PLANAR PARA ALGEBRAS, REFLECTION POSITIVITY

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Abstract. We define the notion of a *planar para algebra*, which arises naturally from combining planar algebras with the idea of $\mathbb{Z}_N$ para symmetry in physics. A subfactor planar para algebra is a Hilbert space representation of planar tangles with parafermionic defects, that are invariant under isotopy. For each $\mathbb{Z}_N$, we construct a family of subfactor planar para algebras which play the role of Temperley-Lieb-Jones planar algebras. The first example in this family is the parafermion planar para algebra. Based on this example, we introduce parafermion Pauli matrices, quaternion relations, and braided relations for parafermion algebras which one can use in the study of quantum information. Two different reflections play an important role in the theory of planar para algebras. One is the adjoint operator; the other is the modular conjugation in Tomita-Takesaki theory. We use the latter one to define the double algebra and to introduce reflection positivity. We give a new and geometric proof of reflection positivity by relating the two reflections through the quantum Fourier transform.

Contents

1. Introduction

We introduce the notion of a *planar para algebra*, which generalizes the concept of a planar algebra introduced by Jones [Jon98]. The idea arises naturally from considering fermions or para symmetry in physics. Mathematically this motivates giving the planar para algebra a $\mathbb{Z}_N$ grading. One might think of planar para algebras as a topological quantum field theory with parafermionic defects [Ati88, Wit88].

The partition function of a planar para algebra is a representation of planar tangles with parafermionic defects on a vector space invariant under isotopy. Usually we require that those tangles without boundary are presented by a scalar multiple of the vacuum vector defined by the empty diagram. When the partition function has the standard positivity property in planar algebra theory with respect to the vertical reflection, we call the planar para algebra a *subfactor* planar para algebra; those planar para algebras are closely related to subfactor theory.

The fundamental planar algebra is known as the Temperley-Lieb-Jones planar algebra. The positivity condition for the Temperley-Lieb-Jones planar algebra was proved by Jones' remarkable result on the rigidity of indices [Jon83]. In Theorem 2.24, we show that for each group $\mathbb{Z}_N$ one can construct a planar para algebra which plays the role of the Temperley-Lieb-Jones planar algebra in the theory of planar para algebras. We prove a similar rigidity result for the positivity condition in Theorem 8.1 and thereby obtain a family of subfactor planar para algebras.

For each $\mathbb{Z}_N$, the subfactor planar algebra in the family that has the smallest index is called a parafermion planar algebra, since it is algebraically isomorphic to the parafermion algebra with infinitely many generators. We explore other planar para algebra properties of parafermion algebras in Sections 4 and 6.

The expectation of a parafermion algebra is a tracial state. We can realize the underlying Hilbert space by the Gelfand-Naimark-Segal construction. The parafermion planar algebra not only gives a pictorial representation of a parafermion algebra, but it also gives a picture of the underlying Hilbert space. We give different models to present parafermion Pauli matrices and the underlying space in terms of diagrams. One model generalizes the representation for Pauli matrices by Majoranas, commonly used in condensed-matter physics.

Furthermore, we extend the isotopy to the three-dimensional space by introducing braids. We prove that parafermion planar para algebras are *half-braided*. The diagrammatic representation of the underlying Hilbert space is compatible with the braided isotopy. In [JLW16a] we discuss an application of the isotopy property to quantum information.

A significant ingredient of planar para algebra is the quantum Fourier transform $\mathfrak{F}$ defined as a rotation of the diagrams. The action of rotation on various defects are described by the para degree. For usual planar algebras, a 2π rotation equals the identity. In the case of planar fermion algebras, the 2π rotation on a fermion has eigenvalues ± 1. In the general case of planar para algebras, the 2π rotation of a $\mathbb{Z}_N$ parafermion has the eigenvalue $e^{\frac{2\pi i}{N}}$. In terms of planar para algebras, we define the quantum Fourier transform $\mathfrak{F}$ on parafermion algebras as a $\frac{\pi}{2}$ rotation of the diagrams. This quantum Fourier transform reduces to the usual quantum Fourier transform on a special subspace, as explained in §6.4.

The graded commutant of the parafermion algebra on the GNS representation can be represented pictorially in the parafermion planar algebra. The modular conjugation in Tomita-Takisaki theory turns out to be a horizontal reflection. In §7 we study reflection-doubled algebras, leading to the study of the *reflection-positivity* property [OS73a, OS73b]. This property is quite important in the context of particle physics and statistical physics, where it has wide use in establishing existence results in quantum field theory, as well as in the study of phase transitions. Reflection positivity of parafermion algebras had been proved in a different context [JP15a, JJ16a, JJ16b], where one finds further references to other papers on reflection positivity.

In Theorem 7.1 we give a new and geometric proof of reflection positivity that applies to parafermion algebras as a special case, and in general to subfactor planar para algebras. In particular, we relate the two notions of positivity mentioned above: C^* positivity and reflection positivity. We show that reflection positivity of a Hamiltonian in a subfactor planar para algebra is a consequence of the C^* positivity of the quantum Fourier transform of the Hamiltonian.

The underlying mechanism that leads to reflection positivity relies on the relation between two different reflections, one is the rotation of the other. In the planar para algebra, a horizontal reflection Θ defines the double. On the other hand a vertical reflection defines the adjoint *. These two reflections are related by a $\frac{\pi}{2}$ rotation, which is how the quantum Fourier transform enters. We combine rotation and reflection with the isotopy invariance of the partition function, in order to obtain the reflection positivity property. For parafermion algebras, we show in Theorem 7.6 that reflection positivity is equivalent to the positivity of the coupling constant matrix J^0 of the Hamiltonian for interaction across the reflection plane.

2. Planar Para Algebras

2.1. **Planar tangles.** Our definition of planar para algebras involves planar tangles. These tangles are similar to the planar tangles in Jones' original definition of planar algebras [Jon12]. However, for readers who are not familiar with planar algebras, we give the definitions here, indicating some main distinctive features in color or boldface[1].

A planar k-tangle T will consist of a smooth closed output disc D_0 in $\mathbb{C}$ together with a finite (possibly empty) set $\mathcal{D} = \mathcal{D}_T$ of disjoint smooth input discs in the interior of D_0. Each input disc $D \in \mathcal{D}$ and the output disc D_0, will have an even number $2k_D \geq 0$ of marked points on its boundary with $k = k_{D_0}$. Inside D_0, but outside the interiors of the $D \in \mathcal{D}$, there is also a finite set of disjoint smoothly embedded curves called strings, which are either closed curves, or the end points of the strings are different marked points of D_0 or of the D's in $\mathcal{D}$. Each marked point is the end-point of some string, which meets the boundary of the corresponding disc transversally.

The connected components of the complement of the strings in $\overset{\circ}{D}_0 \backslash \bigcup_{D\in\mathcal{D}} D$ are called regions. The connected component of the boundary of a disc, minus its marked points, will be called the intervals of that disc. Regions of the tangle are shaded (say in gray), or they are unshaded (say in white). Shading is done in a way that regions whose boundaries meet have different shading. Intervals have a unique shading, as only one side of any interval lies in a region. The shading will be considered to extend to the intervals which are part of the boundary of a region.

To each disc in a tangle there is a distinguished ***point*** on its boundary that is not an end point of a string. The distinguished point is marked by a dollar sign \$, placed to the left of the input disc, or to the right of the output disc. This distinguished point defines a distinguished interval for each disc.

We denote the set of all planar k-tangles for $k \geq 0$ by $\mathcal{T}_k$, and let $\mathcal{T} = \cup_k \mathcal{T}_k$. If the distinguished interval of D_0 for $T \in \mathcal{T}$ is unshaded, T will be called positive; if it is shaded, $\mathcal{T}$ will be called negative. Thus $\mathcal{T}_k$ is the disjoint union of sets of positive and negative planar para tangles: $\mathcal{T}_k = \mathcal{T}_{k,+} \cup \mathcal{T}_{k,-}$.

Definition 2.1. A planar tangle will be called *regular* if the distinguished point of each disc is on the left, and the distinguished points of the input discs are ordered vertically. Let RT denote the set of *regular planar tangles.*

This means that the x coordinate of each disc is the smallest one among all points on the boundary of the disc, and the y coordinates of the input discs are pairwise different. Let $y(D)$ denote the y coordinate of the distinguished point of an input disc D.

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[1] A main difference between planar and planar para algebras is that we mark a distinguished point on the boundary of each disc, within a distinguished interval. This change is necessary, in order to describe the precise height of Jones' symbol \$. This height is significant in the definition of our twisted tensor product.

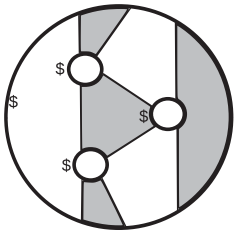


FIGURE 1. A regular planar 3-tangle

In certain situations one can compose two tangles T and S to obtain a tangle $T \circ_D S \in$ RT. To make this possible, the output disc of $S \in$ RT must be identical to one input disc D of a $T \in$ RT. Furthermore D must be lower than all the \$'s above the \$ of D, and it must be higher than the \$'s under the \$ of D. This makes it possible to find a diffeomorphism of the plane that moves each disc in T, other than D, to be completely higher or lower than D. Using this representative of the planar tangle D, one can then define the composition $T \circ_D S$ in the usual way: match the intervals and points of D in S with those of T. Also replace any closed, contractible string formed in this composition by a scalar δ, which we denote as the *circle parameter*.

2.2. Planar para algebras.

Let G be a finite abelian group and ω be a bicharacter of G.

Definition 2.2. A (shaded) (G, ω) planar para algebra $\mathscr{P}_\bullet$ will be a family of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces indexed by the set $\mathbb{N} \cup \{0\}$, having the following properties:

- Let $\mathscr{P}_{n,\pm}$ denote the $\pm$ graded space indexed by n.
- To each regular planar n-tangle T for $n \geq 0$ and $\mathcal{D}_T$ non-empty input discs, there will be a multilinear map
$$Z_T : \times_{i \in \mathcal{D}_T} \mathscr{P}_{D_i} \to \mathscr{P}_{D_0} , \quad \text{(II.1)}$$
where $\mathscr{P}_D$ is the vector space indexed by half the number of marked boundary points of i.
- The $\mathbb{Z}_2$ grading of each $\mathscr{P}_i$ is taken to be $+$ if the distinguished interval of D_i is unshaded, or $-$ if it is shaded, and similarly for $\mathscr{P}_{D_0}$.

Definition 2.3. The map Z_T is called the "partition function" of T and is subject to the following five requirements:

(i) (RT isotopy invariance) If φ is a continuous map from $[0,1]$ to orientation preserving diffeomorphisms of $\mathbb{C}$, such that φ_0 is the identity map and $\varphi_t(T) \in$ RT, then
$$\mathbb{Z}_T = Z_{\varphi_1(T)} ,$$
where the sets of internal discs of T and $\varphi_t(T)$ are identified using φ_t, for $t \in [0,1]$.

(ii) (Naturality) If $T \circ_D S$ exists and $\mathcal{D}_S$ is non-empty

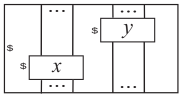
$$Z_{T \circ_D S} = Z_T \circ_D Z_S$$

where D is an internal disc in T.

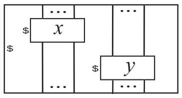
(iii) (Grading) Each vector space $\mathscr{P}_{n,\pm}$ is G graded,

$$\mathscr{P}_{n,\pm} = \oplus_{g \in G} \mathscr{P}_{n,\pm,g} \ , \quad \text{and} \quad Z_T : \otimes_{i \in \mathcal{D}_T} \mathscr{P}_{i,g_i} \to \mathscr{P}_{D_0, \sum_i g_i} \ .$$

(iv) (Para isotopy) Take $\mathscr{P}_g = \oplus_{n,\pm} \mathscr{P}_{n,\pm,g}$ for $g \in G$. We have



$= \omega(g,h)$



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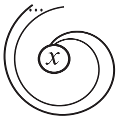
for any $x \in \mathscr{P}_g$ and $y \in \mathscr{P}_h$

(v) (Rotation) The clockwise 2π rotation of any g graded vector x is $\omega(g,g)x$, i.e.,



$= \omega(g,g)x.$

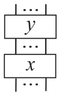
Remark 2.4. "Planar algebras" satisfying conditions (i) and (ii) have their own interests. Conditions (iii), (iv) and (v) are motivated by the discussion of parafermion algebras in [JP15a, JJ16a, JJ16b].

Remark 2.5. One can remove the condition that the $ is on the left, and introduce the rotation isotopy for arbitrary angle, not only 2π. However, this makes the definition and computation more complicated. For convenience, we choose a representative of planar tangles in the isotopy class by fixing the $ sign on the left.

Remark 2.6. When ω is the constant 1, the planar para algebra is a planar algebra. The zero graded planar para subalgebra is a planar algebra.

Definition 2.7. A vector x in $\mathscr{P}_g$ is called homogenous. The grading of x is defined to be g, denoted by $|x|_G$, or $|x|$, if it causes no confusion.

Notation 2.8. Furthermore in case it cannot cause confusion, we omit the output disc and the $ signs. A vector in $\mathscr{P}_{m,\pm}$ is called an m-box. Usually we put m strings on the top and m strings on the bottom. Then the m-box space $\mathscr{P}_{m,\pm}$ forms an algebra, where we denote the multiplication of $x, y \in \mathscr{P}_{m,\pm}$ diagrammatically by



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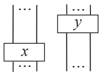
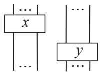
The identity is given by the diagram with m vertical strings, denoted by I_m.

Definition 2.9. We denote the graded tensor product as follows:

$x \otimes_+ y =$ 

, $x \otimes_- y =$ 

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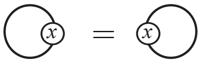
If x and y are homogenous, then we infer from para isotopy that $x \otimes_+ y = \omega(|x|, |y|)\, x \otimes_- y$. Under the multiplication and the graded tensor product $\otimes_+$, one obtains a (G, ω) graded tensor category. The objects are given by zero graded idempotents and the morphisms are given by maps from idempotents to idempotents. We refer the readers to [ENO05, MPS10] for the planar algebra case.

Definition 2.10. A planar para algebra is called unital if the empty disc is a vector in $\mathscr{P}_{0,\pm,0}$, called the vacuum vector.

Definition 2.11. A unital planar para algebra is called spherical, if $\underline{\dim \mathscr{P}_{0,\pm} = 1}$ and



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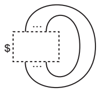
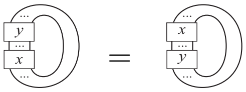
for any 1-box x. Both $\mathscr{P}_{0,+}$ and $\mathscr{P}_{0,-}$ are identified as the ground field.

Proposition 2.12. The linear functional 

on m-boxes is a trace, i.e.,



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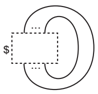
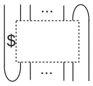
We call it the (unnormalized) Markov trace.

Proof. It is enough to prove the equation for any homogenous x and y. When the grading $|x|+|y|$ is not 0 mod N, both sides are zeros. When the grading $|x|+|y|$ is 0 mod N, applying the para isotopy and the 2π rotation of x, we obtain the equality. □

The normalized Markov trace tr on m-boxes is given by $\frac{1}{\delta^m}$ 

. The inclusion from $\mathscr{P}_{m,\pm}$ to $\mathscr{P}_{m+1,\pm}$ by adding one string to the right preserves the normalized trace.

2.3. **Quantum Fourier transforms.** The quantum Fourier transform is an important ingredient in planar (para) algebras. It behaves as a rotation in planar algebras. The quantum Fourier transform $\mathfrak{F}$ is defined as the action of the following tangle,



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This definition is motivated by the quantum Fourier transform on paragroups introduced by Ocneanu [Ocn88]. We also use other rotations on the m-box space:

- Denote the 2π rotation by $\rho_{2\pi} = \mathfrak{F}^{2m}$.
- Denote the π rotation by $\rho_\pi = \mathfrak{F}^m$, which one also calls the contragredient map. Note
$$\rho_\pi(xy) = \rho_\pi(y)\rho_\pi(x) \ , \quad \text{and} \quad \rho_\pi(x \otimes y) = \rho_\pi(y) \otimes \rho_\pi(x) \ . \tag{II.2}$$
- For even m, denote the $\frac{\pi}{2}$ rotation by $\rho_{\frac{\pi}{2}} = \mathfrak{F}^{\frac{m}{2}}$. This can also be considered as the quantum Fourier transform.

We refer the readers to Section 4 in [Liua] and [JLW16b] on the study of the quantum Fourier transform on subfactor planar algebras.

2.4. **Reflections.** Two reflections that play distinct roles are reflections about a vertical or horizontal line. The vertical reflection defines the usual adjoint in a planar para algebra.

Definition 2.13. We say a planar para algebra $\mathscr{P}_\bullet$ is a *-algebra, if there is an anti-linear involution $* : \mathscr{P}_{m,\pm,g} \to \mathscr{P}_{m,\pm,-g}$, for each m and $g \in G$; and $Z_{T^*}(x^*) = Z_T(x)^*$, for any x in the tensor power of $\mathscr{P}_{n,\pm}$, where the tangle T^* is the vertical reflection of the tangle T.

Definition 2.14. An anti-linear involution Θ on the unshaded planar para algebra $\mathscr{P}_\bullet$ is called a horizontal reflection, if $\Theta : \mathscr{P}_{m,\pm,g} \to \mathscr{P}_{m,\pm,-g}$, for each even m and $g \in G$; $\Theta : \mathscr{P}_{m,\pm,g} \to \mathscr{P}_{m,\mp,-g}$, for each odd m and $g \in G$; and $\Theta(Z_T(x)) = Z_{\Theta(T)}(\Theta(x))$, where the tangle $\Theta(T)$ is the horizontal reflection of the tangle T. In particular, the reflection Θ acts as $\Theta(x \otimes_+ y) = \Theta(y) \otimes_- \Theta(x)$ and $\Theta(xy) = \Theta(x)\Theta(y)$.

Consider the example of the group $G = \mathbb{Z}_N$, and the bicharacter $\omega(j,k) = q^{jk}$, where $q = e^{\frac{2\pi i}{N}}$. Choose ζ to be a square root of q such that $\zeta^{N^2} = 1$. Then

$$\zeta = \begin{cases} -e^{\frac{\pi i}{N}} \ , & \text{if } N \text{ is odd} \\ \pm e^{\frac{\pi i}{N}} \ , & \text{if } N \text{ is even} \end{cases} \ . \tag{II.3}$$

In the odd case with one solution, also $\zeta^N = 1$. In the even case one must choose one of the two solutions throughout, and also $\zeta^N = -1$.

Proposition 2.15. Let ζ be a square root of $q = e^{\frac{2\pi i}{N}}$, such that $\zeta^{N^2} = 1$. Then

$$\left| \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \zeta^{j^2} \right| = 1 \ . \tag{II.4}$$

Proof. The Fourier transform $\mathfrak{F}$ on $\mathbb{Z}_N$ is

$$(\mathfrak{F}f)(j) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} q^{ij} f(i) \;, \quad \text{with inverse} \quad (\mathfrak{F}^2 f)(-i) = f(i) \;. \tag{II.5}$$

Let $\omega = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \zeta^{j^2}$, so we wish to show $|\omega| = 1$. Also let $f(i) = \zeta^{i^2}$ and $g(j) = \zeta^{-j^2}$. Then

$$(\mathfrak{F}f)(j) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} q^{ij} \, \zeta^{i^2} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^{(i+j)^2} \, \zeta^{-j^2} = \omega \, g(j) \;.$$

In the last equality, we use that $\zeta^{N^2} = 1$, so the sum of $\zeta^{(i+j)^2}$ over i is independent of j. Similarly $(\mathfrak{F}g)(i) = \overline{\omega} f(i)$. But using the Fourier inversion identity of (II.5), as well as $f(-i) = f(i)$ in our case, we infer $|\omega|^2 = 1$. □

Recall that the 2π rotation is not the identity on a $(\mathbb{Z}_N, \omega)$ planar para algebra. Also recall that for $\zeta^{2N} = \zeta^{N^2} = 1$, the power $\zeta^{-|x|^2}$ is well-defined.

Definition 2.16. Define the reflection $\Theta = \Theta_\zeta$ as an antilinear extension of the operator on homogeneous elements x given by

$$\Theta(x) = \zeta^{-|x|^2} \rho_\pi(x^*) \;. \tag{II.6}$$

Proposition 2.17. On a $(\mathbb{Z}_N, \omega)$ planar para *-algebra, the map Θ defined in (II.6) is a horizontal reflection.

Proof. The horizontal reflection is the composition of an anti-clockwise π rotation, a vertical reflection and a complex conjugation. Suppose $T(x)$ is a labelled tangle for a regular planar tangle T and $x = \otimes_i x_i$. Assume that the ith label x_i is graded by g_i. Then the label $\Theta(x_i)$ in $\Theta(x)$ is graded by $-g_i$, and $\Theta(x_i) = \zeta^{-g_i^2} \rho_\pi(x_i)$. The para isotopy of each pair of labels contributes a scalar $q^{(-g_i)(-g_j)}$. Therefore

$$\begin{aligned} Z_{\Theta(T)}(\Theta(x)) &= \prod_i \zeta^{-g_i^2} \times \prod_{i,i'} q^{-g_i g_{i'}} \rho_\pi(Z_{T^*}(x^*)) \\ &= \zeta^{-|x|^2} \rho_\pi(Z_T(x)^*) \\ &= \Theta(Z_T(x)) \end{aligned}$$

□

To introduce reflection positivity, the reflection $\Theta(x)$ should be the horizontal reflection of x; it should be represented as a box beside x, namely on the same level, with also the \$ signs on the same horizontal level. But equal levels are not permitted in planar para algebras.

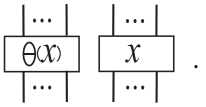
In order to avoid this difficulty, we introduce the twisted tensor product, which plays the same role as the twisted product for parafermion algebras in [JP15b, JJ16b]. For any homogenous x, we have $|\Theta(x)| = -|x|$. By para isotopy,

$$\Theta(x) \otimes_+ x = q^{-|x|^2} \Theta(x) \otimes_- x.$$

Definition 2.18 (Twisted tensor product). Let the twisted tensor product of $\Theta(x)$ and x be

$$\Theta(x) \otimes_t x := \zeta^{|x|^2}\Theta(x) \otimes_+ x = \zeta^{-|x|^2}\Theta(x) \otimes_- x \ , \tag{II.7}$$

pictorially denoted by



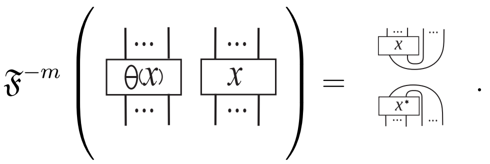
(II.8)

Proposition 2.19. For homogenous x and y in $\mathscr{S}_{m,\pm}$, we have

$$(\Theta(x) \otimes_t x)(\Theta(y) \otimes_t y) = \Theta(xy) \otimes_t xy.$$

Proof. It follows from the equality $\zeta^{|x|^2}\zeta^{|y|^2}q^{|x||y|} = \zeta^{(|x|+|y|)^2}$. □

Proposition 2.20. For any homogenous m-box x, we have



Proof. Note that $\Theta(x) \otimes_t x = \rho_\pi(x^*) \otimes_+ x$. Applying $\mathfrak{F}^{-m}$, we obtain the equality by isotopy. □

2.5. **Subfactor planar para algebras.** The m-box space of a planar para *-algebra has an inner product $tr(x^*y)$ for m-boxes x and y.

Definition 2.21. A subfactor planar para algebra $\mathscr{P}_\bullet$ will be a spherical planar para *-algebra with $\dim \mathscr{P}_{m,\pm} < \infty$ for all m, and such that the inner product is positive.

Remark 2.22. We call it a subfactor planar para algebra, because a subfactor planar para algebra is the graded standard invariant of a G graded subfactor. The general theory will be discussed in a coming paper. Motivated by the deep work of Popa [Pop90, Pop94], we conjecture that strongly amenable graded hyperfinite subfactors of type II_1 are classified by subfactor planar para algebras.

When $\omega = 1$, the subfactor planar para algebra $\mathscr{P}_\bullet$ is a (G graded) subfactor planar algebra. The zero graded part of a subfactor planar para algebra is a subfactor planar algebra.

Many notions of subfactor planar algebras are inherited for subfactor planar para algebras, such as the Jones projections, the basic construction, principal graphs, depths. We refer the readers to [Jon83, Jon98] for the planar algebra case.

Definition 2.23. A subfactor planar para algebra $\mathscr{P}_\bullet$ is called irreducible, if $\dim \mathscr{P}_{1,\pm,0} = 1$.

2.6. **Examples.** Skein theory is a presentation theory for planar algebras in terms of generators and (algebraic and topological) relations. One can study the skein theory for planar para algebras in a similar way. We refer the reader to [Jon98] for the skein theory of planar algebras (in Section 1) and many interesting examples (in Section 2). Also see [BMPS12, Liub] for the skein-theoretic construction of the extended Haagerup planar algera and a new family of planar algebras.

Let us construct a spherical unshaded planar para algebra with the para symmetry $(\mathbb{Z}_N, \omega)$. We take the same bicharacter that we considered in §2.4, namely $\omega(j,k) = q^{jk}$, where $q = e^{\frac{2\pi i}{N}}$ and choose ζ to be a square root of q given in (II.3), such that $\zeta^{N^2} = 1$. This planar para algebra plays the role of the Temperley-Lieb-Jones planar algebra among planar para algebras with para symmetry $(\mathbb{Z}_N, \omega)$.

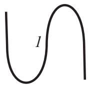
Let $\mathscr{P}_\bullet$ be the unshaded planar algebra over the field $\mathbb{C}(\delta)$ generated by a 1-box c, graded by 1, and satisfying the following relations:

(1) 

=  ,

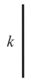
(2) 

= 0, for $1 \le k \le N-1$,

(3) 

$= \zeta$ 

, namely Fourier-parafermion relation,

where δ is the circle parameter and 

denotes a through string with k labels c.

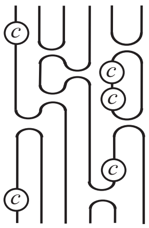
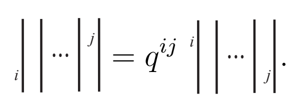


FIGURE 2. A regular planar 6-tangle labelled by c

Precisely, the vectors in $\mathscr{P}_m$ are linear sums of regular planar m-tangles labelled by c modulo the relations. The para isotopy can also be viewed as relations:



Regular planar tangles act on labelled regular planar tangles by gluing the diagrams.

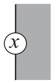
The planar para algebra is called *evaluable* by the relations, if $\dim(\mathscr{P}_0) \leq 1$, i.e., any regular labelled planar 0-tangles is reduced to the ground field; and $\dim(\mathscr{P}_m) < \infty$.

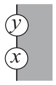
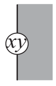
The relations are called *consistent*, if $\dim(\mathscr{P}_0) = 1$, i.e., different processes of evaluating a regular labelled planar 0-tangle give the same value in the ground field. In this case, the map from regular labelled planar 0-tangles to the ground field is called the *partition function*, denoted by Z.

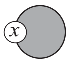
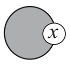
Theorem 2.24. The above relations of the generator c are consistent and the unshaded planar para algebra $\mathscr{P}_\bullet$ is evaluable and spherical over the field $\mathbb{C}(\delta)$.

Proof. See Appendix A. □

When δ is a real number, we introduce the vertical reflection on $\mathscr{P}_\bullet$ mapping c to $c^{-1}(= c^{N-1})$. Note that the involution preserves the relations of c, thus it is well-defined on the planar para algebra $\mathscr{P}_\bullet$. So $\mathscr{P}_\bullet$ is a planar para *-algebra over $\mathbb{C}$. We will prove that the partition function Z is positive semi-definite with respect to * in Sections 3 and 8 and construct subfactor planar para algebras by taking a proper quotient.

Note that the 1-box space of a (G, ω) planar para algebra forms a finite dimensional G graded algebra with a G graded trace. (Here a G graded trace means that the trace of any non-zero graded vector is zero.) On the other hand, given an Abelian group G, a bicharacter ω of G, and any finite dimensional G graded algebra A with a G graded trace τ, we can construct a shaded (G, ω) planar para algebra $\mathscr{P}(A)$ with the circle parameter δ over the field $\mathbb{C}(\delta)$. The generators of $\mathscr{P}(A)$ are 1-boxes , for all $x \in A$. The relations are given by

(1)  = ;

(2)  $= \tau(x)$ and  $= \tau(x)$,

for any $x, y \in A$.

Theorem 2.25. The above relations are consistent and the shaded (G, ω) planar para algebra $\mathscr{P}(A)$ is evaluable and spherical over the field $\mathbb{C}(\delta)$.

Proof. The proof is similar to that of Theorem 2.24. □

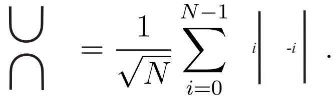
In the case of parafermion planar para algebras, the $\mathbb{Z}_N$ graded algebra A is given by the N dimensional algebra generated by c and $c^N = 1$. The G graded trace is given by $\tau(c^k) = 0$, for $1 \leq k \leq N-1$, and $tr(1) = 1$.

3. Parafermion Planar Para Algebras

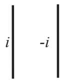
In this section, we take $\delta = \sqrt{N}$ and study the planar para algebra $\mathscr{P}_\bullet$ over the field $\mathbb{C}$. Recall that $\mathscr{P}_\bullet$ is a planar para *-algebra with the vertical reflection * defined as an extension of $c^* = c^{-1}$.

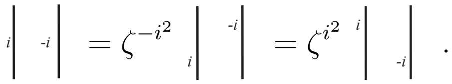
The kernel of the partition function $\ker(Z) = \bigcup_{m,\pm}\{x \in \mathscr{P}_{m,\pm} | Z(tr(xy)) = 0, \forall y \in \mathscr{P}_{m,\pm}\}$ is *an ideal* of $\mathscr{U}(P)$, in the sense that any fully labelled regular planar tangle with a label in $\ker(Z)$ is in $\ker(Z)$. Thus action of regular planar tangles is well defined on the quotient $\mathscr{P}/\ker(Z)$.

We prove that the following relation holds in $\mathscr{P}/\ker(Z)$:



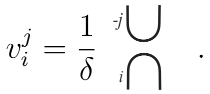
(III.1)

Recall that the twisted tensor product  is defined as



By relation III.1, any fully labelled regular planar tangle is a linear sum of labelled regular planar tangles with only labelled vertical strings. The algebra generated by labelled vertical strings is a parafermion algebra, see Section 6.1 for the definition of parafermion algebras. Therefore we call the planar para algebra $\mathscr{P}/\ker(Z)$ the parafermion planar para algebra, denoted by $PF_\bullet$. We prove that $PF_\bullet$ is a subfactor planar para algebra. We use this C^* positivity condition to prove reflection positivity in Section 7. We give some interesting properties of the parafermion planar para algebra in Section 4 and 6. Further applications in quantum information of these topological isotopy and braided relations in Section 6.5 are discussed in [JLW16a].

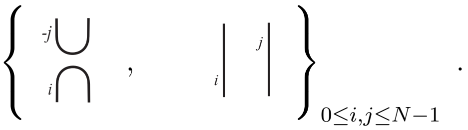
Notation 3.1. Take



Then it is easy to check that $v_i^j v_k^l = \delta_{j,k} v_i^l$, and $(v_i^j)^* = v_j^i$. In particular, $\{v_i^i\}$ are pairwise orthogonal idempotents.

Lemma 3.2. The vector $I_2 - \sum_{i \in \mathbb{Z}_N} v_i^i$ is in the kernel of the partition function of $\mathscr{P}_\bullet$.

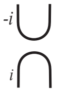
Proof. The 2-box space has a generating set



Take $x = id - \sum_{i \in \mathbb{Z}_N} v_i^i \in \ker(Z)$. It is easy to check that $tr(xy) = 0$ for any 2-box y. By the spherical property, we have that any 0-tangle labelled by x is isotopic to $tr(xy)$ for some 2-box y. So x is in the kernel of the partition function. $\square$

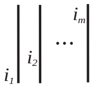
Thus we have the relation $I_2 = \sum_{g \in G} v_g$ in the quotient $(\mathscr{P}/\ker Z)_\bullet$, i.e.,

$$= \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \tag{III.2}$$



Take the quantum Fourier transform $\mathfrak{F}$ on both sides, i.e., the $\frac{\pi}{2}$ rotation. We obtain Equation III.1.

Lemma 3.3. The vectors $c^{i_1} \otimes_+ c^{i_2} \cdots \otimes_+ c^{i_m}$, i.e.,



$$, \quad \text{for} \quad 0 \le n_1, n_2 \cdots n_m \le N-1 \ ,$$

form an orthonormal basis of $(\mathscr{P}/\ker Z)_m$.

Proof. Any m-box is a linear sum of labelled Temperley-Lieb diagrams. Applying the relation III.1, any labelled Temperley-Lieb diagram is a linear sum of the vectors $c^{n_1} \otimes_+ c^{n_2} \cdots \otimes_+ c^{n_m}$, $0 \le n_1, n_2 \cdots n_m \le N-1$. Thus these vectors form a generating set of $\mathscr{P}/\ker Z_\bullet$. It is easy to check that these vectors form an orthonormal basis with respect to the Markov trace. $\square$

Theorem 3.4. When $\delta = \sqrt{N}$, the kernel of the partition function $\ker Z$ is generated by $id - \sum_{i \in \mathbb{Z}_N} v_i^i$, and $\mathscr{P}/\ker Z$ is a subfactor planar para algebra.

Proof. The proof is a consequence of Lemmas 3.2 and 3.3. $\square$

4. PARAFERMION PAULI MATRICES

We define unitary $N \times N$ matrices X, Y, Z that play the role in the parafermion algebra of the 2×2 Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ for fermions. We call the matrices X, Y, Z the *parafermion Pauli matrices.* They act on an N-dimensional Hilbert space with basis vectors indexed by $\mathbb{Z}_N$. In §4.1 we define these matrices and determine some of their properties.

In §4.2 we discuss the relation between the parafermion Pauli matrices and algebra. In §4.3 we give two different ways to realize these matrices as quadratic functions of parafermion matrices $\widehat{X}, \widehat{Y}, \widehat{Z}$, acting on a larger space. We call these *Model I and Model II.* Restricted to a subspace, we obtain a representation of the matrices X, Y, Z. The first model is a generalization of the well-known representation for Pauli matrices by Majoranas, commonly used in condensed-matter physics. The second model is different. In both cases the Pauli matrices are products of parafermion particle operators with their anti-particle operators.

4.1. **Parafermion Pauli matrices.**

Let us use Dirac notation for vectors, and take the ortho-normal basis for an N-dimensional Hilbert space: $\{\, |k\rangle \mid k \in \mathbb{Z}\}$. Choose $q = e^{\frac{2\pi i}{N}}$ and its square root ζ such that $\zeta^{N^2} = 1$, as in (II.3). Define the Pauli matrices X, Y, Z by their action on the basis,

$$X\,|k\rangle = |k-1\rangle \ , \quad Y\,|k\rangle = \zeta^{-2k-1}\,|k+1\rangle \ , \quad \text{and} \quad Z\,|k\rangle = q^k\,|k\rangle \ . \tag{IV.1}$$

Clearly X, Y, Z are unitary. In case $N = 2$, the choices

$$\zeta = -i \ , \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{IV.2}$$

yield the standard representation of the Pauli matrices; the choice $\zeta = i$ yields the complex conjugate representation.

For any $N \in \mathbb{N}$, these matrices satisfy a first set of parafermion Pauli matrix relations,

$$X^N = Y^N = Z^N = 1 \ , \quad YX = q\,XY \ , \quad ZY = q\,YZ \ , \quad \text{and} \quad XZ = q\,ZX \ . \tag{IV.3}$$

They also satisfy a second set of parafermion Pauli matrix relations that involve ζ,

$$XYZ = YZX = ZXY = \zeta^{-1} \ . \tag{IV.4}$$

4.2. **Quaternion relations.**

Here we give the analog of the quaternion algebra, namely the parafermion quaternion relations. Define the three unitary transformations $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ by

$$\boldsymbol{i} = -\zeta^{-1}Y \ , \quad \boldsymbol{j} = -\zeta X \ , \quad \boldsymbol{k} = -\zeta Z \ . \tag{IV.5}$$

Then we have the following relations for $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, which we call the *parafermion quaternion relations*:

$$\boldsymbol{i}^N = \boldsymbol{j}^N = \boldsymbol{k}^N = -1 \ , \quad \boldsymbol{ij} = q\,\boldsymbol{ji} \ , \quad \boldsymbol{jk} = q\,\boldsymbol{kj} \ , \quad \boldsymbol{ki} = q\,\boldsymbol{ik} \ , \quad \text{and} \quad \boldsymbol{i}\,\boldsymbol{j}\,\boldsymbol{k} = -1 \ . \tag{IV.6}$$

The matrices $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ generate the algebra of $N \times N$ matrices.

4.3. **Representation quadratic in parafermions.**

Let c_1, c_2, c_3, c_4 denote four parafermion operators that satisfy the relations

$$c_i c_j = q\,c_j c_i \ , \quad \text{for} \quad i < j \ , \quad \text{and} \quad c_i^N = 1 \ , \quad \text{where} \quad q = e^{\frac{2\pi i}{N}} \ . \tag{IV.7}$$

Let $\zeta = q^{1/2}$ with $\zeta^{N^2} = 1$.

4.3.1. **Model I.** For any $N \in \mathcal{N}$ define the matrices

$$\widehat{X} = \zeta\, c_1^{-1} c_4 \;, \quad \widehat{Y} = \zeta\, c_2 c_4^{-1} \;, \quad \widehat{Z} = \zeta\, c_3^{-1} c_4 \;. \tag{IV.8}$$

These matrices have the property that they satisfy the first set of parafermion Pauli relations given in (IV.3) for X, Y, Z, namely

$$\widehat{X}^N = \widehat{Y}^N = \widehat{Z}^N = 1 \;, \quad \widehat{Y}\widehat{X} = q\, \widehat{X}\widehat{Y} \;, \quad \widehat{Z}\widehat{Y} = q\, \widehat{Y}\widehat{Z} \;, \quad \widehat{X}\widehat{Z} = q\, \widehat{Z}\widehat{X} \;. \tag{IV.9}$$

However they do not identically satisfy the second set of parafermion Pauli relations (IV.4) that involve the square root ζ. The product $\widehat{X}\widehat{Y}\widehat{Z}$ has the form

$$\widehat{X}\widehat{Y}\widehat{Z} = \widehat{Y}\widehat{Z}\widehat{X} = \widehat{Z}\widehat{X}\widehat{Y} = \zeta^{-1}\gamma \;, \quad \text{where} \quad \gamma = q c_1^{-1} c_2 c_3^{-1} c_4 \;, \tag{IV.10}$$

which indicates that γ commutes with $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$. Thus one achieves the correct parafermion Pauli matrix algebra representing (IV.4) on the subspace for which the unitary γ has eigenvalue $+1$. In the $N = 2$ case, this is a well-known transformation in condensed matter physics.

4.3.2. **Model II.** This second model comes from taking

$$\widehat{X} = \zeta\, c_1^{-1} c_2 \;, \quad \widehat{Y} = \zeta\, c_1 c_3^{-1} \;, \quad \widehat{Z} = \zeta\, c_1^{-1} c_4 \;. \tag{IV.11}$$

These matrices also satisfy the first set of parafermion Pauli relations given in (IV.3) for X, Y, Z, namely

$$\widehat{X}^N = \widehat{Y}^N = \widehat{Z}^N = 1 \;, \quad \widehat{Y}\widehat{X} = q\, \widehat{X}\widehat{Y} \;, \quad \widehat{Z}\widehat{Y} = q\, \widehat{Y}\widehat{Z} \;, \quad \widehat{X}\widehat{Z} = q\, \widehat{Z}\widehat{X} \;. \tag{IV.12}$$

In this case one also finds that

$$\widehat{X}\widehat{Y}\widehat{Z} = \widehat{Y}\widehat{Z}\widehat{X} = \widehat{Z}\widehat{X}\widehat{Y} = \zeta^{-1}\gamma \;, \quad \text{where} \quad \gamma = q c_1^{-1} c_2 c_3^{-1} c_4 \;. \tag{IV.13}$$

So the relationship again reduces to the desired one on the $+1$ eigenspace of γ.

5. PICTORIAL REPRESENTATIONS OF PARAFERMION PAULI MATRICES

Here we give three different diagrammatic representations for the matrices X, Y, Z and of $\widehat{X}, \widehat{Y}, \widehat{Z}$ defined in §4, as well as representations for the vectors in the underlining vector space on which they act. The two-string model represents X, Y, Z as $N \times N$ matrices. The four-string models illustrate how to represent $\widehat{X}, \widehat{Y}, \widehat{Z}$ as $N^2 \times N^2$ matrices. These matrices leave invariant a subspace of dimension N, and on that subspace they represent the algebra of the matrices X, Y, Z. The diagrams give a simple interpretation to the matrices $\widehat{X}, \widehat{Y}, \widehat{Z}$ in the four-string models, and show how they leave the appropriate subspace invariant. Throughout this section take $\delta = \sqrt{N}$.

5.1. **The two-string model.** In this model, we deal with the X, Y, Z directly. We represent the vector $|k\rangle$ by the cap diagram

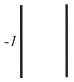
$|k\rangle = N^{-\frac{1}{4}}$  . (V.1)

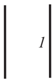
The vertical reflection gives the adjoint, or dual vector $\langle k|$, which we represent as the cup diagram

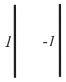
$\langle k| = N^{-\frac{1}{4}}$  , (V.2)

so that $\langle k, k'\rangle = \langle k\,|k'\rangle = \delta_{kk'}$.

The parafermion Pauli matrices X, Y and Z act on these vectors. We represent them as

$X =$  ,

$Y =$  ,

$Z =$  . (V.3)

From the diagrams it is clear that

$$XYZ = \zeta^{-1} \,. \tag{V.4}$$

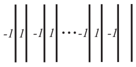
Let us define X_m, Y_m, Z_m to be the diagrams with X, Y, Z on the $(2m-1)^{\text{th}}$ and $2m^{\text{th}}$ strings respectively. Take

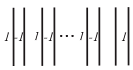
$$\widetilde{X}_m = Z_1^{-1} Z_2^{-1} \cdots Z_{m-1}^{-1} X_m \,, \tag{V.5}$$

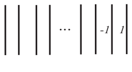
$$\widetilde{Y}_m = Z_1 Z_2 \cdots Z_{m-1} Y_m \,, \tag{V.6}$$

$$\widetilde{Z}_m = Z_m \,. \tag{V.7}$$

They are presented pictorially as $2m$-boxes:

$\widetilde{X}_m =$  , (V.8)

$\widetilde{Y}_m =$  , (V.9)

$\widetilde{Z}_m =$  . (V.10)

Then $\widetilde{X}_m$, $\widetilde{Y}_m$, $\widetilde{Z}_m$ satisfy the same relation as Pauli matrices X, Y and Z. Moreover, they commute for different m. This gives a representation of the parafermion algebras with $2m$ generators by the mth tensor power of the $N \times N$ matrix algebra generated by Pauli matrices.

5.2. **The Type I four-string model.** This model corresponds to the parafermion representation of $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$ given in Model I of §4.3.

Basically the vectors $|k\rangle$ belong to the zero-graded part of the tensor product of two copies of the two-string model. We represent the vector $|k\rangle$ and its adjoint by

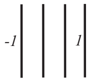
$$|k\rangle = N^{-\frac{1}{2}}$$ 

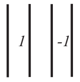
$$, \qquad \text{and} \qquad \langle k| = N^{-\frac{1}{2}}$$ 

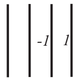
$$. \qquad \text{(V.11)}$$

These vectors map to the corresponding vectors in the two-string model of §5.1. More generally one could denote vectors $|k, k'\rangle$ given by the tensor product of caps $|k\rangle$ and $|k'\rangle$, but we draw the diagrams for vectors in the zero-graded subspace; for these vectors $k' = -k$.

We represent the parafermion Pauli matrices $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$ of Model I by

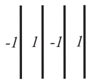
$$\widehat{X} =$$ 

$$, \qquad \widehat{Y} =$$ 

$$, \qquad \widehat{Z} =$$ 

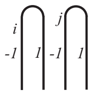
$$. \qquad \text{(V.12)}$$

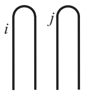
Also

$$\gamma = \zeta \widehat{X}\widehat{Y}\widehat{Z} =$$ 

$$.$$

5.2.1. **The grading operator.** We call γ the *grading operator*, since



$$= q^{-i-j}$$ 

$$.$$

Hence γ has eigenvalue 1 on the zero-graded subspace on which the matrices $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$ satisfy the correct algebraic relations. The diagrams show that $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$ preserve the grading and that they commute with γ.

5.2.2. **The braiding operators.** Introduce the unitary braids b_1 and b_2 defined as

$$b_1 |k\rangle = \zeta^{-k^2} |k\rangle ; \tag{V.13}$$

$$b_2 |k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \zeta^{j^2} |j+k\rangle . \tag{V.14}$$

One represents the braids b_1 and b_2 pictorially (up to a phase) as

$$b_1 =$$ 
$$, \qquad b_2 =$$ 
$$, \qquad \text{(V.15)}$$

while their adjoints are represented as

$$b_1^* =$$ 
$$, \qquad b_2^* =$$ 
$$. \qquad \text{(V.16)}$$

Diagrammatically, one sees that

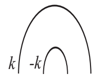
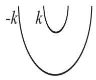
$$b_2\,\widehat{X}\,b_2^* = \widehat{X}\ , \qquad b_1\,\widehat{Z}\,b_1^* = \widehat{Z}\ . \tag{V.17}$$

One can derive the relations

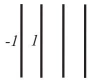
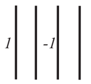
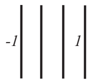
$$b_1\,\widehat{X}\,b_1^* = \widehat{Y}^{-1}\ , \quad b_1\,\widehat{Y}\,b_1^* = \widehat{X}^{-1}\ , \quad b_2\,\widehat{Y}\,b_2^* = \widehat{Z}^{-1}\ , \quad b_2\,\widehat{Z}\,b_2^* = \widehat{Y}^{-1}\ , \tag{V.18}$$

using the braiding relations that we will give in §6.5. We omit this calculation.

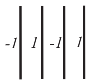
5.3. The Type II four-string model. This model corresponds to the parafermion representation of $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$ given in Model II of §4.3. We represent the vector $|k\rangle$ and its dual $\langle k|$ by

$|k\rangle = N^{-\frac{1}{2}}$ , and $\langle k| = N^{-\frac{1}{2}}$ . (V.19)

The Pauli matrices X, Y and Z are presented by

$\widehat{X} =$ , $\widehat{Y} =$ , $\widehat{Z} =$ . (V.20)

The grading operator γ is represented by

$\gamma =$ . (V.21)

In this case, the matrices b_1 and b_2 defined in Equations (V.13) and (V.14) are presented pictorially (up to a phase) as

$b_1 =$  , $b_2 =$  . (V.22)

One can also derive braiding relations in the Type II model for the parafermion Pauli matrices, such as (V.17)–(V.18) in the Type I model, by using the braiding relations in §6.5.

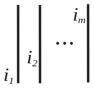
6. A Pictorial Interpretation of Parafermion Algebras

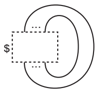
6.1. Parafermion algebras. The parafermion algebra is defined by generators: c_i, $i = 1, 2, \cdots$ and relations,

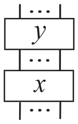
$$c_i^N = 1\ , \quad c_i\,c_j = q\,c_j\,c_i\ , \quad \text{for} \quad i < j\ , \quad \text{with} \quad q = e^{\frac{2\pi i}{N}}\ . \tag{VI.1}$$

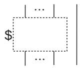
Denote the parafermion algebra generated by c_i, $1 \leq i \leq m$ as PF_m. It has a basis $C_I = c_1^{i_1} c_2^{i_2} \cdots c_m^{i_m}$, for $0 \leq i_1, i_2, \cdots, i_m \leq N-1$. The expectation on PF_m is defined as $tr(1) = 1$, $tr(C_I) = 0$, if $C_I \neq 1$. It is a tracial state. The inclusion from PF_m to PF_{m+1} is trace preserving.

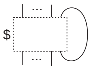
Remark 6.1. If we apply the Gelfand-Naimark-Segal construction to the inductive limit $\lim_{m\to\infty} PF_m$ with respect to the tracial state, then we obtain a hyperfinite factor $\mathcal{R}$ of type II_1. The Bernoulli shift $c_i \to c_{i+1}$ is an endomorphism ρ of the factor $\mathcal{R}$. The graded standard invariant of the corresponding subfactor $\mathcal{R} \supset \rho(\mathcal{R})$ is exactly the subfactor planar para algebra for parafermions with $\delta = \sqrt{N}$.

6.2. **Actions of planar tangles on parafermion algebras.** In planar para algebras, the labelled regular planar m-tangle  is presented by the vector $c_1^{i_1} c_2^{i_2} \cdots c_m^{i_m}$ in PF_m.

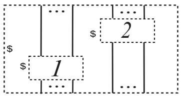
The Markov trace $\frac{1}{\delta^m}$  is the expectation on PF_m.

The multiplication tangle gives the usual multiplication on PF_m: $xy =$  .

The tangle $\frac{1}{\delta}$  is the trace preserving inclusion from PF_m to PF_{m+1}.

The tangle $\frac{1}{\delta}$  is the trace preserving conditional expectation from PF_m to PF_{m-1}.

We also have the graded tensor products from $PF_m \hat{\otimes} PF_n$ to PF_{m+n} given by

 .

6.3. **Temperley-Lieb subalgebras.** Take

$$E_i = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{\frac{k^2}{2}} c_i^k c_{i+1}^{-k} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{-\frac{k^2}{2}} c_{i+1}^{-k} c_i^k \ .$$

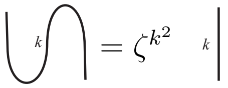
From Equation (III.1), we infer that E_i is presented by . The E_i satisfy the following relations and generate a Temperley-Lieb subalgebra in the parafermion algebra:

(1) $E_i = E_i^* = \frac{1}{\sqrt{N}} E_i^2$.
(2) $E_i E_j = E_j E_i$, for $|i - j| \geq 2$.
(3) $E_i E_{i\pm 1} E_i = E_i$.

One can check the following joint relations for E_i and c_i algebraically,

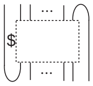
$$E_i\, c_i^k = q^{-\frac{k^2}{2}} E_i\, c_{i+1}^k \ ,$$

$$c_i^k E_i = q^{\frac{1^2}{2}} c_{i+1}^k E_i \ .$$

They can also be derived from the relation  .

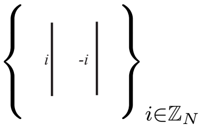
6.4. **Quantum Fourier transform.**

The quantum Fourier transform is an important ingredient in subfactor planar (para) algebras. Since the parafermion algebra forms a subfactor planar para algebra. we can introduce the quantum Fourier transform on parafermion algebras. Its algebraic definition is complicated, but its topological definition is simply a rotation. The quantum Fourier transform $\mathfrak{F}$ is given by the action of the following tangle,

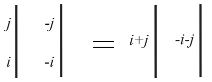
 .

Algebraically the quantum Fourier transform on PF_m is defined as follows: we first embed PF_m in PF_{m+1} by mapping c_i to c_{i+1}. The inclusion is denoted by ι_l. Let Φ_r be the trace preserving conditional expectation from PF_{m+1} to the subalgebra PF_m generated by $c_1, c_2, \cdots, c_m$. Then the quantum Fourier transform of $x \in PF_m$ is defined as $\mathfrak{F}(x) = \sqrt{N}\Phi_r(E_m E_{m-1} \cdots E_1 \iota_l(x))$.

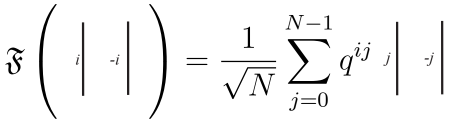
In particular, the zero graded part of the 2-box space has a basis

 .

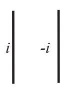
Moreover, the basis forms the group $\mathbb{Z}_N$:

 .

Proposition 6.2. The restriction of the quantum Fourier transform on the zero graded part of 2-box space is the discrete Fourier transform on the group $\mathbb{Z}_N$:

 . (VI.2)

Proof. Diagrammatically,

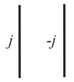
$\mathfrak{F}\Bigg($ 

$\Bigg) =$ 

, by Proposition 2.20,

$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1}$ 

, by Equation III.1,

$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} q^{ij}$ 

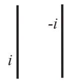
, by para isotopy.

□

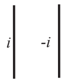
Note that the 2-box space forms an N by N matrix algebra. Thus we can extend the Fourier transform on the group $\mathbb{Z}_N$ to the quantum Fourier transform on $N \times N$ matrices.

6.5. **Braided relations.** In this section, we construct braids for parafermion algebras which behave well in a diagrammatical way, so that the strings can act over the parafermion planar para algebra $PF_\bullet$ in the 3-dimensional space.

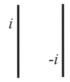
Take $\omega = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^{i^2}$, so $|\omega| = 1$ by Proposition 2.15. Let $\omega^{\frac{1}{2}}$ be a square root of ω. Let us construct the braids as

 $= \frac{\omega^{\frac{1}{2}}}{\sqrt{N}} \sum_{i=0}^{N-1}$ 

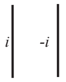
(VI.3)

$= \frac{\omega^{\frac{1}{2}}}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^{-i^2}$ 

, (VI.4)

 $= \frac{\omega^{-\frac{1}{2}}}{\sqrt{N}} \sum_{i=0}^{N-1}$ 

(VI.5)

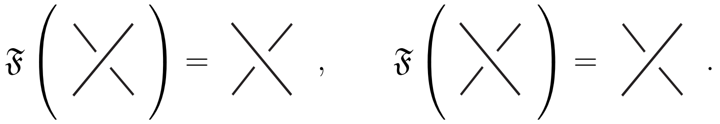
$= \frac{\omega^{\frac{1}{2}}}{\sqrt{N}} \sum_{i=0}^{N-1} \zeta^{i^2}$ 

. (VI.6)

Since $\zeta^{i^2} = \zeta^{(-i)^2}$, the two braids behave well under the vertical reflection $*$ and also under the horizontal reflection Θ:

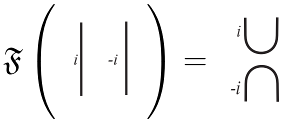
$\Bigg($  $\Bigg)^* =$  , $\quad \Theta\Bigg($  $\Bigg) =$  . (VI.7)

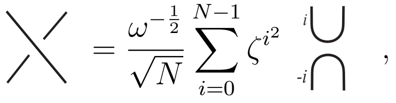
Proposition 6.3. With the above notion of $\omega^{\frac{1}{2}}$, the two braids behave well under $\frac{\pi}{2}$ rotation:



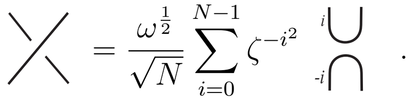
(VI.8)

Proof. The computation has been done in the proof of Proposition 2.15 ☐

Recall that  . Thus

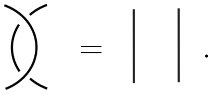


(VI.9)



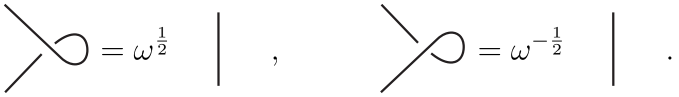
(VI.10)

Therefore the braids are unitary and we have the Reidemeister move of type II:

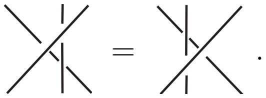


(VI.11)

Moreover, we have the following Reidemeister moves of type I:

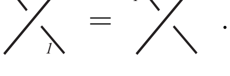


The Reidemeister move of type III is also known as the Yang-Baxter equation:



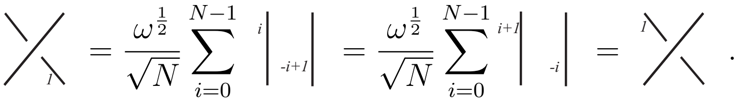
This is a consequence of:

Theorem 6.4 (Braid-Parafermion Relation). We have the relation:



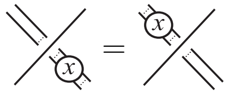
(VI.12)

Proof. By Equation VI.5,



Here we translate the sum in $\mathbb{Z}_N$. $\square$

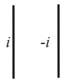
The braid-parafermion relation VI.12 says that the generator c can move under the string. Combining this with the Reidemeister move of type II in (VI.11), any m-box x can move under the string:

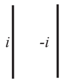


(VI.13)

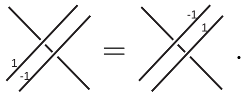
Therefore the strings can be lifted to the three dimensional space acting over the planar para algebra. We call this property of the parafermion planar para algebra $PF_\bullet$ the *half-braided* property.

Definition 6.5. An unshaded planar (para) algebra is called half braided, if there are (zero-graded) 2-boxes  and  , such that Equations (VI.8), (VI.11) hold, and for any m-box x Equation (VI.13) holds.

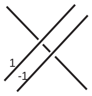
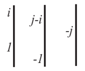
Remark 6.6. The zero graded part of the parafermion planar para algebra $PF_\bullet$ is the group $\mathbb{Z}_N$ subfactor planar algebra $P^{\mathbb{Z}_N}$. It is generated by 2-boxes $\Big\{$  $\Big\}_{i\in\mathbb{Z}_N}$ which form

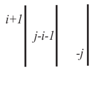
the group $\mathbb{Z}_N$. The bosonic generator  is decomposed as the twisted tensor product of the parafermion c^i and its antiparticle $\Theta(c^i)$. We interpret the decomposition as the parasymmetry of the parafermion planar para algebra $PF_\bullet$. The proof the Yang-Baxter equation takes advantage of the parasymmetry.

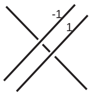
Theorem 6.7. The string moves under the zero-graded planar subalgebra $P^{\mathbb{Z}_N}$ of $PF_\bullet$ as a $\mathbb{Z}_2$ flip:

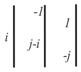


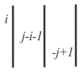
Proof. By Equation VI.6,

 $= \dfrac{\omega^{-1}}{N}\displaystyle\sum_{i,j=0}^{N-1}$ 

$= \dfrac{\omega^{-1}}{N}\displaystyle\sum_{i,j=0}^{N-1} q^{j-i}$  .

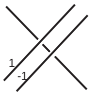


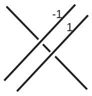
$= \frac{\omega^{-1}}{N}\sum_{i,j=0}^{N-1}$


$= \frac{\omega^{-1}}{N}\sum_{i,j=0}^{N-1} q^{j-i}$


.

Substitute i, j by $i+1, j+1$ in the above equation. Then we have that

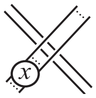


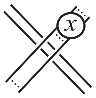
$=$


.

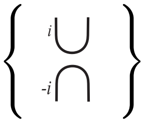
$\square$

Corollary 6.8. Any element x in $P^{\mathbb{Z}_N}$ can move above double strings.



$=$


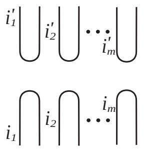
.

Therefore the even, zero-graded part of $PF_\bullet$ can move both above and under double strings, and we recover a well-known modular tensor category. The simple objects are given by projections


$_{i\in\mathbb{Z}_N}$. The morphisms are given by zero-graded elements of $PF_\bullet$. The braids are derived from

. The multiplication and the tensor product are given by the action of corresponding tangles in Section 2.2. Moreover, $PF_\bullet$ turns out to be a module category over the modular tensor category. We refer the readers to [LR95, Xu98, BE98, Ocn00, Ost03] on the general theory of module categories over modular tensor categories.

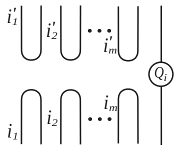
6.6. **Matrix units.** With the help of the pictures, we construct matrix units of para fermion algebras. The matrix units of PF_{2m} are given by

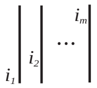
$N^{-\frac{m}{2}}$


,

for $0 \leq i_1, i'_1, i_2, i'_2, \cdots, i_m, i'_m \leq N-1$. Note that PF_1 is the group algebra for $\mathbb{Z}_N$. The N minimal projections of PF_1 are given by $Q_i = \frac{1}{N}\sum_{j=0}^{N-1} q^{ij} c_1^j$, for $0 \leq i \leq N-1$.

The matrix units of PF_{2m+1} are given by

$N^{-\frac{m}{2}}$  ,

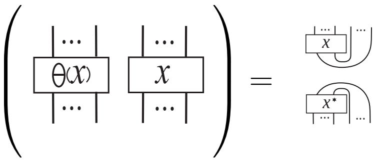
for $0 \leq i, i_1, i_1', i_2, i_2', \cdots, i_m, i_m' \leq N-1$. If we apply the relation III.1, then the matrix units can be expressed in terms of the usual basis  of the parafermion algebra PF_m.

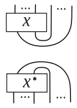
7. Reflection positivity

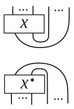
In this section, we will apply the quantum Fourier transform on subfactor planar para algebras to prove the reflection positivity.

7.1. **General case.** Suppose $\mathscr{S}$ is a $(\mathbb{Z}_N, \omega)$ subfactor planar para *-algebra, where $\omega(i,j) = q^{ij}$, $q = e^{\frac{2\pi i}{N}}$. Recall that ζ is a square root of q and $\zeta^{N^2} = 1$. Then $\zeta^{|x|^2}$ is well-defined for any homogenous x. By Proposition 2.17, the map $\Theta(x) = \zeta^{-|x|^2}\rho_\pi(x^*)$ extends anti-linearly to a horizontal reflection on a subfactor planar para algebra.

In Proposition 2.20, we proved that

$\mathfrak{F}^{-m}$  ,

for any homogenous m-box x. Note that the lower half of  is the adjoint of the upper half. Thus

 ≥ 0

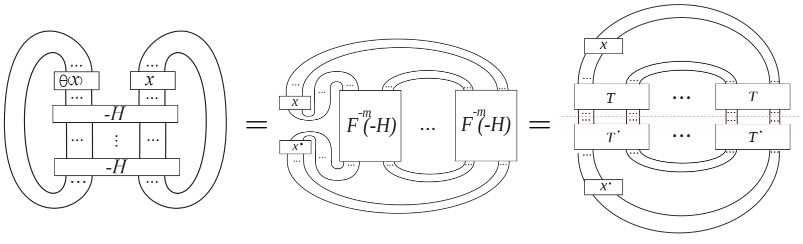
as an operator in the C^* algebra $\mathscr{S}_{2m,\pm}$. Reflection positivity is related to the C^* positivity by the quantum Fourier transform $\mathfrak{F}^{-m}$, the anti-clockwise $\frac{\pi}{2}$ rotation.

Theorem 7.1 (**Reflection Positivity: General Case**)**.** Consider a subfactor planar para algebra $\mathscr{S}$, and a Hamiltonian $H \in \mathscr{S}_{2m,\pm,0}$. Let $\mathfrak{F}^{-m}(-H)$ be a positive operator in $\mathscr{S}_{2m,\pm}$. Then H has reflection positivity on $\mathscr{S}_{m,\pm}$, for all $\beta \geq 0$. That is

$$tr(e^{-\beta H}(\Theta(x) \otimes_t x)) \geq 0,$$

for any homogenous $x \in \mathscr{S}_{m,\pm}$.

Proof. If $\mathfrak{F}^{-m}(-H)$ is positive, then we take its square root $T = T^* = (\mathfrak{F}^{-m}(-H))^{\frac{1}{2}}$. For any homogenous $x \in \mathscr{S}_{m,\pm}$, $\Theta(x) \otimes_t x$ is zero graded. Applying anti-clockwise $\dfrac{\pi}{2}$ rotation, we have



$$\geq 0. \tag{VII.1}$$

The last inequality holds, since the lower half is adjoint of the upper half. Algebraically, for any $k \geq 0$,

$$\delta^{2m} tr((-H)^k \Theta(x) \otimes_t x) \geq 0.$$

For any $\beta > 0$, we have that

$$tr(e^{-\beta H}(\Theta(x) \otimes_t x)) = \sum_{k=0}^{\infty} \beta^k tr((-H)^k \Theta(x) \otimes_t x) \geq 0$$

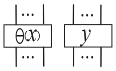
and H has reflection positivity. □

Remark 7.2. The quantum Fourier transform as the anti-clockwise $\frac{\pi}{2}$ rotation changes the trace to vacuum state, the multiplication to the convolution. The positivity of the convolution positive operators is known as the Schur product theorem, proved in [Liua] for subfactor planar algebras. For the parafermion algebra case, the Schur product of $\mathfrak{F}^{-m}(-H)$ corresponds to the Hadamard product of the coupling constant matrix of H.

7.2. **Quantized vectors.** The homogenous condition for x in Theorem 7.1 is not necessary. Let us extend the twisted tensor product for any $\Theta(x)$ and y.

Definition 7.3. Suppose $x = \sum_{i=0}^{N-1} x_i$ and $y = \sum_{i=0}^{N-1} y_i$, and x_i, y_i are graded by i. We define the twisted tensor product

$$\Theta(x) \otimes_t y = \sum_{i,j=0}^{N-1} \zeta^{ij} \Theta(x_i) \otimes_+ y_j.$$

We use 

to denote $\Theta(x) \otimes_t y$.

Proposition 7.4. For x, y in $\mathscr{S}_{m,\pm}$, we have

$$\Theta(\Theta(x) \otimes_t y) = \Theta(y) \otimes_t x \ , \qquad \text{and} \qquad (\Theta(x) \otimes_t y)^* = \Theta(x^*) \otimes_t y^*.$$

Proof. Since Θ is a horizontal reflection, we have that

$$\begin{aligned}\Theta(\Theta(x) \otimes_t y) &= \Theta(\zeta^{|x|_+|y|_+}\Theta(x) \otimes_+ y) \\ &= \zeta^{-|x|_+|y|_+}\Theta(y) \otimes_- x \\ &= \Theta(y) \otimes_t x.\end{aligned}$$

Since $*$ is a vertical reflection, we have that

$$\begin{aligned}(\Theta(x) \otimes_t y)^* &= (\zeta^{|x|_+|y|_+}\Theta(x) \otimes_t y)^* \\ &= \zeta^{-|x|_+|y|_+}\Theta(x^*) \otimes_- y^* \\ &= \Theta(x^*) \otimes_t y^* \ .\end{aligned}$$

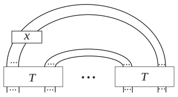
□

For a Hamiltonian $H \in \mathscr{S}_{2m,\pm}$, we define an inner product

$$< x, y >_\Theta = tr(e^{-\beta H}(\Theta(x) \otimes_t y)),$$

for $x, y \in \mathscr{S}_{m,\pm}$. If H has reflection positivity, then $\mathscr{S}_{m,\pm}$ forms a Hilbert space with respect to the inner product $< \cdot, \cdot >_\Theta$, called the quantized space. The image of x in the quantized space is denoted by $\hat{x}$. We give a presentation of the quantized vector $\hat{x}$ in the subfactor planar para algebra $\mathscr{S}$.

Theorem 7.5. Suppose $\mathfrak{F}^{-m}(-H)$ is positive, and T is its square root. We construct the quantized vector

$$\hat{x} := \oplus_{k=0}^{\infty} \frac{\beta^{\frac{k}{2}}}{\delta^k}$$


.

(There are k copies of T in the diagram.) Then

$$\langle x, x \rangle_\Theta = \hat{x}^* \hat{x} \geq 0.$$

Proof. Suppose $x = \displaystyle\sum_{i=0}^{N-1} x_i$ and x_i is graded by i. Then $\langle x_i, x_i \rangle_\Theta = \hat{x}_i{}^* \hat{x}_i$ by Equation VII.1. Since $\langle x_i, x_j \rangle_\Theta$ and $\hat{x}_i{}^* \hat{x}_j$ are graded by $j - i$, we infer that they are zero if $i \neq j$. Therefore

$$< x, x >_\Theta = \sum_{i=0}^{N-1} < x_i, x_i > = \sum_{i=0}^{N-1} \hat{x}_i{}^* \hat{x}_i = \hat{x}^* \hat{x} \geq 0.$$

□

7.3. **Parafermion algebras.** Recall that the basis of PF_m is given by $c_1^{i_1}c_2^{i_2}\cdots c_m^{i_m}$, $0 \leq i_1, i_2 \cdots i_m \leq N-1$.

Let A_+ be the sub algebra of PF_{2m} that consists of $I_m \otimes x$, for $x \in PF_m$. Let A_- be the sub algebra of PF_{2m} that consists of $y \otimes I_m$, for $y \in PF_m$. Then the graded tensor product $A = A_- \hat{\otimes} A_+$ is PF_{2m}.

Note that $\Theta(c) = \zeta\mathfrak{F}^{-1}(c^*) = c^{-1}$. The reflection Θ from $A_\pm \cong PF_m$ to $A_\mp \cong PF_m$ is the anti-linear extension of $\Theta(c^{i_1} \otimes_+ c^{i_2} \cdots \otimes_+ c^{i_m}) = c^{-i_m} \otimes_- \cdots \otimes_- c^{-i_2} \otimes_- c^{-i_1}$. Therefore $A = \theta(A_+)\hat{\otimes}A_+$. We call the graded tensor product A the double algebra of A_+.

Take the Hamiltonian H in PF_m. In terms of the basis C_I, we have

$$-H = \sum_{I,I'} J_I^{I'} \Theta(C_I) \otimes_t C_{I'}$$

for some coupling constants $J_I^{I'}$. The Hamiltonian H is called reflection invariant, if $\Theta(H) = H$, or equivalently $J_I^{I'} = \overline{J_{I'}^{I}}$ for all I, I', or equivalently J is a Hermitian matrix.

Let J_0 be the sub matrix of J, whose coordinates I and I' are both non-empty, i.e., the matrix of coupling constants crossing the reflection plane. The following theorem is formulated and proved in [JJ16b] by a different method. Here we give a diagrammatic interpretation that gives special insight and understanding.

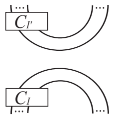
Theorem 7.6 (**Reflection Positivity for Parafermions).** Suppose the Hamiltonian H is reflection invariant and $|H|_+ = 0$. Then H has reflection positivity, i.e.,

$$tr(e^{-\beta H}(\Theta(x) \otimes_t x)) \geq 0,$$

for any $x \in PF_m$, for all $\beta \geq 0$, if and only if $J_0 \geq 0$.

Proof. Take

$$v_I^{I'} = N^{-\frac{m}{2}}$$



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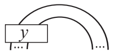
Then $v_I^{I'}$ are matrix units acting on the Hilbert space $V = \{$  $|y \in PF_m\}$. By Proposition 2.20,

$$\mathfrak{F}^{-m}(-H) = N^{\frac{m}{2}} \sum_{I,I'} J_I^{I'} v_I^{I'}. \tag{VII.2}$$

Note that $e^{-\beta(H+rI_{2m})} = e^{-\beta r}e^{-\beta H}$, so the scalar r will not affect the reflection positivity condition of H. Without loss of generality, we assume that $J_\emptyset^\emptyset = 0$.

When $J_0 \geq 0$, for any $s > 0$, take

$$-H(s) = -H + s \sum_{I,I' \neq \emptyset} J_I^{\emptyset} J_{\emptyset}^{I'} \Theta(C_I) \otimes_t C_{I'}) + s^{-1} I_{2m} \ .$$

Since J is Hermitian, we have

$$\begin{aligned}\mathfrak{F}^{-m}(-H(s)) = N^{\frac{m}{2}} \sum_{I \neq \emptyset, I' \neq \emptyset} J_I^{I'} v_I^{I'} + N^{\frac{m}{2}} s^{-1} (v_{\emptyset}^{\emptyset} + s \sum_{I \neq \emptyset} J_I^{\emptyset} v_I^{\emptyset})(v_{\emptyset}^{\emptyset} + s \sum_{I \neq \emptyset} J_I^{\emptyset} v_I^{\emptyset})^* \\ \geq 0\end{aligned}$$

By Theorem 7.5, $H(s)$ has reflection positivity,

$$tr(e^{-\beta H(s)}(\Theta(x) \otimes_t x)) \geq 0,$$

so does $H(s) - s^{-1} I_{2m}$. Take $s \to 0$. This shows that H has reflection positivity,

$$tr(e^{-\beta H}(\Theta(x) \otimes_t x)) \geq 0 \ .$$

On the other hand, if H has reflection positivity for all $\beta \geq 0$, then for any homogenous x in PF_m orthogonal to I_m, we have

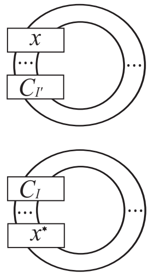
$$tr(e^{-\beta H}(\Theta(x) \otimes_t x)) \geq 0 \ ,$$

and the equality holds when $\beta = 0$. Take the first derivative with respect to β. Then we have

$$tr(-H(\Theta(x) \otimes_t x)) \geq 0 \ . \tag{VII.3}$$

Apply the anti-clockwise $\frac{\pi}{2}$ rotation to Equation VII.3, and use Equation VII.2. This shows that we have

$$\sum_{I,I'} J_I^{I'}$$



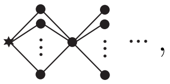
$$\geq 0 \ ,$$

for any m-box x orthogonal to I_m. Therefore the matrix J_0 as the restriction of J on the subspace $V \setminus \mathbb{C}\{$  $\}$ is positive. □

8. Positivity for the General Circle Parameter

We constructed the planar para algebra $\mathscr{P}_\bullet$ over the field $\mathbb{C}(\delta)$ in Section 2.6. The m-box space has a sub algebra generated by labelled tangles with only vertical strings which is isomorphic to the parafermion algebra PF_m.

Similar to the Temperley-Lieb-Jones planar algebra case, we can construct matrix units of $\mathscr{P}_m$ over the field $\mathbb{C}(\delta)$ inductively by the matrix units of parafermion algebras constructed in Section 6, the basic construction and the general Wenzl's formula [Wen87, Liub]. If a labelled tangle is not in the basic construction ideal, then it is in the parafermion algebra. Therefore, the principal graph of the planar para algebra $\mathscr{P}_\bullet$ is the same as the Bratteli diagram of parafermion algebras, i.e.,



assuming the quantum dimensions of vertices in the principal graph are non-zero. This assumption can be avoided by the bi-induction argument in [Liub]. Moreover, we obtain the formula of the quantum dimensions of these vertices. There is one depth $2m$ vertex. Its quantum dimension is $\sqrt{N}[2m]$. There are N depth $2m+1$ vertices. Any of them has quantum dimension $\frac{\sqrt{N}}{N}[2m+1]$. Here $[m]$ is the quantum number $\frac{q^m-q^{-m}}{q-q^{-1}}$, and $\delta=\sqrt{N}[2]$.

Jones' remarkable rigidity theorem [Jon83] says that all possible values of the circle parameter of a subfactor planar algebra are given by

$$\{2\cos\frac{\pi}{n}|n=3,4,\cdots\}\cup[2,\infty).$$

These values are realized by Temperley-Lieb-Jones subfactor planar algebras.

To obtain the positivity for the planar para algebra $\mathscr{P}_\bullet$, δ has to be positive. In this case, we can define the (unique) vertical reflection $*$ on the planar algebra induced by $c^*=c^{-1}$.

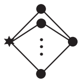
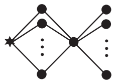
Theorem 8.1. The planar para algebra $\mathscr{P}_\bullet$ has positivity if and only if $\frac{\delta}{\sqrt{N}}$ is in

$$\{2\cos\frac{\pi}{k}|k=3,4,\cdots\}\cup[2,\infty).$$

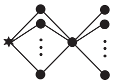
Proof. The matrix units of $\mathscr{P}_m$ are constructed over the field $\mathbb{C}(\delta)$. When δ is a scalar, the matrix units of $\mathscr{P}_m$ are well-defined by Wenzl's formula, if the Markov trace is non-degenerated on $\mathscr{P}_{m-1}$.

If $2\cos\frac{\pi}{k-1}<\delta<2\cos\frac{\pi}{k}$, then $[i]>0$ for all $i<k$. Thus the matrix units of $\mathscr{P}_k$ are still well-defined. Since $[k]<0$, the positivity fails.

If $\delta=2\cos\frac{\pi}{k}$, then $[i]>0$ for all $i<k$. Thus the matrix units of $\mathscr{P}_k$ are still well-defined. Since $[k]=0$, any minimal idempotent orthogonal to the basic construction ideal has trace 0. Thus it is in the kernel of the partition function. Therefore, $\mathscr{P}_\bullet$ modulo the kernel of the partition function is a depth $k-1$ subfactor planar para algebra. It has the following principal graph for $k=3,4,5,\cdots$

 ;  ;  ; ⋯

If $\delta \geq 2$, then $[i] > 0$ for all i. Thus the matrix units of $\mathscr{P}_\bullet$ are still well-defined. Moreover, $\mathscr{P}_\bullet$ is a subfactor planar para algebra with the following principal graph

 ⋯ .

□

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Appendix A. The Construction of Planar Para Algebras for Parafermions

Since the generators are 1-boxes, any labelled 0-tangle is a disjoint union of closed strings labelled by generators. For an innermost closed string, we can move all its labels toward on point by isotopy. Then we can reduce the labelled closed string to a scalar by the relations given by the multiplication and the trace. Note that only the zero graded part on the closed string is evaluated as a non-zero scalar, since the trace is graded. The para isotopy and the 2π rotation reduce to the usual isotopy of planar algebras on zero-graded part. Thus the evaluation of different labeled closed strings are independent modulo para isotopy. Essentially we only need the consistency condition on a single labelled closed string which indicates the associativity of the multiplication and the tracial condition of the expectation.

The above argument can be formalized by the method in Section 5 in [Liub] which was motivated by the work of Kauffman [Kau90]. The idea is first constructing the planar algebras generated by the generators without relations, namely the *universal planar algebra*. Then one can define a partition function on the universal planar para algebra as the average of complexity reducing evaluations and prove that the relations are in the kernel of the partition function.

Proof of Theorem 2.6. For the group $\mathbb{Z}_N$ and a bicharacter $\omega(i,j) = q^{ij}$, $q = e^{\frac{2\pi i}{N}}$, first let us construct a shaded $(\mathbb{Z}_N, \omega)$ planar para algebra generated by the 1-box c with grading 1 and relations $c^N = 1$, 

$= 0$, for $1 \leq k \leq N-1$. The para isotopy and the 2π rotation for the generator c can also be viewed as relations of c.

Let $\mathscr{U}$ be the $(\mathbb{Z}_N, \omega)$ universal planar para algebra generated by c. Let us define the partition function Z inductively by the number of labelled circles of labelled 0-tangles as follows.

The partition function of the empty diagram is 1. We assume that the partition function for diagrams with at most $n-1$ labelled circles is defined. Let us define the partition function of a labelled 0-tangle T with n labelled circles. Let IC be the set of innermost (labelled) circles of T. Take one circle L in IC, let us define $Z(T,L)$.

If L has no label c, then $Z(T,L) := \delta Z(T \setminus L)$. If the number of labels of L is not divisible by N, then $Z(T,L) := 0$. If L has Nk labels, we count the labels in L anti-clockwise starting from the top label c, denoted by c_i, $0 \leq i \leq Nk-1$. Let us move c_i clockwise to c_0 one by one by RT isotopy and para isotopy. While applying the para isotopy to c_i and another label, we obtain a scalar q or q^{-1} each time. While moving c_i to c_0, if c_i is rotated clockwise by $2k_i\pi$, then we obtain a scalar q^{k_i}. Let q^L be the multiplication of all these scalars. Then $Z(T,L) := q^L \delta Z(T \setminus L)$.

Let us define

$$Z(T) = \frac{1}{|IC|} \sum_{L \in IC} Z(T,L).$$

By an inductive argument and the fact that $q^N = 1$, it is easy to check that $Z(T)$ is well-defined on the universal planar para algebra. The most complex case is to show the $Z(T,L)$ is well-defined while applying the para isotopy to c_0 and c_1. Under this isotopy, the top label becomes c_1. In this case, we need to move c_0 clockwise along L. We obtain $Nk-1$ scalars q from the para isotopy, and one scalar q from the 2π rotation of c_0. Their multiplication is 1. So $Z(T,L)$ does not change.

Moreover, it is easy to check that all the relations are in the kernal of the partition function Z. Therefore the relations are consistent. The identity is the only 0 graded 1-box, so $\mathscr{P}/\mathscr{I}$ is a spherical planar para algebra.

Take ζ to be a square root of q such that $\zeta^{N^2} = 1$. Note that $\zeta\mathfrak{F}(c)$ satisfies the relations as c. Therefore, we can lift the shading of $\mathscr{P}/\mathscr{I}$ and by introducing the relation $\mathfrak{F}(c) = q^{\frac{1}{2}}c$. Then $\mathscr{P}/\mathscr{I}$ is an unshaded planar algebra. □

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