-space inner products on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ respectively as

$$
\langle f_1, f_2\rangle_{-1} = \langle f_1, C\, f_2\rangle_{L^2(\mathbb{R}^d)}\ , \quad \text{and} \quad \langle h_1, h_2\rangle_{-\frac{1}{2}} = \left\langle h_1, \frac{1}{2\mu}\, h_2\right\rangle_{L^2(\mathbb{R}^{d-1})}\ . \qquad (3.21)
$$

In this case $\mu = \sqrt{-\vec{\nabla}^2 + m^2}$. We also denote $\mu(\vec{p}) = \sqrt{\vec{p}^{\,2}+m^2}$ as the multiplication operator in Fourier space given by μ.

Start by choosing a real, spatial test function $h(\vec{x}) \in \mathcal{S}(\mathbb{R}^{d-1})$, whose Fourier transform $\widetilde{h}(\vec{p})$ has compact support. (Reality only requires $\widetilde{h}(\vec{p}) = \overline{\widetilde{h}(-\vec{p})}$.) Define the family of functions

$$
h_T(\vec{x}) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{T\mu(\vec{p})}\, h(\vec{p})\, e^{i\vec{p}\cdot\vec{x}}\, d\vec{p} \in \mathcal{S}(\mathbb{R}^{d-1})\ , \qquad (3.22)
$$

with $h_0 = h$, and $0 \leqslant T$. Let δ_S denote the one-dimensional Dirac measure with density $\delta_S(t) = \delta(t-S)$. For $0 \leqslant S < T$ define a space-time one-particle function $f(x) = f_{S,T}(x)$ by

$$
f = h_S \otimes \delta_S - h_T \otimes \delta_T\ . \qquad (3.23)
$$

Note that f is in the null space of the RP form defined by C, relative to the time-reflection ϑ. In fact

$$\begin{aligned}\langle f,\vartheta f\rangle_{-1} &= \langle f,\vartheta Cf\rangle_{L^2}\\ &= \left\langle h_S, e^{-2S\mu}\, h_S\right\rangle_{-\frac12} + \left\langle h_T, e^{-2T\mu}\, h_T\right\rangle_{-\frac12}\\ &\qquad - \left\langle h_S, e^{-(S+T)\mu}\, h_T\right\rangle_{-\frac12} - \left\langle h_T, e^{-(S+T)\mu}\, h_S\right\rangle_{-\frac12}\\ &= \langle h,h\rangle_{-\frac12} + \langle h,h\rangle_{-\frac12} - \langle h,h\rangle_{-\frac12} - \langle h,h\rangle_{-\frac12} = 0\,. \end{aligned} \tag{3.24}$$

Hence our test of RP relies whether $\langle f,f\rangle_{\mathcal H}$ is non-negative, where

$$\langle f,f\rangle_{\mathcal H} = \langle f,\vartheta D_\lambda f\rangle_{L^2} = -\left\langle \vartheta f, e^{-\lambda\left(-\Delta+m^2\right)} Cf\right\rangle_{L^2}\ . \tag{3.25}$$

Expanding f according to (3.23) yields four terms, each proportional to

$$\begin{aligned} F(t_1,t_2) &= \left\langle \left(h_{t_1}\otimes\delta_{-t_1}\right), e^{-\lambda(-\Delta+m^2)}\, C\ \left(h_{t_2}\otimes\delta_{t_2}\right)\right\rangle_{L^2}\\ &= \left\langle \left(g_{t_1}\otimes\delta_{-t_1}\right), X\ \left(g_{t_2}\otimes\delta_{t_2}\right)\right\rangle_{L^2}\ . \end{aligned} \tag{3.26}$$

Here $g_t = e^{\frac12\lambda\vec\nabla^2}\, h_t \in \mathcal S(\mathbb R^{d-1})$, and $X = \left(e^{-\lambda(-\frac{\partial^2}{\partial t^2}+m^2)}\, C\right)$. Note that $\vartheta\delta_{t_1} = \delta_{-t_1}$, as $(\vartheta\delta_{t_1})(u) = (\vartheta\delta)(u-t_1) = \delta(-u-t_1) = \delta(u+t_1) = \delta_{-t_1}(u)$. However ϑ does not affect the time in h_{t_1}.

The integral kernel for X is real and has the form

$$X(x,x') = (4\pi\lambda)^{-1/2} e^{-\lambda m^2}\int_{\mathbb R} du\,\left(\frac{1}{2\mu}\, e^{-\frac{(t-t'-u)^2}{4\lambda}-|u|\mu}\right)(\vec x-\vec x')\ . \tag{3.27}$$

Thus

$$F(t_1,t_2) = (4\pi\lambda)^{-1/2} e^{-\lambda m^2}\int_{-\infty}^{\infty} du\, e^{-\frac{(t_1+t_2-u)^2}{4\lambda}}\left\langle e^{-|u|\mu} g_{t_1}, e^{-|u|\mu} g_{t_2}\right\rangle_{-\frac12}\ . \tag{3.28}$$

Since f is real and the kernels are real, $F(t_1,t_2) = F(t_2,t_1)$ is real. Also it is clear from inspection that the compact support of $\widetilde h$ ensures that $F(t_1,t_2)$ extends in a neighborhood of $(t_1,t_2)=(0,0)$ to a complex analytic function of (t_1,t_2) with a convergent power series at the origin.

Combining these remarks,

$$\langle f,f\rangle_{\mathcal H} = -\left(F(S,S)+F(T,T)-F(S,T)-F(T,S)\right)\ . \tag{3.29}$$

As in §3.2.3, we take $S=0$ and $0<T$. Define $F(T)$ by

$$\langle f,f\rangle_{\mathcal H} = -(4\pi\lambda)^{-1/2} e^{-\lambda m^2} F(T)\ . \tag{3.30}$$

Then

$$F(T) = (4\pi\lambda)^{1/2} e^{\lambda m^2} \left(F(0,0) + F(T,T) - 2F(0,T) \right) \ . \tag{3.31}$$

The function f provides a counterexample to RP if for any T one has both $0 < F(T)$ and $\lambda, m^2 < \infty$. Expand $F(T)$ as a power series at $T = 0$. We claim that $F(0) = F'(0) = 0$, so

$$F(T) = \frac{T^2}{2} F''(0) + O(T^3) \ . \tag{3.32}$$

Clearly $F(0) = 0$. Also

$$\begin{aligned} F'(T) \quad = \quad & -\frac{1}{\lambda} \int_{-\infty}^{\infty} du\, (2T-u) e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\ & + 2 \int_{-\infty}^{\infty} du\, e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\ & + 2\, \frac{1}{2\lambda} \int_{-\infty}^{\infty} du\, (T-u) e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\ & - 2 \int_{-\infty}^{\infty} du\, e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \ . \end{aligned} \tag{3.33}$$

Taking $T = 0$, the second and last terms cancel, leaving an integrand that is an odd function of u. Therefore $F'(0) = 0$. Likewise the second derivative equals

$$\begin{aligned}
F''(T) \;=\; & -\frac{2}{\lambda}\int_{-\infty}^{\infty} du\, e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& +\frac{1}{\lambda^2}\int_{-\infty}^{\infty} du\,(2T-u)^2 e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& -\frac{2}{\lambda}\int_{-\infty}^{\infty} du\,(2T-u) e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& -\frac{2}{\lambda}\int_{-\infty}^{\infty} du\,(2T-u) e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& +4\int_{-\infty}^{\infty} du\, e^{-\frac{(2T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_T, \mu^2\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& +\frac{1}{\lambda}\int_{-\infty}^{\infty} du\, e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& -\frac{1}{2\lambda^2}\int_{-\infty}^{\infty} du\,(T-u)^2 e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& +\frac{1}{\lambda}\int_{-\infty}^{\infty} du\,(T-u) e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& +\frac{4}{\lambda}\int_{-\infty}^{\infty} du\,(T-u) e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, \mu\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \\
& -2\int_{-\infty}^{\infty} du\, e^{-\frac{(T-u)^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, \mu^2\, e^{-|u|\mu} g_T \right\rangle_{-\frac{1}{2}} \; .
\end{aligned} \tag{3.34}$$

Thus for $T=0$,

$$F''(0) = \int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{4\lambda}} \left\langle e^{-|u|\mu} g_0, \left(2\mu^2 - \frac{1}{\lambda} + \frac{u^2}{2\lambda^2}\right) e^{-|u|\mu} g_0 \right\rangle_{-\frac{1}{2}} \; . \tag{3.35}$$

The positivity of $F''(0)$ would be a consequence of the expectation of the operator $2\mu^2 - \frac{1}{\lambda} + \frac{u^2}{2\lambda^2}$ being positive in the vectors $e^{-|u|\mu}\, g_0$ under consideration.

The integral of the third term, $u^2/2\lambda^2$, is strictly positive for all $\lambda < \infty$. Furthermore μ acts in Fourier space as multiplication by $\mu(\vec{p})$, so $m \leqslant \mu$, and $0 \leqslant 2\mu^2 - \lambda^{-1}$ if $\frac{1}{2} \leqslant \lambda m^2$. This agrees with the conclusion of §3.2.3. But as $1 < d$, we can assume that the support of $\widetilde{h}$ (which is also the support of $e^{-|u|\mu(\vec{p})}\widetilde{g_0}$) lies outside the ball of radius $(2\lambda)^{-1/2}$. This entails $\lambda^{-1} \leqslant 2\mu(\vec{p})^2$ on the support of $\widetilde{h}$, and $0 \leqslant 2\mu^2 - \lambda^{-1}$ on the domain of functions h we consider.

Therefore we infer for such h that $0 < F''(0)$. Consequently for small, strictly positive T, one has $0 < F(T)$. Assuming $\lambda, m^2 < \infty$, the relation

(3.30) shows that $\langle f, f\rangle_{\mathcal{H}} < 0$. Hence we conclude that RP fails in $1 < d$ for all $0 < \lambda, m^2 < \infty$.

4. An Interesting Fundamental Question. When constructing a non-Gaussian $d\mu(\Phi)$ from a SPDE, how might one establish RP? The answer to this question is essential, for only with RP can one make the connection between probability theory and relativistic quantum field theory. And it is difficult to imagine in a situation where one does not have an explicit form for the answer (as in the Gaussian case of the free field), that one can establish a positivity condition unless it holds in each approximation. The result presented here clearly generalize to non-Gaussian measures whose moments depend continuously on the non-linearity. So can one modify the SPDE procedure in order to preserve RP for every intermediate λ?

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