Introduction to Quantum Field Theory

Arthur Jaffe
Harvard University
Cambridge, MA 02138, USA
©by Arthur Jaffe. Reproduction only with permission of the author.

24 May, 2005 at 7:26

Contents

Part I

Life of a Single Particle

Chapter 1

Introduction

Our goal is to present a brief and self-contained introduction to quantum field theory from the constructive point of view. We try to motivate some basic results and relate them to interesting open problems.

One should mention right at the start that one still does not understand whether quantum mechanics and special relativity are compatible at a fundamental level in our Minkowski four-space world. One generally assumes that this means finding a complete Yang-Mills gauge theory or the interaction of gauge fields with fermionic matter fields, the simplest form being quantum chromodynamics (QCD). Associated with this picture is the belief that the fundamental vector meson excitations are massive (as opposed to photons, which arise in the limiting case of an abelian gauge symmetry. The proof of the existence of a "mass gap" appears a necessary integral part of solving the entire puzzle.

This question remains one of the deepest open issues in theoretical physics, as well as in mathematics. Basically the question remains: can one give a mathematical foundation to the theory of fields in four-dimensions? In other words, can do quantum mechanics and special relativity lie on the same footing as the classical physics of Newton, Maxwell, Einstein, or Schrödinger—all of which fits into a mathematical framework that we describe as the language of physics. This glaring gap in our fundamental knowledge even dwarfs questions of whether there are other more complicated and sophisticated approaches to physics—those that incorporate gravity, strings, or branes—for understanding their fundamental significance lies far in the future. In fact, one believes that stringy proposals, if they can be fully implemented, have limiting cases that appear as relativistic quantum fields, just as relativistic quantum fields describe non-relativistic quantum theory and classical physics in various limiting cases.

We begin with the quantum mechanical treatment of a particle of a given mass. If we assume that the symmetry of the quantum theory includes the transformations of special relativity, then much of the structure follows naturally. We then develop the basic Euclidean point of view, that arises from attempting to analytically continue Lorentz symmetry to Euclidean symmetry. This provides also the natural connection with path integrals. We specialize the case of a single, free, bosonic particle; this illustrates many of the main ideas.

Each method gives a route to quantization. In the path integral framework we encounter classical fields defined on Euclidean space (with a positive metric and Euclidean symmetry). One encounters a condition known as reflection (or Osterwalder-Schrader) positivity that allows one obtain a quantum theory (on Hilbert space) from a path integral. The quantum theory that one finds agrees with the usual picture of canonical quantization that one learns in standard field theory. The quantum theory also comes with a representation of the inhomogeneous Lorentz group (the Poincaré group) that arises from an analytic continuation of the quantization of the Euclidean group.

Thus the two fundamental points of view mesh to one. We first investigate a special case that relates to the Gaussian path integral and the free quantum field. We then give the general construction that applies for bosonic non-linear fields.

Chapter 2

Life of a Particle in Real Time

We introduce quantum theory for a single, spinless particle of mass $m > 0$. We assume that the particle moves in Euclidean space with coordinates $\vec{x}$ and of dimension $s = d - 1$. The usual case is $s = 3$, but for until we encounter interactions we also allow for arbitrary integer values of s.

2.1 Quantum Theory

The quantum state of a particle is described by a wave function $\mathfrak{f}$. We deal concretely with some concepts that appear in more abstract form in later chapters. A particle follows the usual rules of quantum theory:

- The wave function of a quantum system is a vector $\mathfrak{f}$ in a Hilbert space $\mathcal{H}$, comprising possible wave functions.
- Quantum mechanical observables (such as the energy H or the momentum $\vec{P}$) are self-adjoint linear transformations on $\mathcal{H}$.
- The value of an observable T in the state $\mathfrak{f}$ is its expectation $\langle \mathfrak{f}, T\mathfrak{f}\rangle_{\mathcal{H}}$.
- A group $\mathfrak{G}$ of physical symmetries is described by a unitary representation $U(\mathfrak{G})$ of $\mathfrak{G}$ on $\mathcal{H}$.
- The self-adjoint generator of a one-parameter subgroup of symmetries $\mathfrak{G}$ is identified with a specific physical observable.

There are two alternative ways in which one views the action of a symmetry group $\mathfrak{G}$.

HP. In the *Heisenberg picture,* one considers that the symmetry acts on the observables. So an observable T transforms under a symmetry element $g \in \mathfrak{G}$ as

$$T \to T^g = U(g)TU(g)^* \ . \tag{2.1}$$

The value of the transformed observable in the state $\mathfrak{f}$ is given by the expectation

$$\langle \mathfrak{f},\, T^g\, \mathfrak{f}\rangle_{\mathcal{H}} = \langle \mathfrak{f},\, U(g)TU(g)^*\, \mathfrak{f}\rangle_{\mathcal{H}} \ . \tag{2.2}$$

SP. Alternatively, in the *Schrödinger picture*, one considers that the symmetry acts on the states. In this case the state vector $\mathfrak{f}$ transforms under the symmetry according to the anti-representation

$$\mathfrak{f} \to \mathfrak{f}^g = U(g)^* \mathfrak{f} \;, \tag{2.3}$$

for which $U(g_1)^* U(g_2)^* = U(g_2 g_1)^*$. The value of the observable T in the transformed state $\mathfrak{f}^g$ also equals (2.2).

The one-parameter group of time-translations defines the dynamics of quantum theory, and this group has a special significance in quantum theory. The time-translation group $U(t) = e^{itH}$ is generated by the energy observable H (also called the Hamiltonian). Its action in the Schrödinger picture is

$$\mathfrak{f} \to \mathfrak{f}^t = U(t)^* \mathfrak{f} = e^{-itH} \mathfrak{f} \;, \tag{2.4}$$

and this gives the solution to the Schrödinger equation.

$$i\hbar \frac{\partial \mathfrak{f}^t}{\partial t} = H \mathfrak{f}^t \;, \qquad \text{with initial data } \mathfrak{f}^0 = \mathfrak{f} \;, \tag{2.5}$$

in units where Planck's constant $\hbar = 1$. Therefore one often calls e^{-itH} the Schrödinger group.

2.2 Poincaré Symmetry

We are concerned here with quantum theory that is compatible with special relativity. So we expect that the symmetry group of relativity has a unitary representation on $\mathcal{H}$. This group of symmetries is sometimes called the Poincaré group. An element of the Poincaré group $\{\Lambda, a\}$ comprises both a Lorentz transformation Λ and a translation a of Minkowski space, which we now define.

The coordinates of d-dimensional Minkowski space-time $\mathbb{M}^d$ are $x = (\vec{x}, t)$. The metric g in Minkowski space is a $d \times d$ diagonal matrix with entries $g^{\mu\nu}$, and with eigenvalues $\{-1, -1, -1, \ldots, -1, 1\}$. The Minkowski square of x is[1]

$$x_{\mathbb{M}}^2 = \sum_{\mu\nu} x_\mu g^{\mu\nu} x_\nu = t^2 - \vec{x}^2 \;. \tag{2.6}$$

Time-like vectors have positive squares, space-like vectors have negative squares, and light-like vectors have square zero.

Lorentz transformations act linearly as $x \to \Lambda x$, and they are specified by real $d \times d$ matrices Λ, chosen to preserve the Minkowski square of x. Thus the Lorentz matrices satisfy

$$\Lambda^T g \Lambda = g \;, \tag{2.7}$$

[1] For simplicity of notation we generally will suppress the subscript $\mathbb{M}$ in denoting the square of the Minkowski length. In this chapter all squares or inner products of Minkowski-space vectors will be assumed to be Minkowski scalar products. On the other hand, inner products of spatial components $\vec{x}$ of vectors will be assumed to have Euclidean (positive) signature.

where Λ^T denotes the transpose of the matrix Λ. This condition is equivalent to the preservation of the Minkowski squared length. For one can write in matrix notation $x^2 = x^T g x$, so

$$(\Lambda x)^2 = x^T \Lambda^T g \Lambda x = x^T g x = x^2 \ . \tag{2.8}$$

Translations in the Poincaré group act in an affine manner, $x \to x + a$. One defines the Poincaré transformation $\{\Lambda, a\}$ to act as

$$\{\Lambda, a\}\, x = \Lambda x + a \ . \tag{2.9}$$

The multiplication law for the Poincaré group (2.9) is

$$\{\Lambda_1, a_1\}\{\Lambda_2, a_2\} = \{\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1\} \ , \tag{2.10}$$

and in particular,

$$\{\Lambda, a\} = \{I, a\}\{\Lambda, 0\} \ . \tag{2.11}$$

The inverse $\{\Lambda, a\}^{-1} = \{\Lambda^{-1}, -\Lambda^{-1} a\}$ acts on Minkowski space as

$$\{\Lambda, a\}^{-1}\, x = \Lambda^{-1}(x - a) \ . \tag{2.12}$$

In the case $\Lambda = I$, this multiplication law ensures that all space-time translations commute, for

$$U(I, a_1) U(I, a_2) = U(I, a_2 + a_1) = U(I, a_2) U(I, a_1) \ . \tag{2.13}$$

The basic physical assumption of relativistic quantum theory is that one can identify the self-adjoint generators of the one-parameter subgroups of the Poincaré group with the following physical observables.

- One identifies the d generators of the space-time translations $U(I, a)$ with the components of the momentum vector $\vec{P}$ and the energy H,

  $$U(I, a) = e^{i a_d H - i \vec{a} \cdot \vec{P}} = e^{i a \cdot P} \ , \tag{2.14}$$

  where $P = (\vec{P}, H)$ denotes the momentum-energy vector.

- Likewise one identifies the $(d-1)(d-2)/2$ infinitesimal generators L_{ij} of rotations in the planes $x_i\, x_j$ as angular momentum. One identifies the $(d-1)$ self-adjoint generators M_i generating hyperbolic rotations in the planes $x_i\, x_d$ as Lorentz boosts.

The commutativity of the space-time translation subgroup means that the components P_μ of the momentum-energy vector are mutually commuting operators,

$$[P_\mu, P_\nu] = 0 \ , \qquad \text{for all } \ 1 \le \mu, \nu \le d \ . \tag{2.15}$$

2.3 Stability

In quantum theory, one also generally assumes that there is a state of lowest energy—the vacuum state. Without this assumption the world would be unstable and under a perturbation it could collapse. Thus a fundamental assumption of quantum theory is that one can add a constant to the Hamiltonian to make it positive. One generally writes,

$$0 \leq H \ , \tag{2.16}$$

although in certain circumstances the absolute zero of energy can play a role.

2.4 Special Features of a Single Particle

Furthermore in special relativity, a particle of mass m satisfies the energy-momentum relation,

$$P^2 = H^2 - \vec{P}^2 = m^2 \ . \tag{2.17}$$

Equivalently, if we can write

$$H = \left(\vec{P}^2 + m^2\right)^{1/2} \ , \tag{2.18}$$

so this H must be defined as the positive square root. Also $m^2 \geq 0$, so we can define the mass operator M as the positive square root of P^2,

$$M = \left(P^2\right)^{1/2} = \left(H^2 - \vec{P}^2\right)^{1/2} \ . \tag{2.19}$$

The mass operator commutes with the entire representation $U(\Lambda, a)$,

$$U(\Lambda, a)M = MU(\Lambda, a) \ . \tag{2.20}$$

The spectrum of the mass operator M labels the hyperboloids in the spectrum of the representation $U(\Lambda, a)$, and the group maps each hyperboloid into itself. If $\mathcal{H}$ is the space of quantum-mechanical states for a single particle of mass m, then we require that every vector in $\mathcal{H}$ be an eigenvector of the mass operator M with eigenvalue m,

$$M\,\mathfrak{f} = m\,\mathfrak{f} \ . \tag{2.21}$$

2.5 The Configuration Space Representation

One obtains further structure by assuming a particular representation of the wave functions $\mathfrak{f}$ as functions $\mathfrak{f}(\vec{x})$ on configuration space $\vec{x} \in \mathbb{R}^s$. Here $s = d-1$ denotes that we take an s-dimensional time slice of $\mathbb{M}^d$. According to the picture above, the symmetries of quantum theory (including Poincaré space-time symmetry) act on the Hilbert space of functions $\mathfrak{f}$ defined on a time slice.

According to the description of symmetries in (2.4), the group $\mathfrak{G}$ acts on state vectors in the Schrödinger picture according to an anti-representation. If a symmetry $g \in \mathfrak{G}$ acts on $\mathbb{R}^s$ by

$$\vec{x} \to g\vec{x} \ , \tag{2.22}$$

then the natural anti-representation is

$$\left(U(g)^* \mathfrak{f}\right)(\vec{x}) = \mathfrak{f}(g\vec{x}) \ , \tag{2.23}$$

for which

$$\left(U(g_1)^* U(g_2)^* \mathfrak{f}\right)(\vec{x}) = \left(U(g_2)^* \mathfrak{f}\right)(g_1\vec{x}) = \mathfrak{f}(g_2 g_1 \vec{x}) = \left(U(g_2 g_1)^* \mathfrak{f}\right)(\vec{x}) \ . \tag{2.24}$$

In case $\mathfrak{G}$ is also measure-preserving in $\mathcal{H}$, then $U(g)$ defined in this way is also unitary.

2.5.1 The Momentum and Energy Operators

Consider the spatial translation subgroup $T_{\vec{a}} = \{I, (\vec{a}, 0)\}$ of the Poincaré group, which acts on $\mathbb{R}^s$ by

$$T_{\vec{a}}\vec{x} = \vec{x} + \vec{a} \ . \tag{2.25}$$

Then

$$\left(U(T_{\vec{a}})^* \mathfrak{f}\right)(\vec{x}) = \mathfrak{f}(\vec{x} + \vec{a}) \ . \tag{2.26}$$

But according to the rules of quantum theory, the group $U(T_{\vec{a}})$ is the same as the subgroup $e^{-i\vec{a}\cdot\vec{P}}$ in (2.14) generated by the momentum. Therefore,

$$\left(U(T_{\vec{a}})^* \mathfrak{f}\right)(\vec{x}) = \left(e^{i\vec{a}\cdot\vec{P}} \mathfrak{f}\right)(\vec{x}) = \mathfrak{f}(\vec{x} + \vec{a}) \ . \tag{2.27}$$

Therefore one infers that the momentum operator $\vec{P}$ in configuration space has the usual representation in quantum theory for wave functions defined on configuration-space,

$$\boxed{\vec{P} = -i\vec{\nabla}_x} \ , \tag{2.28}$$

where $\vec{\nabla}_x$ denotes the gradient. We conclude further that in this representation, the one-particle Hamiltonian H defined in (2.18) has the form

$$\boxed{H = \left(-\vec{\nabla}_x^2 + m^2\right)^{1/2}} \ . \tag{2.29}$$

Thus the solution to the Schrödinger equation

$$\mathfrak{f}^t(\vec{x}) = \left(e^{-itH} \mathfrak{f}\right)(\vec{x}) \ , \qquad \text{with } \mathfrak{f}^0 = \mathfrak{f} \ , \tag{2.30}$$

introduced in (2.5), also satisfies the second-order wave equation called the Klein-Gordon equation. Denoting the wave operator by

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 \ , \tag{2.31}$$

the Klein-Gordon equation for mass m is the equation

$$\left(\Box + m^2\right) \mathfrak{f}^{\,t}(\vec{x}) = 0 \ . \tag{2.32}$$

Note that as the square root of a differential operator, this one-particle Hamiltonian operator H is non-local. The solution (2.30) to the Klein-Gordon equation spreads instantaneously over all all of space, $\mathbb{R}^s$. This fact can be illustrated by the (non-normalizable) configuration-space wave initial value $\mathfrak{f} = \delta$ for the equation, which illustrates the basic point. One can compute the solution

$$\mathfrak{f}^{\,t}(\vec{x}) = \left(e^{-itH}\mathfrak{f}\right)(\vec{x}) \ , \qquad \text{with } \ \mathfrak{f}^{\,0}(\vec{x}) = \delta(\vec{x}) \ , \tag{2.33}$$

in closed form. For example in case $s = 3$ and $|\vec{x}| = r > t \geq 0$, one finds

$$\mathfrak{f}^{\,t}(\vec{x}) = \frac{i}{2\pi^2 r} \int_m^\infty \zeta e^{-\zeta r} \sinh\left(t\sqrt{\zeta^2 - m^2}\right) d\zeta \ , \tag{2.34}$$

which is nonzero.

Exercise 2.5.1. *Solutions to the Klein-Gordon wave equation propagate with finite speed. But* $\mathfrak{f}^{\,t}(\vec{x})$ *instantly spreads from its localization at the origin (at* $t = 0$*) to all space (for any* $t > 0$*), as for* $s = 3$ *in* (2.34)*. Does this fact not contradict the laws of special relativity that influence cannot propagate faster than the speed of light?*

There is an interesting scalar product defined on solutions to the Schrödinger equation, or on *positive energy* solutions to the Klein-Gordon equation of the form (2.30). Consider

$$\langle \mathfrak{f}, 2H\mathfrak{g}\rangle_{L^2(\mathbb{R}^s)} = i\left(\left\langle \mathfrak{f}^{\,t}, \frac{\partial}{\partial t}\mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)} - \left\langle \frac{\partial}{\partial t}\mathfrak{f}^{\,t}, \mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)}\right) \ . \tag{2.35}$$

As a consequence of the Schrödinger equation, the right side of (2.35) equals $\langle \mathfrak{f}^{\,t}, 2H\mathfrak{g}^{\,t}\rangle_{L^2(\mathbb{R}^s)}$. Furthermore the same equation shows that the this expression does not depend on t, as its time derivative is

$$\begin{aligned} \frac{\partial}{\partial t}\left\langle \mathfrak{f}^{\,t}, 2H\mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)} &= i\left(\left\langle \mathfrak{f}^{\,t}, \frac{\partial^2}{\partial t^2}\mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)} - \left\langle \frac{\partial^2}{\partial t^2}\mathfrak{f}^{\,t}, \mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)}\right) \\ &= -i\left(\left\langle \mathfrak{f}^{\,t}, H^2\mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)} - \left\langle H^2\mathfrak{f}^{\,t}, \mathfrak{g}^{\,t}\right\rangle_{L^2(\mathbb{R}^s)}\right) = 0 \ . \end{aligned} \tag{2.36}$$

In the final step, we use the fact that H is self adjoint on $L^2(\mathbb{R}^s)$. Thus one can evaluate $\langle \mathfrak{f}^{\,t}, 2H\mathfrak{g}^{\,t}\rangle_{L^2(\mathbb{R}^s)}$ at $t = 0$, and as $m \leq H$, one infers that the expectation $\langle \mathfrak{f}, 2H\mathfrak{f}\rangle_{L^2(\mathbb{R}^s)}$ defines an inner product on solutions of the Schrödinger equation. (Note that the right side of (2.35) is negative when $\mathfrak{f}^{\,t} = \mathfrak{g}^{\,t}$ is a negative-energy solution to the Klein-Gordon equation.)

2.6 The Momentum Space Representation

The momentum representation is defined by the Fourier transformation of the configuration space representation. It has the feature that the momentum operator $\vec{P}$ acts as multiplication by a coordinate $\vec{p}$. Define the Fourier transform as

$$(\mathfrak{F}\mathfrak{f})(\vec{p}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \mathfrak{f}(\vec{x}) e^{-i\vec{p}\vec{x}} d\vec{x} \,. \tag{2.37}$$

Sometimes it is less cumbersome to write[2]

$$\widetilde{\mathfrak{f}}(p) = (\mathfrak{F}\mathfrak{f})(p) \,. \tag{2.38}$$

Clearly Fourier transformation $\mathfrak{F}$ is a linear transformation, for if the Fourier transform of $\mathfrak{f}$ and $\mathfrak{g}$ exist, then both $\mathfrak{F}(\mathfrak{f}+\mathfrak{g}) = \mathfrak{F}\mathfrak{f} + \mathfrak{F}\mathfrak{g}$, and $\mathfrak{F}\lambda\mathfrak{f} = \lambda\mathfrak{F}\mathfrak{f}$ for any $\lambda \in \mathbb{C}$.

We also claim that $\mathfrak{F}$ is a unitary transformation on the Hilbert space $L^2(\mathbb{R}^s)$, namely every $L^2(\mathbb{R}^s)$ function has a Fourier transform and

$$\mathfrak{F}\mathfrak{F}^* = \mathfrak{F}^*\mathfrak{F} = I \,. \tag{2.39}$$

The outline of the argument is that Plancherel's formula states that $\mathfrak{F}$ preserves $L^2(\mathbb{R}^s)$ inner products,

$$\left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}} \right\rangle_{L^2(\mathbb{R}^s)} = \langle \mathfrak{f}, \mathfrak{g} \rangle_{L^2(\mathbb{R}^s)} \,, \qquad \text{for all } \mathfrak{f}, \mathfrak{g} \in L^2(\mathbb{R}^s) \,. \tag{2.40}$$

Furthermore the Fourier inversion theorem says that $\mathfrak{F}$ is invertible, and hence it is unitary. Actually the simplest way to show that $\mathfrak{F}$ is unitary on L^2 is to exhibit an orthogonal basis of eigenfunctions for $\mathfrak{F}$.[3] The forumla for the inverse of $\mathfrak{F}$ is the Fourier inversion formula

$$\mathfrak{f}(\vec{x}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \widetilde{\mathfrak{f}}(\vec{p}) e^{i\vec{p}\vec{x}} d\vec{p} \,, \tag{2.41}$$

and an expression of its validity for all square-integrable functions.

The Fourier representation is called the momentum representation. In fact this is natural because we saw in (2.28) that the quantum-mechanical momentum operator $\vec{P}$ the form $\vec{P} = -i\vec{\nabla}_x$. Thus in the Fourier representation the momentum operator $\vec{P}$ acts as multiplication by the coordinate $\vec{p}$. In particular,

$$\vec{P}\,\mathfrak{f}(\vec{x}) = -i\nabla_x \mathfrak{f}(\vec{x}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \vec{p}\,\widetilde{\mathfrak{f}}(\vec{p}) e^{i\vec{p}\vec{x}} d\vec{p} \,. \tag{2.42}$$

[2]One must be careful with this notation in our context. One must guard against confusing $\widetilde{\mathfrak{f}}$ with our $\hat{\mathfrak{f}}$, that we introduce in a later chapter to denote something very different—the quantization of $\mathfrak{f}$.

[3]In fact, the Gaussian function $\Omega_0 = \pi^{-s/4} \exp\left(-\vec{x}^2/2\right)$ is an eigenvector of $\mathfrak{F}$ with eigenvalue 1. The Hermite functions, given by products of polynomials in each coordinate times Ω_0 complete an orthogonal basis of eigenfunctions, and each has an eigenvalue of either ± 1 or $\pm i$. Thus the proof that $\mathfrak{F}$ is unitary is equivalent to the proof that the Hermite functions are a basis for L^2. See Appendix Appendix **??**, for a complete proof.

Write the self adjoint operators for each component of the momentum are multiplication by the coordinate of $\vec{p}$, or simply $\vec{P} = \vec{p}$. In other words, for all $\widetilde{\mathfrak{f}} \in \widetilde{\mathcal{H}}$,

$$\vec{P}\,\widetilde{\mathfrak{f}}(\vec{p}) = \vec{p}\,\widetilde{\mathfrak{f}}(\vec{p}) \ . \tag{2.43}$$

Likewise, the Hamiltonian H acts in the momentum space representation as the operator $\widetilde{H}$ of multiplication by the function $\omega(\vec{p})$, defined as the positive square root

$$\widetilde{H} = \omega(\vec{p}) = \left(\vec{p}^2 + m^2\right)^{1/2} \ . \tag{2.44}$$

Hence

$$\left(e^{-itH}\mathfrak{f}\right)(\vec{x}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{-it\omega(\vec{p})}\,\widetilde{\mathfrak{f}}(\vec{p}) e^{i\vec{p}\vec{x}} d\vec{p}\,, \tag{2.45}$$

or

$$\left(e^{-it\widetilde{H}}\,\widetilde{\mathfrak{f}}\right)(\vec{p}) = e^{-it\omega(\vec{p})}\,\widetilde{\mathfrak{f}}(\vec{p}) \ . \tag{2.46}$$

2.7 The Lorentz-Invariant Scalar Product

One can now arrive easily at the correct scalar product by considering the momentum representation of the wave functions $\mathfrak{f}$. The wave functions $\widetilde{\mathfrak{f}}(\vec{p})$ only depend on the spatial components of the momenta $\vec{p}$. Let us suppose that the Hilbert space norm can be defined by an integral over all d components of p. Then one obtains a Lorentz-invariant measure on the mass-m hyperboloid $p^2 = p_d^2 - \vec{p}^2 = m^2$ as follows: restrict the Lorentz-invariant Lebesgue measure $dp = d\vec{p}\,dp_d$ to the mass-m hyperboloid by multiplying it with the Lorentz invariant Dirac measure $\delta(p^2 - m^2)$. Furthermore, one wants positive energies, so also restrict the integral to the positive-p_d. Since the wave functions do not depend on p_d, the p_d integral can be done separately, namely

$$\int_{p_d>0} \delta(p^2 - m^2) dp_d = \int_{p_d>0} \delta((p_d - \omega(\vec{p}))(p_d + \omega(\vec{p})))\, dp_d = \frac{1}{2\omega(\vec{p})} \ . \tag{2.47}$$

Thus we obtain a natural Lorentz-invariant scalar product by multiplying this density with $\overline{\widetilde{\mathfrak{f}}(\vec{p})}\,\widetilde{\mathfrak{g}}(\vec{p})$ and integrating over $d\vec{p}$. Let

$$\left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}} \right\rangle_{\widetilde{\mathcal{H}}} = \int_{\mathbb{R}^{d-1}} \overline{\widetilde{\mathfrak{f}}(\vec{p})}\,\widetilde{\mathfrak{g}}(\vec{p})\, \frac{d\vec{p}}{2\omega(\vec{p})} \ . \tag{2.48}$$

This defines a Hilbert space $\widetilde{\mathcal{H}}$ of functions $\widetilde{\mathfrak{f}}(\vec{p})$ such that

$$\int \left|\widetilde{\mathfrak{f}}(\vec{p})\right|^2 \frac{d\vec{p}}{2\omega(\vec{p})} < \infty \ . \tag{2.49}$$

The operator of multiplication by $(2\omega(\vec{p}))^{-1}$ in Fourier space can also be expressed in configuration space. It is given by the operator $(2H)^{-1}$, with H the non-local (pseudo-differential) operator

(2.29). This is the special Hamiltonian for a single free particle of mass m, so we also denote the one-particle Hamiltonian H by

$$H = \omega = \left(-\vec{\nabla}_x^2 + m^2\right)^{1/2} . \tag{2.50}$$

By relating both spaces to $L^2(\mathbb{R}^{d-1})$, one can write the configuration-space Hilbert space $\mathcal{H}$ in terms of the momentum representation $\widetilde{\mathcal{H}}$. In particular, regard $\mathfrak{F}$ as a map from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, and $\mathfrak{F}^*$ as the backwards map from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$. Then define $\mathcal{H}$ with the inner product

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathcal{H}} = \left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}} \right\rangle_{\widetilde{\mathcal{H}}} . \tag{2.51}$$

Define the operator $G = (2\omega)^{-1}$ on $L^2(\mathbb{R}^s)$ with integral kernel $G(\vec{x} - \vec{y})$. It is given by

$$(G\mathfrak{f})\,(\vec{x}) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^s} (2\omega(\vec{p}))^{-1}\, \widetilde{\mathfrak{f}}(\vec{p})\; e^{i\vec{p}\cdot\vec{x}}\, d\vec{p} = \int_{\mathbb{R}^s} G(\vec{x}-\vec{y})\mathfrak{f}(\vec{y}) d\vec{y} , \tag{2.52}$$

where $G(\vec{x} - \vec{y})$ is the generalized function

$$G(\vec{x}-\vec{y}) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^s} \frac{1}{2\omega(\vec{p})} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} d\vec{p} . \tag{2.53}$$

The Hilbert space of generalized functions with the inner product (2.51) occurs frequently in analysis and is known as the Sobolev space $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^s)$. In summary,

$$\boxed{\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathcal{H}} = \langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^s)} = \left\langle \mathfrak{f}, (2\omega)^{-1}\ \mathfrak{g} \right\rangle_{L^2(\mathbb{R}^s)} = \langle \mathfrak{f}, G\mathfrak{g} \rangle_{L^2(\mathbb{R}^s)} = \left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}} \right\rangle_{\widetilde{\mathcal{H}}}} . \tag{2.54}$$

One can also write the scalar product (2.35) on solutions to the Schrödinger equation in the form

$$\boxed{\langle \mathfrak{f}, \mathfrak{g} \rangle_{L^2(\mathbb{R}^s)} = i\left(\left\langle \mathfrak{f}^t, \tfrac{\partial}{\partial t}\mathfrak{g}^t \right\rangle_{\mathcal{H}} - \left\langle \tfrac{\partial}{\partial t}\mathfrak{f}^t, \mathfrak{g}^t \right\rangle_{\mathcal{H}}\right)} . \tag{2.55}$$

The index $-1/2$ on the Sobolev space $\mathfrak{H}_{-1/2}$ means that the space includes not only all $L^2(\mathbb{R}^s)$ functions, but also generalized functions which when acted on by $\omega^{-1/2}$ are square integrable. Functions in $\mathcal{H}$ are said to include all functions that are one-half a derivative of an $L^2(\mathbb{R}^s)$ function.[4] In Fourier space, the function $\omega(\vec{p})^{-1/2}$ decays as $|\vec{p}|^{-1/2}$ for large $|\vec{p}|$, so the corresponding Fourier transforms when multiplied by an inverse half power of $|\vec{p}|$ for large $|\vec{p}|$ are square integrable.

2.8 The Poincaré Group on $\mathcal{H}$

It is now straightforward to write down the representation $U(\Lambda, a)$ of the Poincaré group. Let us start by finding the representation $\widetilde{U}(\Lambda, a)$ on the Hilbert space $\widetilde{\mathcal{H}}$ in the momentum representation.

[4] One can define similar spaces $\mathfrak{H}_p(\mathbb{R}^s)$ by replacing the transformation ω^{-1} in the inner product by the transformation ω^{-2p}. In the case of positive p, the Sobolev space does not include all square integrable functions, only those in the domain of ω^p as an operator on $L^2(\mathbb{R}^s)$.

Although the wave function $\widetilde{\mathfrak{f}}(\vec{p})$ depends on $\vec{p} \in \mathbb{R}^{d-1}$, it is convenient to regard it as a function of a Minkowski-space momentum variable p with d components, that lies on the hyperboloid $p^2 = m^2$, and with $p_d > 0$. There is a unique correspondence between $(d-1)$-vectors $\vec{p}$ and such d-vectors $p = (\vec{p}, \omega(\vec{p}))$. Define $\widetilde{\mathfrak{f}}(p)$ by

$$\widetilde{\mathfrak{f}}(p) = \widetilde{\mathfrak{f}}(\vec{p}) \; . \tag{2.56}$$

With this notation, it is clear that

$$\left(\widetilde{U}(I,a)^*\widetilde{\mathfrak{f}}\right)(p) = e^{-ia_d\omega(\vec{p})+i\vec{a}\cdot\vec{p}}\;\mathfrak{f}(p) = e^{-ia\cdot p}\;\mathfrak{f}(p) \; . \tag{2.57}$$

Since $\widetilde{U}(I,a)$ multiplies $\widetilde{\mathfrak{f}}(p)$ by a phase, it acts on $\widetilde{\mathcal{H}}$ as a unitary transformation.

Furthermore, the matrix Λ maps the mass-m hyperboloid into itself. So define the anti-representation $\widetilde{U}(\Lambda,0)^*$ in a fashion similar to the rule (2.23), giving

$$\left(\widetilde{U}(\Lambda,0)^*\widetilde{\mathfrak{f}}\right)(p) = \widetilde{\mathfrak{f}}(\Lambda p) \; . \tag{2.58}$$

Then using the composition law $\widetilde{U}(\Lambda,a)^* = \widetilde{U}(\Lambda,0)^*\widetilde{U}(I,a)^*$, we find

$$\boxed{\left(\widetilde{U}(\Lambda,a)^*\widetilde{\mathfrak{f}}\right)(p) = e^{-ia\cdot\Lambda p}\;\widetilde{\mathfrak{f}}(\Lambda p)} \; . \tag{2.59}$$

Proposition 2.8.1. *The transformation* (2.59) *defines the adjiont of a unitary representation* $\widetilde{U}(\Lambda,a)$ *of the Poincaré group on the one-particle momentum-space Hilbert space* $\widetilde{\mathcal{H}}$. *Under Fourier transformation, it gives the unitary representation*

$$\boxed{U(\Lambda,a) = \mathfrak{F}^*\widetilde{U}(\Lambda,a)\mathfrak{F}} \; , \tag{2.60}$$

on on the configuration-space Hilbert space $\mathcal{H}$.

Proof. The transformation $\widetilde{U}(\Lambda,a)^*$ satisfies the multiplication law for an anti=representation, as this is true of both $\widetilde{U}(I,a)$ and $\widetilde{U}(\Lambda,0)$. Therefore we need only show that $\widetilde{U}(\Lambda,a)^*$ is unitary.

We have already seen that $\widetilde{U}(I,a)^*$ is unitary, so we only need to verify that $\widetilde{U}(\Lambda,0)^*$ is unitary. We see this by expressing the $\widetilde{\mathcal{H}}$ inner product in invariant form,

$$\begin{aligned}\left\langle \widetilde{U}(\Lambda,0)^*\widetilde{\mathfrak{f}}, \widetilde{U}(\Lambda,0)^*\widetilde{\mathfrak{g}}\right\rangle_{\widetilde{\mathcal{H}}} &= \int_{\mathbb{R}^d} \overline{\widetilde{\mathfrak{f}}(\Lambda p)}\,\widetilde{\mathfrak{g}}(\Lambda p)\,\delta(p^2-m^2)\,dp \\ &= \int_{\mathbb{R}^d} \overline{\widetilde{\mathfrak{f}}(p)}\,\widetilde{\mathfrak{g}}(p)\,\delta(p^2-m^2)\,dp = \left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}}\right\rangle_{\widetilde{\mathcal{H}}} \; .\end{aligned} \tag{2.61}$$

Therefore $\widetilde{U}(\Lambda,0)^*$ is a unitary anti-representation on $\widetilde{\mathcal{H}}$, and $U(\Lambda,a)$ is a unitary representation.

The statement about the representation on $\mathcal{H}$ follows from the fact that $\mathfrak{F}^*\mathfrak{F} = I$ on $\widetilde{\mathcal{H}}$ and $\mathfrak{F}\mathfrak{F}^* = I$ on $\mathcal{H}$. That $\mathfrak{F}^*\widetilde{U}(\Lambda,a)\mathfrak{F}$ is a representation then follows from the fact that $\widetilde{U}(\Lambda,a)$ is a representation. In order to show that the representation is unitary, it is sufficient to prove that it preserves scalar products. The fundamental relation (2.51) shows that

$$\langle U(\Lambda,a)\mathfrak{f}, U(\Lambda,a)\mathfrak{g}\rangle_{\mathcal{H}} = \left\langle \widetilde{U}(\Lambda,a)\mathfrak{F}\mathfrak{f}, \widetilde{U}(\Lambda,a)\mathfrak{F}\mathfrak{g}\right\rangle_{\widetilde{\mathcal{H}}} = \left\langle \widetilde{U}(\Lambda,a)\widetilde{\mathfrak{f}}, \widetilde{U}(\Lambda,a)\widetilde{\mathfrak{g}}\right\rangle_{\widetilde{\mathcal{H}}} = \left\langle \widetilde{\mathfrak{f}}, \widetilde{\mathfrak{g}}\right\rangle_{\widetilde{\mathcal{H}}} = \langle \mathfrak{f}, \mathfrak{g}\rangle_{\mathcal{H}} \; . \tag{2.62}$$

Thus $U(\Lambda,a) = \mathfrak{F}^*\widetilde{U}(\Lambda,a)\mathfrak{F}$ is unitary, and the proof is complete.

Chapter 3

Life of a Particle at Imaginary Time

In Chapter 2 we described such a particle by a quantum-mechanical wave function $\mathfrak{f}(\vec{x})$, with $\vec{x} \in \mathbb{R}^{d-1}$. These wave functions were chosen to lie in the one-particle Hilbert space $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$, namely the Sobolev space $\mathfrak{H}_{-1/2}$ that contains all square integrable functions as well as vectors which have finite norm in the inner product

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathcal{H}} = \langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})} = \left\langle \mathfrak{f}, (2\omega)^{-1} \mathfrak{g} \right\rangle_{L^2(\mathbb{R}^{d-1})} , \tag{3.1}$$

with $\omega = (-\nabla^2 + m^2)^{1/2}$ equal to the one-particle Hamiltonian. By using this description, rather than the usual L^2 wave functions, we have an easy way to describe quantum theory in a Lorentz covariant fashion, and we found a representation of the Poincaré group on $\mathcal{H}$.

In this chapter we give a different perspective on the ordinary quantum theory of a single spinless, positive msss-m particle on $\mathbb{R}^{d-1}$. Here we switch to Euclidian space-time $\mathbb{R}^d$, where space and time enjoy the same geometry—although we still distinguish a special time direction in order to make a connection with ordinary quantum theory. Space-time points are vectors

$$x = (\vec{x}, x_d) \in \mathbb{R}^d , \qquad \text{with Euclidean length squared } x^2 = \vec{x}^2 + x_d^2 . \tag{3.2}$$

The last coordinate x_d, which is the imaginary time. It corresponds in many cases to the analytic continuation from Minkowski space to purely imaginary times, $x_d = it$.

Euclidean wave functions will be functions $f(x)$ on Euclidean space-time, that are elements of the Euclidean Hilbert space $\mathcal{E}$. We choose $\mathcal{E}$ so the inner product is naturally invariant under all rotations and translations (Euclidean transformations) of $\mathbb{R}^d$. A straight-forward choice for the space of wave functions might be $L^2(\mathbb{R}^d)$, with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx . \tag{3.3}$$

However, this is **not** the space we use. Just as in the quantum theory of Chapter 2, the space of Lebesgue square-integrable functions is not the natural choice for the Euclidean Hilbert space.

Rather, we choose for $\mathcal{E}$ the somewhat larger Sobolev space $\mathcal{E} = \mathfrak{H}_{-1}(\mathbb{R}^d)$. Elements of this space are generalized functions with inner product equal to

$$\langle f, g\rangle_{\mathcal{E}} = \langle f, g\rangle_{\mathfrak{H}_{-1}(\mathbb{R}^d)} = \left\langle f, (-\Delta + m^2)^{-1} g\right\rangle_{L^2(\mathbb{R}^d)} , \tag{3.4}$$

where $\Delta = \sum_{j=1}^{d} \partial^2/\partial x_j^2$ is the Laplacian on $\mathbb{R}^d$.

Life in Euclidean space is different from life in Minkowski space, at least for non-zero time. So before considering that issue, let us mention how one can identify the Euclidean picture with the Minkowski picture on the time-zero hyperplane.

A very nice property of the Euclidean wave functions that we have chosen is that they have a localization to a sharp time, and this is one reason for the choice $\mathcal{E} = \mathfrak{H}_{-1}(\mathbb{R}^d)$. In fact, this space includes sharp-time wave functions that have the spatial dependence $\mathfrak{f}(\vec{x}) \in \mathcal{H}$, namely the one-particle wave functions introduced in Chapter 2. At time zero, these special wave functions have the form

$$f(\vec{x}, t) = \mathfrak{f}(\vec{x})\, \delta(t) . \tag{3.5}$$

We suppress the variables and write such a product wave-function f as $f = \mathfrak{f} \otimes \delta$.[1]

These functions are vectors in the Hilbert space $\mathcal{E}$, namely

$$(\mathfrak{f} \otimes \delta) \in \mathfrak{H}_{-1}(\mathbb{R}^d) , \qquad \text{if } \mathfrak{f} \in \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1}) . \tag{3.6}$$

The inner product between two such special wave-functions localized at time zero is

$$\langle \mathfrak{f} \otimes \delta, \mathfrak{g} \otimes \delta\rangle_{\mathcal{E}} = \langle \mathfrak{f}, \mathfrak{g}\rangle_{\mathcal{H}} . \tag{3.7}$$

This gives the real justification for our choice. It can be interpreted as a way to identify certain Euclidean wave functions in $\mathcal{E} = \mathfrak{H}_{-1}(\mathbb{R}^d)$ that behave exactly as the ordinary one-particle wave functions $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. These special Euclidean wave-functions are a subspace of $\mathcal{E}$. We have chosen the normalization of the inner product so that we obtain the elementary relation (3.7) for this imbedding.

There are many other nice consequences of this relationship between $\mathcal{E}$ and $\mathcal{H}$ that one can see by going away from $x_d = 0$ to the Euclidean point $(\vec{x}, x_d)$. This corresponds to a Minkowski point $(\vec{x}, it)$ that is analytically continued to imaginary time,

$$(\vec{x}, x_d) \leftrightarrow (\vec{x}, it) . \tag{3.8}$$

In order to illustrate this point, let us consider another elementary example. Consider the time translation transformation $T_s(\vec{x}, x_d) \to (\vec{x}, x_d - s)$. This transformation acts on $L^2(\mathbb{R}^d) \in \mathcal{E}$ as a unitary group. It is defined on smooth functions $g(x) \in L^2(\mathbb{R}^d)$ by

$$(T_s g)(\vec{x}, x_d) = g(\vec{x}, x_d - s) . \tag{3.9}$$

[1] We do not explain the notation $\otimes$ for "tensor product" here, but return at length to this topic in Chapter 4.

The Laplacian Δ commutes with time translations, so T_s also acts as a unitary transformation on $\mathcal{E} = \mathfrak{H}_{-1}(\mathbb{R}^d)$,

$$T_s^* T_s = T_s T_s^* = I \ . \tag{3.10}$$

Denote the translated delta function by $\delta_s(t) = \delta(t - s)$, so one can write

$$T_s (\mathfrak{g} \otimes \delta) = \mathfrak{g} \otimes \delta_s \ . \tag{3.11}$$

As T_s is unitary, one could carry out the calculation (3.7) at any time s; there is nothing special about time zero. Hence one finds that for any $s \in \mathbb{R}$,

$$\langle \mathfrak{f} \otimes \delta_s, \mathfrak{g} \otimes \delta_s \rangle_{\mathcal{E}} = \langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathcal{H}} \ . \tag{3.12}$$

We conclude that the scalar product of sharp time vectors can only depend on the difference in the time of their localizations.

If we consider two different times, the matrix elements of T_s give the interesting relation that generalizes (3.7). In §3.4 we show that for $s \geq 0$,

$$\begin{aligned} \langle (\mathfrak{f} \otimes \delta), T_s (\mathfrak{g} \otimes \delta) \rangle_{\mathcal{E}} &= \langle \mathfrak{f} \otimes \delta, \mathfrak{g} \otimes \delta_s \rangle_{\mathcal{E}} \\ &= \left\langle \mathfrak{f}, e^{-s\omega} \mathfrak{g} \right\rangle_{\mathcal{H}} \ . \end{aligned} \tag{3.13}$$

Here ω is the single-particle energy operator introduced earlier. Furthermore, the inner product in $\mathcal{E}$ is invariant under time-reflection, so (3.13) is unchanged if we replace s by $-s$, so

$$\langle (\mathfrak{f} \otimes \delta), T_s (\mathfrak{g} \otimes \delta) \rangle_{\mathcal{E}} = \left\langle \mathfrak{f}, e^{-|s|\omega} \mathfrak{g} \right\rangle_{\mathcal{H}} \ . \tag{3.14}$$

Observe that on the left one has the matrix elements of a unitary operator T_s acting on $\mathcal{E}$. On the right side, one obtains the corresponding matrix elements on $\mathcal{H}$ of the operator

$$R(s) = e^{-|s|\omega} \ , \tag{3.15}$$

which is the self adjoint contraction that one obtains by analytically continuing the Schrödinger group $e^{-is\omega}$ to purely imaginary time s in the upper or lower complex half-plane, depending on whether the time is positive or negative. On the other hand, the operator T_s itself does not have an analytic continuation to complex s. But the equalities (3.13)–(3.14) show that certain matrix elements of T_s do have analytic continuations.

More generally, each unitary unitary Euclidean transformation ($\mathbb{R}^d$-rotation or space-time translation) acting on $\mathcal{E}$ corresponds in a 1-1 fashion with the analytic continuation of a unitary representation of the Poincaré transformations (Lorentz transformations and space-time translations) acting on the space $\mathcal{H}$.

In order to assure the analytic continuation of matrix elements of the Euclidean transformations on $\mathcal{E}$, one must restrict consideration to a subspace of $\mathcal{E}$. The functions that one studies belong to the subspace of "positive time functions," namely those what vanish for negative times, or $\mathfrak{H}_{-1}(\mathbb{R}^d_+)$, where the subscript designates the positive time half space.

A more general correspondence between the spaces $\mathcal{H}$ and $\mathcal{E}$ arises from considering time reflection between positive-time and negative-time functions, and using the property of *reflection positivity*, explained in §3.3. This general approach gives a relationship between the Euclidean and real-time pictures which one can interpret as a procedure for quantization.

3.1 Wave Functions

The Schwartz space functions $\mathcal{S}(\mathbb{R}^d)$ in d-dimensions, is the linear vector space equipped with the countable family of norms arising from the Hilbert-space inner products,

$$\langle f, g\rangle_{r,s} = \left\langle (1+x^2)^r(1-\Delta)^s f, (1+x^2)^r(1-\Delta)^s g\right\rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.16}$$

One takes all possible non-negative integer values for r, s, which one writes $r, s \in \mathbb{Z}_+$. Here $x^2 = x_1^2 + \cdots + x_d^2$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$.

The Fourier transform operator $\mathfrak{F}$ on $L^2(\mathbb{R}^d)$ has spectrum $\{\pm 1, \pm i\}$, and the eigenfunctions of $\mathfrak{F}$ are the Hermite functions, namely Hermite polynomials times a Gaussian. These eigenfunctions are elements of $\mathcal{S}(\mathbb{R}^d)$, so

$$\mathfrak{F}\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) \ . \tag{3.17}$$

The proper Euclidean group $\{R, a\}$ on $\mathbb{R}^d$ consists of rotation matrices $R \in SO(d)$ and space-time translations $a \in \mathbb{R}^d$. We are also interested in reflections, especially the time-reflection $\Theta\colon (\vec{x}, x_d) \to (\vec{x}, -x_d)$ and the spatial reflection $\Pi\colon (\vec{x}, x_d) \to (-\vec{x}, x_d)$. The total reflection is given by $\Theta\Pi x = -x$.

We represent each of these Euclidean transformations by a unitary transformation on $L^2(\mathbb{R}^d)$, and we denote these unitaries by $T(R, y), \Theta$, and Π.

$$\left(T(R;y)f\right)(x) = f(R^{-1}x + y) \ , \qquad \text{and} \quad \left(\Theta f\right)(x) = f(\Theta x) \ . \tag{3.18}$$

We also abbreviate $T(I, x)$ by T_x and $T(R, 0)$ by $T(R)$.

For a subset $\mathcal{O} \subset \mathbb{R}^d$ define the functions

$$\mathcal{S}(\mathcal{O}) = \mathcal{S}(\mathbb{R}^d) \cap C^\infty(\mathcal{O}) \ , \tag{3.19}$$

where $C^\infty(\mathcal{O})$ denotes the space of smooth functions supported in $\mathcal{O}$. Likewise, define $L^2(\mathcal{O})$. The space $\mathcal{S}(\mathcal{O})$ is a dense subspace of $L^2(\mathcal{O})$. The decomposition

$$L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d_+) \oplus L^2(\mathbb{R}^d_-) \tag{3.20}$$

plays a special role.

3.2 The Euclidean Laplacian and its Green's Function

The most fundamental operator for the free particle in Euclidean space is the Laplacian, Δ on $\mathbb{R}^d$,

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \ . \tag{3.21}$$

We reserve the symbol Δ for the Laplacian on $\mathbb{R}^d$ and ∇^2 for the Laplacian on $\mathbb{R}^{d-1}$. This is not a perverse way to make things complicated, but simplifies notation when both appear. This Laplacian is defined on all Euclidean space, without any boundary conditions. We take Δ to be a self-adjoint operator on $L^2(\mathbb{R}^d)$.

One can consider the Green's operator or resolvent $C = (-\Delta + m^2)^{-1}$. Here $m > 0$ is the *mass*, a given constant. The operator C acts on $L^2(\mathbb{R}^d)$, and the matrix elements $C(x;y)$ of C can be defined as the kernel of an integral operator by the identity

$$(Cf)(x) = \int C(x;y)f(y)dy \,, \qquad \text{for } f \in L^2(\mathbb{R}^d) \,. \tag{3.22}$$

One also calls $C(x;y)$ a Green's function, because it satisfies the equation

$$\left(-\Delta_x + m^2\right) C(x;y) = \delta^d(x-y) \,. \tag{3.23}$$

Here we use a subscript x on Δ to denote that it acts on the x variable. This notations

$$\Delta_x C(x;y) = (\Delta C)(x;y) \,, \tag{3.24}$$

are equivalent and mean the same thing. Likewise,

$$\Delta_y C(x;y) = (C\Delta)(x;y) \,. \tag{3.25}$$

By translation invariance of the Laplacian, this Green's function only depends on the Euclidean difference of x and y, so

$$C(x;y) = C(x-y) = C(R(x-y)) \,, \tag{3.26}$$

for any $R \in O(d)$. One can also interpret $C(x-y)$ as the potential at x due to a unit test charge at y. In particular, the orthogonal invariance of $C(x-y)$ ensures the reciprocity law $C(x-y) = C(y-x)$.

Set $r = |x-y|$. For d=1, the Green's function is continuous on the diagonal ($r = 0$) and one easily computes

$$C(x-y) = \frac{1}{2m} e^{-mr} \,. \tag{3.27}$$

For $d = 2$ the singularity of the Green's function for small r is logarithmic, and

$$C(x-y) \simeq -\frac{1}{2\pi} \ln(mr) \,. \text{ as } r \to 0 \,, \tag{3.28}$$

On the other hand, in $d = 2$ the Green's function decays for large r as

$$C(x-y) \simeq \frac{1}{(8\pi m)^{1/2}} \frac{1}{r} e^{-mr} \,, \text{ as } r \to \infty \,. \tag{3.29}$$

For $d = 3$, the Green's function again has an elementary form; it equals the Yukawa potential both at short and at long distances,

$$C(x-y) = \frac{1}{4\pi r} e^{-mr} \,. \tag{3.30}$$

In general, the Green's function can be expressed in terms of Hankel functions. But the singularity of $C(x-y)$ for $r \to 0$ is always given by the Coulomb potential. For $d \geq 3$,

$$C(x-y) \simeq \alpha_d \frac{1}{r^{d-2}} \ , \ \text{as} \ r \to 0 \ , \qquad \text{where} \ \alpha_d = \frac{1}{4\pi^{d/2}} \Gamma\left(\frac{d-2}{2}\right) \ , \tag{3.31}$$

while at long distances the exponential decay is modified by another power or r,

$$C(x-y) \simeq \beta_d \frac{1}{r^{(d-1)/2}} e^{-mr} \ , \ \text{as} \ r \to \infty \ , \qquad \text{where} \ \beta_d = 2^{-(d+1)/2} \pi^{-(d-1)/2} m^{(d-3)/2} \ . \tag{3.32}$$

Without knowing such details about the Green's function, we have the important facts:

Proposition 3.2.1. *Let $m > 0$ and $r = |x-y| > 0$. Then $0 < C \leq m^{-2}$. Also $C(x-y)$ is a strictly positive, real-analytic function of x and y, which is also monotone decreasing in r.*

Proof. The operator bounds follow from considering C in Fourier space, where

$$\langle f, Cf \rangle_{L^2(\mathbb{R}^d)} = \left\langle \tilde{f}, \frac{1}{p^2+m^2} \tilde{f} \right\rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.33}$$

Thus

$$0 \leq \langle f, Cf \rangle_{L^2(\mathbb{R}^d)} \leq \frac{1}{m^2} \left\langle \tilde{f}, \tilde{f} \right\rangle_{L^2(\mathbb{R}^d)} = \frac{1}{m^2} \langle f, f \rangle_{L^2(\mathbb{R}^d)} \ , \tag{3.34}$$

with $\langle f, Cf \rangle_{L^2(\mathbb{R}^d)}$ only vanishing for $f = 0$. Euclidean invariance of $C(x-y)$ shows that $C(x-y)$ is a function only of r for $r = |x-y| > 0$. Choose a rotation so that $R(x-y) = (\vec{0}, r)$, yielding

$$C(x-y) = \frac{1}{(2\pi)^d} \int \left(\int \frac{1}{p^2+m^2} e^{-irp_d} dp_d \right) d\vec{p} \ . \tag{3.35}$$

Write $\omega = (\vec{p}^2 + m^2)^{1/2}$, and use $p^2 + m^2 = (p_d + i\omega)(p_d - i\omega)$. Using the Cauchy residue formula at the pole $p_d = i\omega(\vec{p})$, one obtains the representation

$$C(x-y) = \frac{1}{(2\pi)^{d-1}} \int \frac{1}{2\omega(\vec{p})} e^{-r\omega(\vec{p})} d\vec{p} \ > \ 0 \ . \tag{3.36}$$

This shows that $C(x-y)$ is monotone decreasing in r, and also real analytic in r for $r > 0$. This yields real analyticity in $x - y \neq 0$.

3.3 Reflection Positivity

Define a form $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ on $L^2(\mathbb{R}^d_+) \times L^2(\mathbb{R}^d_+)$ by the formula

$$\langle f, g \rangle_{\mathcal{H}_1} = \langle f, \Theta C g \rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.37}$$

This form is conjugate linear in the first factor and linear in the second factor, which is called *sesquilinear*. We use this form to define a new inner product space, namely the Hilbert space $\mathcal{H}_1$ of one-particle, quantum theory wave functions associated with the classical Euclidean space wave functions $L^2(\mathbb{R}^d_+)$.

Definition 3.3.1. *The operator C on $L^2(\mathbb{R}^d)$ is said to be* **reflection positive** *with respect to Θ if the form $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ is positive semi-definite, restricted to the subspace $L^2(\mathbb{R}^d_+) \subset L^2(\mathbb{R}^d)$. In other words*

$$0 \le \langle f, f\rangle_{\mathcal{H}_1} = \langle f, \Theta C f\rangle \ , \qquad \text{for all } f \in L^2(\mathbb{R}^d_+) \ . \tag{3.38}$$

Proposition 3.3.2. *The operator C is reflection positive.*

Proof. To show $0 \le \langle f, f\rangle_{\mathcal{H}_1}$ for $f \in L^2(\mathbb{R}^d_+)$, we evaluate this inner product. Set $t = x_d$. The Fourier transform of f in the t direction is the boundary value of a function of the variable p_d that has an analytic continuation throughout the upper half plane, and this continuation vanishes along the semicircle of constant $|p_d|$ in the complex p_d-plane as $|p_d| \to \infty$. In fact take $p = \{\vec{p}, p_d\}$, with $\vec{p} = \{p_1, \ldots, p_{d-1}\}$, so the Fourier transform of $f \in L^2(\mathbb{R}^d_+)$ for $x = \{\vec{x}, t\}$ is

$$(\mathfrak{F} f)(p) = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} f(x)\, e^{i\vec{p}\cdot\vec{x} + ip_d t} d\vec{x} \right) dt \ . \tag{3.39}$$

Then the analytic continuation to $p_d = iE$, with $E \ge 0$ is

$$(\mathfrak{F} f)(\vec{p}, iE) = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} f(x)\, e^{i\vec{p}\cdot\vec{x} - Et} d\vec{x} \right) dt \ . \tag{3.40}$$

Likewise the complex conjugate of the Fourier transform of the time-reflected function is

$$\overline{(\mathfrak{F}(\Theta f))(p)} = \overline{(\mathfrak{F}(\Theta f))(\vec{p}, p_d)} = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} \overline{f(x)}\, e^{-i\vec{p}\cdot\vec{x} + ip_d t} d\vec{x} \right) dt \ . \tag{3.41}$$

As a function of p_d, this also has an analytic continuation into the upper half plane. Continuing to $p_d = iE$ with $E > 0$, one sees from the representations (3.40)–(3.41) that

$$\begin{aligned} \overline{(\mathfrak{F}(\Theta f))(\vec{p}, iE)} &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} \overline{f(x)}\, e^{-i\vec{p}\cdot\vec{x} - Et} d\vec{x} \right) dt \\ &= \overline{(\mathfrak{F} f)(\vec{p}, iE)} \ . \end{aligned} \tag{3.42}$$

One can complete the dp_d integral along a semicircle in the upper half plane at infinity and use the Cauchy residue formula to evaluate the p_d-integral at the pole $p_d = iE = i\omega(\vec{p})$ to give

$$\begin{aligned} \langle f, f\rangle_{\mathcal{H}_1} &= \int \overline{(\mathfrak{F}(\Theta f))(p)}\ (\mathfrak{F} f)(p) \frac{1}{p^2 + m^2}\, dp \\ &= \int \left(\int \overline{(\mathfrak{F}(\Theta f))(p)}\ (\mathfrak{F} f)(p) \frac{1}{(p_d + i\omega)(p_d - i\omega)} dp_d \right) d\vec{p} \\ &= \pi \int |(\mathfrak{F} f)(\vec{p}, i\omega(\vec{p}))|^2 \frac{1}{\omega(\vec{p})}\, d\vec{p} \ \ge\ 0 \ . \end{aligned} \tag{3.43}$$

Hence the form $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ is positive semi-definite as claimed.

Exercise 3.3.1. *Show that $0 \le C(x - y)$ is also a consequence of Proposition 3.3.2.*

3.4 Osterwalder-Schrader Quantization

We started from one-particle quantum theory in §**??**. Our one-particle Hilbert space was the Sobolev space $\mathcal{F}_1 = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$ with the inner product (**??**). In this section we construct construct a Hilbert space $\mathcal{H}_1$ arising from the reflection-positive inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ on $L^2(\mathbb{R}^d_+)$ defined in (3.37). We see shortly in Proposition 3.4.4 that these two constructions are two different ways of looking at the same thing, and

$$\mathcal{H}_1 = \mathcal{F}_1 \, . \tag{3.44}$$

This is the simplest case of what we call OS-quantization. It provides a method to give a quantum-mechanical Hilbert space, as well as natural operators acting on this space. We continue here to analyze one particle states, and operators that act on such states.

Definition 3.4.1. *The null space $\mathcal{N} \subset L^2(\mathbb{R}^d_+)$ of the form $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ are those $f \in L^2(\mathbb{R}^d_+)$ for which $\langle f, f \rangle_{\mathcal{H}_1} = 0$.*

The null space $\mathcal{N}$ is a linear vector space. In fact, we claim that $\mathcal{N}$ is exactly the vector space of those $f \in L^2(\mathbb{R}^d_+)$ such that $\langle f, g \rangle_{\mathcal{H}_1} = 0$ for all $g \in L^2(\mathbb{R}^d_+)$. In fact, given $\langle f, g \rangle_{\mathcal{H}_1} = 0$ for all $g \in L^2(\mathbb{R}^d_+)$, one can choose $g = f$, and in this case $\langle f, f \rangle_{\mathcal{H}_1} = 0$ ensures $f \in \mathcal{N}$. Conversely, if $\langle f, f \rangle_{\mathcal{H}_1} = 0$ and $g \in L^2(\mathbb{R}^d_+)$, then $\left| \langle f, g \rangle_{\mathcal{H}_1} \right| \leq \langle f, f \rangle_{\mathcal{H}_1}^{1/2} \langle g, g \rangle_{\mathcal{H}_1}^{1/2} = 0$.

Definition 3.4.2. *The Hilbert space $\mathcal{H}_1$ is the completion of the equivalence classes $\hat{f} \in L^2(\mathbb{R}^d_+)/\mathcal{N}$ of the form*

$$\hat{f} = \{ f + g : f \in L^2(\mathbb{R}^d_+) \, , \; g \in \mathcal{N} \} \, , \tag{3.45}$$

It has the inner product (3.37)*,*

$$\left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_1} = \langle f, \Theta C g \rangle_{L^2(\mathbb{R}^d)} = \pi \int \overline{(\mathfrak{F} f)\,(\vec{p}, i\omega(\vec{p}))}\; (\mathfrak{F} g)\,(\vec{p}, i\omega(\vec{p})) \frac{1}{\omega(\vec{p})} d\vec{p} \, . \tag{3.46}$$

Remark 1. It should not cause confusion that we use the notation $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ in two senses:

$$\langle f, g \rangle_{\mathcal{H}_1} = \langle f, \Theta C g \rangle_{L^2(\mathbb{R}^d)} \text{ on } L^2(\mathbb{R}^d_+) \times L^2(\mathbb{R}^d_+) \, , \qquad \text{and } \left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_1} \text{ on } \mathcal{H}_1 \times \mathcal{H}_1 \, . \tag{3.47}$$

Exercise 3.4.1. *Check the following three properties $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$.*

i. For $d \geq 1$ the subspace $\mathcal{N} \subset L^2(\mathbb{R}^d_+)$ is infinite dimensional.

ii. For $d \geq 2$ the space $\mathcal{H}_1$ is infinite dimensional.

iii. In case $d = 1$, verify that

$$\langle f, g \rangle_{\mathcal{H}_1} = \frac{\pi}{m} \overline{(\mathfrak{F} f)\,(im)}\, (\mathfrak{F} g)\,(im) \, . \tag{3.48}$$

This inner product has a limit for non-square integrable functions of the form $f =$ const. δ. In this limit

$$\langle \delta, \delta \rangle_{\mathcal{H}_1} = \frac{1}{2m} \, . \tag{3.49}$$

What is the dimension of $\mathcal{H}_1$ in this case?

3.4.1 The Sobolev Space $\mathfrak{H}_{-1}(\mathcal{O})$

Define the Sobolev space $\mathfrak{H}_{-1}\left(\mathcal{O};\mathbb{R}^d\right)$, for $\mathcal{O}\subset\mathbb{R}^d$, as the space of generalized functions

$$\mathfrak{H}_{-1}(\mathcal{O}) = \mathfrak{H}_{-1}\left(\mathcal{O};\mathbb{R}^d\right) = \{f : C^{1/2}f \in L^2(\mathbb{R}^d)\ ,\ \text{ and support } f\subset\mathcal{O}\}\ . \tag{3.50}$$

This space is a Hilbert space with the inner product that can be expressed in terms of the functions f or their Fourier transforms $\tilde f = \mathfrak{F}f$ in several equivalent ways:

$$\begin{aligned}\langle f,g\rangle_{\mathfrak{H}_{-1}(\mathcal{O})} &= \left\langle C^{1/2}f, C^{1/2}g\right\rangle_{L^2(\mathbb{R}^d)}\\ &= \langle f, Cg\rangle_{L^2(\mathbb{R}^d)} = \int \overline{f(x)}\,C(x-y)\,g(y)\,dxdy\\ &= \int \overline{\tilde f(p)}\,\frac{1}{p^2+m^2}\,\tilde g(p)\,dp\ .\end{aligned} \tag{3.51}$$

Proposition 3.4.3. *The quantization map* $\wedge\colon L^2(\mathbb{R}^d_+)\mapsto\mathcal{H}_1$ *of* §*3.4 extends uniquely to a contraction*

$$\wedge\colon \mathfrak{H}_{-1}(\mathbb{R}^d_+)\mapsto\mathcal{H}_1\ , \tag{3.52}$$

Proof. The map $\wedge$ an elementary identity,

$$\left\|\hat f\right\|_{\mathcal{H}_1} = \left\|\ f\ \right\|_{\mathfrak{H}_{-1}(\mathbb{R}^d_+)}\ , \qquad \text{for all } f\in L^2(\mathbb{R}^d_+)\ . \tag{3.53}$$

This is a consequence of the definition (3.37), the unitarity of Θ on $L^2(\mathbb{R}^d)$, along with $[\Theta, C]=0$. This means that the map $\wedge$ extends by continuity, and therefore uniquely, from $L^2(\mathbb{R}^d_+)$ to the larger space $\mathfrak{H}_{-1}(\mathbb{R}^d_+)$, of which $L^2(\mathbb{R}^d_+)\subset\mathfrak{H}_{-1}(\mathbb{R}^d_+)$ is dense. The range of this extended map (that we also denote by $\wedge$) is in the space $\mathcal{H}_1$. The extension is a contraction by virtue of the identity (3.53).

Part (iii) of Exercise 3.4.1 gives a special case of this extension for $d=1$. In arbitrary dimension d, we claim that

$$\mathfrak{H}_{-1}(\mathbb{R}^d_+)\supset\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})\otimes\delta\ , \tag{3.54}$$

as $f=\mathfrak{f}\otimes\delta$, with $\mathfrak{f}\in\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. In terms of coordinates these are functions of the form

$$f(\vec x, x_d) = (\mathfrak{f}\otimes\delta)(\vec x, x_d) = \mathfrak{f}(\vec x)\delta(x_d)\ ,\ \text{ with }\ \omega^{1/2}\mathfrak{f}\in L^2(\mathbb{R}^s)\ . \tag{3.55}$$

For such a function,

$$\begin{aligned}\langle\mathfrak{f}\otimes\delta,\mathfrak{f}\otimes\delta\rangle_{\mathfrak{H}_{-1}(\mathbb{R}^d_+)} &= \frac{1}{2\pi}\int\left|\tilde{\mathfrak{f}}(\vec p)\right|^2\frac{1}{p^2+m^2}\,dp\\ &= \int\left|\tilde{\mathfrak{f}}(\vec p)\right|^2\frac{d\vec p}{2\omega(\vec p)}\\ &= \langle\mathfrak{f},\mathfrak{f}\rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})}\ ,\end{aligned} \tag{3.56}$$

where we use the Cauchy residue formula to evaluate the p_d integral. This justifies (3.54).

3.4.2 Why "Quantization"?

We now identify the map $\wedge$ that we have been calling a "quantization map," as the map from $\mathfrak{H}_{-1}(\mathbb{R}^d_+)$ to $\mathcal{F}_1$. This justifies calling the map by this name.

Proposition 3.4.4 (Identification of Quantization). *After extension by continuity, the Hilbert space of one-particle quantum theory in §??, and the Hilbert space of one-particle quantum theory in Proposition 3.4.3 are the same. In particular*

$$\mathcal{H}_1 = \mathcal{F}_1 \ , \qquad \text{which one can write} \qquad \left(\mathfrak{H}_{-1}(\mathbb{R}^d_+)\right)^\wedge = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1}) \ . \tag{3.57}$$

Proof. By definition $\mathcal{F}_1 = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. After extension $\mathcal{H}_1 = \left(\mathfrak{H}_{-1}(\mathbb{R}^d_+)\right)^\wedge$, which coincides with the definition of $\mathcal{H}_1$ as the completion of the pre-Hilbert space $L^2(\mathbb{R}^d_+)/\mathcal{N}$. The inner product (3.46) on $L^2(\mathbb{R}^d_+)$ shows that

$$\left(L^2(\mathbb{R}^d_+)\right)^\wedge \subset \left(\mathfrak{H}_{-1}(\mathbb{R}^d_+)\right)^\wedge \subset \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1}) \ . \tag{3.58}$$

On the other hand, the computation (3.56) shows that every function in $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$ is obtained by the quantization of $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$,

$$\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1}) \subset \left(\mathfrak{H}_{-1}(\mathbb{R}^d_+)\right)^\wedge \ . \tag{3.59}$$

Therefore $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1}) = \left(\mathfrak{H}_{-1}(\mathbb{R}^d_+)\right)^\wedge$ which completes the proof.

3.4.3 Quantization of Operators

Denote the quantization map for the Green's operator C that takes classical functions $f \in L^2(\mathbb{R}^d_+)$ to quantum state vectors $\hat{f} \in \mathcal{H}_1$ by,

$$\wedge\colon f \mapsto \hat{f} \ . \tag{3.60}$$

This map extends naturally to give a quantization map on certain linear operators T on $L^2(\mathbb{R}^d)$. We begin with operators that satisfy the following

Assumptions The bounded transformation T defined on $L^2(\mathbb{R}^d)$ is such that:

1. The transformation T maps $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$.

2. The transformation T maps $\mathcal{N}$ into $\mathcal{N}$, where $\mathcal{N}$ is given by Definition 3.4.1.

Definition 3.4.5. *Define the quantization $\hat{T}$ of a transformation T satisfying the assumptions above by*

$$\hat{T}\hat{f} = \widehat{Tf} \ . \tag{3.61}$$

Alternatively one says the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}^d_+) & \xrightarrow{T} & L^2(\mathbb{R}^d_+) \\ \wedge\downarrow & & \downarrow\wedge \\ \mathcal{H}_1 & \xrightarrow{\widehat{T}} & \mathcal{H}_1 \end{array} \quad . \tag{3.62}$$

Exercise 3.4.2. *Show that if $T\mathcal{N} \not\subset \mathcal{N}$, this procedure does not give a well-defined operator $\hat{T}$.*

Exercise 3.4.3. *Suppose that T_1, T_2 both transform $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$, both transform $\mathcal{N}$ into $\mathcal{N}$. Then is it necessarily the case that $\widehat{T_1T_2} = \hat{T}_1\hat{T}_2$? If in addition $T_1T_2 = T_2T_1$, is $\hat{T}_1\hat{T}_2 = \hat{T}_2\hat{T}_1$?*

Proposition 3.4.6 (A Quantizability Condition). *Let T be a bounded transformation on $L^2(\mathbb{R}^d)$. Suppose that*

- *Both T and $\Theta T^*\Theta$ map $L^2(\mathbb{R}^d_+)$ to $L^2(\mathbb{R}^d_+)$, and*
- *$CT = TC$.*

Then T satisfies the Assumptions 1–2 above, and the quantization $\hat{T}$ of T exists.

Proof. For $f \in \mathcal{N}$ and arbitrary $g \in L^2(\mathbb{R}^d_+)$, consider

$$\begin{aligned} \langle g, Tf\rangle_{\mathcal{H}_1} &= \langle \Theta g, CTf\rangle_{L^2(\mathbb{R}^d)} = \langle \Theta g, TCf\rangle_{L^2(\mathbb{R}^d)} = \langle T^*\Theta g, Cf\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \Theta\left(\Theta T^*\Theta\right)g, Cf\rangle_{L^2(\mathbb{R}^d)} = \langle \left(\Theta T^*\Theta\right)g, f\rangle_{\mathcal{H}_1} \, . \end{aligned} \tag{3.63}$$

Here we use the commutativity of T with C and also the fact that $\Theta T^*\Theta$ acts on $L^2(\mathbb{R}^d_+)$. Applying the Schwarz inequality on $\mathcal{H}_1$, we infer

$$\left|\langle g, Tf\rangle_{\mathcal{H}_1}\right| \leq \|\left(\Theta T^*\Theta\right)g\|_{\mathcal{H}_1}\|f\|_{\mathcal{H}_1} = 0 \, . \tag{3.64}$$

Thus $Tf \in \mathcal{N}$ as desired.

Proposition 3.4.7 (Multiple Reflection Bound). *Let T satisfy the hypotheses of Proposition 3.4.6. Then the norm of the quantization of T is bounded by the original norm,*

$$\left\|\hat{T}\right\|_{\mathcal{H}_1} \leq \|T\|_{L^2(\mathbb{R}^d)} \, . \tag{3.65}$$

Proof. For any $f \in L^2(\mathbb{R}^d_+)$. Setting $g = Tf$ in (3.63), we have

$$\begin{aligned} \left\|\hat{T}\hat{f}\right\|^2_{\mathcal{H}_1} &= \langle \left(\Theta T^*\Theta\right)Tf, f\rangle_{\mathcal{H}_1} \\ &\leq \|\left(\Theta T^*\Theta\right)Tf\|_{\mathcal{H}_1}\,\|f\|_{\mathcal{H}_1} = \|\left(\Theta T^*\Theta\right)Tf\|_{\mathcal{H}_1}\left\|\hat{f}\right\|_{\mathcal{H}_1} \, . \end{aligned} \tag{3.66}$$

Set $S = \left(\Theta T^*\Theta\right)T$. Then S maps $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$ and also $\Theta S^*\Theta = S$. Furthermore, the self-adjointness of C and the fact that $CT = TC$ ensures $CS = SC$. We therefore have checked that the hypotheses on T also apply to S, so one can iterate the above bound. After n steps, we obtain

$$\left\|\hat{T}\hat{f}\right\|_{\mathcal{H}_1} \leq \left\|S^{2^{n-1}}f\right\|_{\mathcal{H}_1}^{2^{-n}} \left\|\hat{f}\right\|_{\mathcal{H}_1}^{1-2^{-n}} . \tag{3.67}$$

We bound $\left\|S^{2^{n-1}}f\right\|_{\mathcal{H}_1}$ using the fact that for any $f \in L^2(\mathbb{R}^d_+)$,

$$\|f\|_{\mathcal{H}_1}^2 = \langle \Theta f, Cf\rangle_{L^2(\mathbb{R}^d)} = \left\langle C^{1/2}\Theta f, C^{1/2}f\right\rangle_{L^2(\mathbb{R}^d)} = \left\langle \Theta C^{1/2} f, C^{1/2}f\right\rangle_{L^2(\mathbb{R}^d)} . \tag{3.68}$$

As Θ is unitary on $L^2(\mathbb{R}^d)$, we infer that

$$\|f\|_{\mathcal{H}_1} \leq \left\|C^{1/2}f\right\|_{L^2(\mathbb{R}^d)} . \tag{3.69}$$

Hence

$$\begin{aligned}\left\|S^{2^{n-1}}f\right\|_{\mathcal{H}_1} &\leq \left\|C^{1/2}S^{2^{n-1}}f\right\|_{L^2(\mathbb{R}^d)} = \left\|S^{2^{n-1}}C^{1/2}f\right\|_{L^2(\mathbb{R}^d)} \\ &\leq \|T\|_{L^2(\mathbb{R}^d)}^{2^n} \left\|C^{1/2}f\right\|_{L^2(\mathbb{R}^d)} . \end{aligned} \tag{3.70}$$

Inserting this bound into (3.67) gives

$$\left\|\hat{T}\hat{f}\right\|_{\mathcal{H}_1} \leq \|T\|_{L^2(\mathbb{R}^d)} \left\|C^{1/2}f\right\|_{L^2(\mathbb{R}^d)}^{2^{-n}} \left\|\hat{f}\right\|_{\mathcal{H}_1}^{1-2^{-n}} . \tag{3.71}$$

Taking the $\limsup_n$ as $n \to \infty$, we obtain

$$\left\|\hat{T}\hat{f}\right\|_{\mathcal{H}_1} \leq \|T\|_{L^2(\mathbb{R}^d)} \left\|\hat{f}\right\|_{\mathcal{H}_1} , \tag{3.72}$$

as claimed.

3.4.4 Some Examples of Quantized Operators

Quantization of Space-Time Translations As an elementary example, we quantize a subset of the space-time translation group T_x, namely the entire group of spatial translation and the semi-group of time translations by positive time. We find that the quantized time-translation is no longer unitary; it is self-adjoint. Its infinitesimal generator is equal to $-\omega$, the relativistic energy operator, and a non-local operator on the one-particle space. The infinitesimal generator of space translations is the usual local, self-adjoint momentum operator $-i\nabla$.

Proposition 3.4.8. *Let T_t denote time translation for positive times $t \geq 0$, and let $T_{\vec{x}}$ denote translation in the spatial direction. Then these maps have quantizations and*

$$\widehat{T}_t = \widehat{T}_t^* = e^{-t\omega} , \qquad \text{and } \widehat{T}_{\vec{x}} = \widehat{T}_{\vec{x}}^{*-1} = e^{i\vec{x}\cdot\vec{p}} , \tag{3.73}$$

where $\omega = \left(\vec{p}^2 + m^2\right)^{1/2}$ is the one-particle Hamiltonian, and where $\vec{p} = -i\nabla$ is the standard momentum operator on the one-particle space $\mathcal{H}_1$. Also

$$\pm|\vec{p}| \leq h . \tag{3.74}$$

Proof. First we check that the space-time translation operators which we wish to quantize satisfy the quantization condition of Proposition 3.4.6. Each operator T_x is unitary, Clearly the chosen operators T_x map $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$. Furthermore, $\Theta T^*_{(\vec{x},t)}\Theta = T_{(-\vec{x},t)}$, so this operator also maps $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$. Finally, note that Δ is translation-invariant. Therefore C is translation-invariant, which means that it commutes with T_x. Hence the hypotheses are satisfied and the T_x with positive times have quantizations. Furthermore, Proposition 3.4.7 ensures that the quantization of each T_x considered must be a contraction on $\mathcal{H}_1$.

In order to evaluate the operator $\hat{T}_t$, one extends the proof of Proposition 3.3.2 by using

$$(\mathfrak{F}T_t g)(\vec{p}, iE) = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} g(x)\, e^{i\vec{p}\cdot\vec{x}} d\vec{x} \right) e^{-E(x_d+t)} dx_d \,. \tag{3.75}$$

Then

$$\begin{aligned} \left\langle f, \hat{T}_t g \right\rangle_{\mathcal{H}_1} &= \langle f, \Theta T_t g \rangle_{L^2(\mathbb{R}^d)} \\ &= \pi \int \overline{(\mathfrak{F}f)(\vec{p}, i\omega(\vec{p}))}\, (\mathfrak{F}g)(\vec{p}, i\omega(\vec{p})) \frac{e^{-t\omega(\vec{p})}}{\omega(\vec{p})} d\vec{p} \\ &= \left\langle f, e^{-t\omega} g \right\rangle_{\mathcal{H}_1} \,. \end{aligned} \tag{3.76}$$

Clearly $\hat{T}_t = e^{-t\omega} = e^{-th}$ is a self-adjoint, contraction semi-group.

Likewise, the spatial translation operator $T_{\vec{x}}$ on $L^2(\mathbb{R}^d_+)$ maps $\mathcal{N}$ into $\mathcal{N}$. Its quantization $\hat{T}_{\vec{a}}$ on $\mathcal{H}_1$ is the unitary generated by the infinitesimal operator $\vec{p} = -i\nabla$, and in Fourier space it acts as $e^{i\vec{a}\cdot\vec{p}}$. The relation (3.74) follows also from simultaneous diagonalization of these operators in Fourier space.

Exercise 3.4.4. *Can one redefine C in a way that is reflection positive, but such that the quantization of time translation is e^{-th}, with h given by the non-relativistic Hamiltonian $h = \frac{1}{2m}\vec{p}^2$ in place of the Hamiltonian $h = \omega$?*

Quantization of Purely-Spatial Rotations The group of $SO(d)$ matrices that determine Euclidean rotations has $d(d-1)/2$ real parameters, corresponding to the one-parameter groups $R_{ij}(\theta)$ of rotations by the angle θ about an axis orthogonal to the $x_i x_j$-plane in $\mathbb{R}^d$, where $1 \le i < j \le d$.

In case of purely-spatial rotations, $i < j < d$. There are $(d-1)(d-2)/2$ such planes, and the action of each such rotation has the form

$$R_{ij}(\theta)\colon (\vec{x}, x_d) \to (R_{ij}(\theta)\vec{x}, x_d) \,, \tag{3.77}$$

leaving the time coordinate fixed. Thus $T(R_{ij}(\theta))$ acts as a unitary on all of $L^2(\mathbb{R}^d_+)$, mapping $L^2(\mathbb{R}^d_+)$ to itself. Just as for the analysis of the spatial translations $T_{\vec{x}}$ in Proposition 3.4.8, one can quantize $T(R_{ij}(\theta))$ to obtain a unitary transformation $(T(R_{ij}(\theta)))^\wedge$ acting on $\mathcal{H}_1$.

What about Coordinates? The coordinates x_j for $1 \leq j \leq d$ are natural candidates for quantization. Clearly multiplying a function f by a bounded function of the coordinate does not change the support of f. So such a multiplication operator maps $L^2(\mathbb{R}^d_+)$ to $L^2(\mathbb{R}^d_+)$.

One might guess that at least the quantization of the spatial coordinates gives canonical variables in our one particle theory. However this expectation is incorrect. In the case of the inner product (3.37), we *cannot* quantize the coordinates! The answer to this apparent mystery is that multiplication by a coordinate does not leave the null space $\mathcal{N}$ invariant. On reconsideration, one might expect this; for the commutator $[x_j, C] \neq 0$, and therefore one cannot use the criteria of Proposition 3.4.6. Nevertheless, in later chapters we *do* recover the ordinary canonical coordinates of quantum theory from the properties of the Green's operator C. It was only an illusion that we might be able to do it now. Nevertheless, our extended study of this example gives us many insights that we use throughout our study of quantum fields.

Exercise 3.4.5. *Find a specific counterexample to the possibility to quantize x_j with the inner product (3.37) on $L^2(\mathbb{R}^d_+)$. Namely find a function $f \in \mathcal{N}$ such that $x_j f$ does not lie in $\mathcal{N}$. (For simplicity, here we deal directly with the coordinate, rather than with a bounded function of the coordinate. In the next section we justify the treatment of unbounded operators.)*

3.4.5 Unbounded Operators on $\mathcal{H}_1$

We also want to quantize some operators T which only map a subspace of $L^2(\mathbb{R}^d_+)$ to $L^2(\mathbb{R}^d_+)$. This situation might arise if T is an unbounded operator on $L^2(\mathbb{R}^d)$, and therefore cannot be defined everywhere. Alternatively, the operator T may be bounded on $L^2(\mathbb{R}^d_+)$, but may only map a subspace of $L^2(\mathbb{R}^d_+)$ into $L^2(\mathbb{R}^d_+)$. In either case, we might expect to find an unbounded quantization $\hat{T}$ acting on $\mathcal{H}_1$. Let $\mathfrak{D}(T)$ denote the domain of T, let $\mathfrak{D}(T)_+ = \mathfrak{D}(T) \cap L^2(\mathbb{R}^d_+)$. For the purpose of quantization, we restrict T to a subdomain $\mathfrak{D}(T)_0 \subset \mathfrak{D}(T)_+$. Let $\mathcal{N}_T = \mathfrak{D}(T)_0 \cap \mathcal{N}$.

Domain Assumptions We require certain conditions:

1. The operator T is densely defined on $L^2(\mathbb{R}^d)$.
2. There is a subdomain $\mathfrak{D}(T)_0 \subset \mathfrak{D}(T) \cap L^2(\mathbb{R}^d_+)$ whose quantization ia a domain for $\hat{T}$: namely $\widehat{\mathfrak{D}(T)}_0$ is dense in $\mathcal{H}_1$.
3. T preserves positive times on the subdomain: $T\,\mathfrak{D}(T)_0 \subset L^2(\mathbb{R}^d_+)$.
4. T preserves the null space of the quantization: $T\mathcal{N}_T \subset \mathcal{N}$.

In case the three assumptions above hold, the quantization of an unbounded operator proceeds as in Definition 3.4.5, but with the domain $\mathfrak{D}(\hat{T})$ of $\hat{T}$ equal to $\widehat{\mathfrak{D}(T)}_0$. There is also an analog of Proposition 3.4.6 giving a condition that ensures the domain assumptions above.

Proposition 3.4.9 (A Quantizability Condition in the Unbounded Case). *Suppose that*

- *The operators T and $T^+ = \Theta T^* \Theta$ have a common dense domain $\mathfrak{D} \subset L^2(\mathbb{R}^d)$.*
- *There is a common subdomain $\mathfrak{D}_0 \subset \mathfrak{D} \cap L^2(\mathbb{R}^d_+)$ whose quantization $\widehat{\mathfrak{D}}_0$ is dense in $\mathcal{H}_1$.*
- *Both T and T^+ map $\mathfrak{D}_0$ into $L^2(\mathbb{R}^d_+)$.*
- *$CT = TC$.*

Then T satisfies the Assumptions 1–4 above, and the quantization $\hat{T}$ of T exists.

Exercise 3.4.6. *Show that we can quantize the differentiation operators $\frac{\partial}{\partial x_j}$ on $L^2(\mathbb{R}^d_+)$ yielding $\widehat{\frac{\partial}{\partial x_j}}$, for $1 \leq j \leq d$. In each case:*

i. Verify the domain assumptions above.

ii. Identify each of the the operators on $\mathcal{H}_1$ in how they act on quantum mechanical wave functions.

3.4.6 Quantization Domains

Definition 3.4.10. *A quantization domain $\mathfrak{D}$ is a subspace of $L^2(\mathbb{R}^d_+)$ whose quantization $\hat{\mathfrak{D}}$ is dense in $\mathcal{H}_1$.*

Proposition 3.4.11. (Euclidean Reeh-Schlieder Property) *For any open subset $\mathcal{O} \subset \mathbb{R}^d_+$, the set $\mathfrak{D} = C^\infty(\mathcal{O})$ is a quantization domain.*

Proof. We show that any vector in $\mathcal{H}_1$ perpendicular to $\hat{\mathfrak{D}}$ is zero. This is equivalent to showing that any vector $f \in L^2(\mathbb{R}^d_+)$ perpendicular to $\mathfrak{D}$ in the inner product given by (3.37) is an element of $\mathcal{N}$.

Suppose that $f \in L^2(\mathbb{R}^d_+)$ with $\hat{f} \perp \mathfrak{D}$. For $x \neq 0$, we saw in Proposition 3.2.1 that $C(x)$ is real-analytic. Let g converge to a Dirac measure localized at $x \in \Delta \subset \mathbb{R}^d_+$, and consider the limiting function

$$F(x) = \langle f, \delta_x \rangle_{\mathcal{H}_1} = \langle \Theta f, C\delta_x \rangle_{L^2(\mathbb{R}^d)} \ . \tag{3.78}$$

As $x \in \mathbb{R}^d_+$, and Θf is supported in the negative-time space $\mathbb{R}^d_-$, the function $F(x)$ is real-analytic throughout $x \in \mathbb{R}^d_+$. The hypotheses ensure that

$$\int F(x) g(x) dx = 0 \ , \tag{3.79}$$

for all $g \in L^2(\mathcal{O})$. It follows that $F(x) = 0$ for $x \in \mathcal{O}$. Since $F(x)$ is real analytic for $x \in \mathbb{R}^d_+$, we conclude that $F(x) = 0$ for all $x \in \mathbb{R}^d_+$. As a consequence, for all $g \in L^2(\mathbb{R}^d_+)$,

$$\int F(x) g(x) dx = \langle f, g \rangle_{\mathcal{H}_1} = 0 \ . \tag{3.80}$$

In other words, $f \in \mathcal{N}$. Therefore, $\mathcal{D}(\mathcal{O})$ is dense in $\mathcal{H}_1$ as claimed.

3.4.7 Quantization of Space-Time Rotations

The $d-1$ space-time planes in $\mathbb{R}^d$ have the form $x_j x_d$, with $1 \leq j \leq d-1$. The corresponding rotations $T(R_{jd}(\theta))$ in each of these planes do not leave $\mathbb{R}^d_+$ invariant. However, for each open subset $\mathcal{O} \in \mathbb{R}^d_+$, there is an angle $\theta_0(\mathcal{O}) > 0$ such that all rotations by angles θ for which $|\theta| < \theta_0(\mathcal{O})$ leave $\mathcal{O}$ inside $\mathbb{R}^d_+$. For these operators, one could quantize them on a domain $L^2(\mathcal{O})$, with appropriate $\mathcal{O} \in \mathbb{R}^d_+$.

An alternative approach is to quantize the infinitesimal generators of these transformations. These generators can be treated as unbounded transformations with domain $\mathfrak{D} = C^\infty(\mathbb{R}^d)$ that also leave the subdomain $\mathfrak{D}_0 = C^\infty(\mathbb{R}^d_+)$ invariant.

$$M_{jd} = -i \left(x_j \frac{\partial}{\partial t} - t \frac{\partial}{\partial x_j} \right) . \tag{3.81}$$

Even though this generator is a linear function of the coordinates, which do not have quantizations, the combination generating a rotation does have a quantization. The key fact is that C is invariant under all Euclidean rotations, so M_{jd} commutes with C, and anti-commutes with Θ. Thus we can take the domain of M_{jd} to be $\mathfrak{D} = C_0^\infty(\mathbb{R}^d)$ and the domain $\mathfrak{D}_0 = C^\infty(\mathbb{R}^d_+)$. Both

$$M_{jd} \,, \qquad \text{and } \Theta M_{jd}^* \Theta = -M_{jd} \tag{3.82}$$

map $\mathfrak{D}_0$ into $\mathfrak{D}_0$. Thus M_{jd} has a quantization.

Exercise 3.4.7. *Compute $\widehat{M}_{jd}$. Show that this quantization is identical to the generator M_j found in Exercise* **??***, as one might expect!*

3.5 Poincaré Symmetry from Euclidean Symmetry

In the various sections above, we have studied the Euclidean group $\{R, a\}$ of rotations and translations on $\mathbb{R}^d$. We have quantized its action $T(R, a)$ as a unitary group on $L^2(\mathbb{R}^d)$. The quantization of rotation and translations in the spatial coordinates $\mathbb{R}^{d-1}$ are unitary operators on $\mathcal{H}_1 = \mathcal{F}_1$. On the other hand the quantizations involving space-time rotations or time translation are self adjoint. We recover a unitary group by analytically continuing the time evolution semigroup $e^{-t\omega}$ on $\mathcal{H}_1$ to the unitary time-translation group $e^{it\omega}$ of a particle on $\mathcal{F}_1$. Likewise, we analytically continue the self-adjoint quantization $e^{\theta M_{jd}}$ to the unitary Lorentz boost operator $e^{i\theta M_{jd}}$. In this way a unitary representation of the Poincaré group arises as the analytic continuation on $\mathcal{H}_1$ of the quantization of the unitary representation of the Euclidean group on $L^2(\mathbb{R}^d)$.

3.6 Properties of Matrices and Operators

In this section we review a few properties, mainly associated with inequalities, for linear transformations acting on a Hilbert space. The statement $a \leq b$ has an elementary meaning for real numbers. However the corresponding statement for hermitian matrices has several different possible interpretations, and we emphasize that in general *monotonicity is a tricky business.* One must be very careful about what one means, for monotonic relations of matrices contain many unexpected twists.

In many cases, the central issue arises in the finite-dimensional case and can be illustrated by matrices. Often, similar relations hold for linear transformations on an infinite dimensional Hilbert space, and in many cases these relations follow from finite-dimensional approximation to the relations in infinite dimension. This is a huge subject that not only enters the theory of quantum fields, but related questions occur in partial differential equations, in probability theory, in statistical physics, in information theory, and in the theory of quantum computing. In many cases one wishes to compare notions of entropy or information content. Because of this tremendous diversity, we restrict attention here to questions related to those that arise in this chapter, and also in later chapters.

3.6.1 Operator Monotonicity

Let us begin by stating three possible meanings that two $n \times n$ self-adjoint matrices A and B satisfy $A \leq B$. Each of these notions of monotonicity is useful. However, the three meanings are quite different!

1. **(Monotonicity of Expectations)** Expectations are monotonic in the sense that

$$\langle f, Af \rangle \leq \langle f, Bf \rangle \ , \ \text{ or equivalently } \ 0 \leq \langle f, (B-A)f \rangle \ , \qquad \text{for all } \ f \in \mathcal{H} \ . \tag{3.83}$$

2. **(Spectral Monotonicity)** The ordered eigenvalues $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ satisfy

$$\lambda_i(A) \leq \lambda_i(B) \ , \ \text{ for each } \ i \ . \tag{3.84}$$

3. **(Pointwise Monotonicity)** In a particular basis, every matrix element satisfies

$$A_{ij} \leq B_{ij} \ . \tag{3.85}$$

If the Hilbert space is finite dimensional, or in case the operators A, B are compact, then the minimax principle shows that **Monotonicity of Expectations** ensures **Spectral Monotonicity**. However, the converse of this statement is not true, even for positive matrices! Were it true, any monotone increasing function $h(s)$ would satisfy $h(A) \leq h(B)$, but this is *not* necessarily the case. This fact lies in territory where one can easily jump to an incorrect conclusion. In fact, the following exercise shows that one can find a 2-dimensional counterexample to this statement. It is instructive to keep this fact in mind.

Exercise 3.6.1. *Find an example of* 2×2 *self-adjoint matrices* A, B *for which* $0 \leq A \leq B$, *but for which* $A^2 \not\leq B^2$, *and also for which* $e^{rA} \not\leq e^{rB}$ *for any* $r > 0$.

Hint. Consider $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \epsilon & \epsilon \\ \epsilon & 1+\epsilon \end{pmatrix}$ for certain $0 < \epsilon$ close to zero.

The third meaning of monotonicity **Pointwise Comparison** stands aside from the other two meanings in the following sense: it is a basis-dependent notion, while the other two meanings are basis-independent. Furthermore, Pointwise Comparison does not entail Monotonicity of Expectations. This is clear from the 2×2 matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{3.86}$$

which has eigenvalues ± 1. The matrix elements are positive $0 \leq \sigma_{ij}$, but $0 \not\leq \sigma$.

Pointwise Comparison also has merit. For example, the famous theorem of Perron and Frobenius states: if $0 < A_{ij}$, then there a positive eigenvalue λ_{PF} strictly larger than than the magnitude of any other eigenvalue. The eigenvalue λ_{PF} has multiplicity one, and the corresponding eigenvector f_i can be chosen so $0 < f_i$. One can use this result to establish uniqueness of the ground states of certain quantum theory Hamiltonians.

For bounded operators on $L^2(\mathbb{R}^d)$ one can replace the matrix elements A_{ij} by the kernel $A(x;y)$ of A considered as an integral operator. Thus statements such as $0 < C(x;y)$ in the preceding section state that C is positive in this third sense.

In this work we adapt the following:

Definition 3.6.1. *Let A, B be bounded, self-adjoint transformations acting on a Hilbert space $\mathcal{H}$. Then the statement that A, B are monotonically related $A \leq B$ (without further qualification) means that monotonicity of expectations* (3.83) *holds.*

If A, B are unbounded, self adjoint transformations on $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively, then $A \leq B$ means that $\mathcal{D}(B) \subset \mathcal{D}(A)$ and (3.83) *hold for all $f \in \mathcal{D}(B)$.*

3.6.2 Two Monotonicity Preserving Functions

In spite of the caution above and the explicit counter-examples of Exercise 3.6.1, there are certain monotonicity preserving functions h for matrices! In other words,

$$\text{if } A \leq B , \qquad \text{then } h(A) \leq h(B) . \tag{3.87}$$

While these functions are convex, that property is not sufficient. Two important examples of such monotonicity preserving functions are: $h(t) = -t^{-1}$ (yielding the "monotonicity reversing" property of the inverse), and $h(t) = t^{\alpha}$, for $0 \leq \alpha \leq 1$.

Proposition 3.6.2. *If the spectrum of A is strictly positive, then we have the "monotonicity-reversing" inequality*

$$B^{-1} \leq A^{-1} , \tag{3.88}$$

and also

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (A + \lambda I)^{-1} \, d\lambda , \qquad \text{for } 0 < \alpha < 1 . \tag{3.89}$$

Remark. A useful particular case of this identity in case $0 \leq H$ is

$$(H+I)^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \left(H + I + \lambda I\right)^{-1} d\lambda \, . \tag{3.90}$$

Proof. In case either A or B is the identity, the monotonicity-reversing inequality is an immediate consequence of the spectral representation theorem for the other operator. A self-adjoint transformation with spectrum on the interval $(0,1]$ has an inverse with spectrum on the interval $[1,\infty)$. Our assumption (suppressing ϵ) can be formulated as saying the self-adjoint transformation $B^{-1/2}AB^{-1/2} \leq I$. Thus monotonicity reversing in the special case yields $I \leq B^{1/2}A^{-1}B^{1/2}$. This can also be written $B^{-1} \leq A^{-1}$, so monotone reversing holds in the general case as stated.

Write the spectral resolution of the self-adjoint operator A^{-1} as

$$A^{-1} = \int_\epsilon^\infty \zeta^{-1} dE(\zeta) \, , \tag{3.91}$$

where $dE(\zeta)$ is the spectral measure. The function $\zeta^{-\alpha}$ is an analytic function of ζ in the complex plane with the exception of a cut, that we place along the negative real axis. Then Cauchy's integral formula allows one to evaluate $\zeta^{-\alpha}$ for $\zeta > 0$ as an integral along the cut, yielding

$$\zeta^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} \left(\zeta + \lambda\right)^{-1} d\lambda \, . \tag{3.92}$$

Here $\sin(\pi\alpha)$ comes from the change in the phase of $\lambda^{-\alpha}$ across the cut. Integrating (3.92) with the spectral measure $dE(\zeta)$, we obtain (3.89), completing the proof of the lemma.

Proposition 3.6.3. *Let A, B be self-adjoint operators satisfying $0 \leq A \leq B$. Then*

$$A^\alpha \leq B^\alpha \, , \qquad \textit{for all } 0 \leq \alpha \leq 1 \, . \tag{3.93}$$

Proof. It is no loss of generality to assume $0 < \alpha < 1$, as the case $\alpha = 1$ is given, and the case $\alpha = 0$ is trivial. If A is not strictly positive, replace A by $A(\epsilon) = A + \epsilon$, and $B(\epsilon) = B + \epsilon$, with $0 < \epsilon$. We begin by proving (3.93) for the modified operators.

Using Proposition 3.6.2 for $0 < \alpha < 1$, one can write

$$A^{-\alpha} - B^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} \left((A + \lambda I)^{-1} - (B + \lambda I)^{-1} \right) d\lambda \geq 0 \, . \tag{3.94}$$

Here we use $\sin(\pi\alpha) > 0$, and we also use monotonicity reversing applied to $A + \lambda$ and $B + \lambda$, with $\lambda \geq 0$. Alternatively write (3.94) as $B^{-\alpha} \leq A^{-\alpha}$. Applying monotonicity reversing once more to the inverse of this inequality, we infer $A^\alpha \leq B^\alpha$ as desired. We can rewrite this monotonicity in terms of the spectral resolutions of $A(\epsilon)$ and $B(\epsilon)$, from which the $\epsilon \to 0$ limit of the monotonicity follows. This completes the proof.

3.6.3 The Perron-Frobenius Theorem

Let $0 \leq A$ be a positive self-adjoint transformation. Assume in addition that in some orthonormal basis $\{e_i\}$, A also is pointwise strictly-positive, namely

$$0 < A_{ij} = \langle e_i, Ae_j \rangle \ . \tag{3.95}$$

One says that a vector f is pointwise-positive in this basis $\{e_i\}$, if $0 \leq f_i$ for each i. Likewise f is strictly pointwise-positive if $0 < f_i$ for each i. In case the transformation A is pointwise strictly-positive, then applied to any pointwise-positive vector f, one gets a pointwise strictly-positive vector Af. One also says that A is positivity-increasing.

Analogously, we also consider the case in which $\mathcal{H}$ is an L^2-space of functions $f(x)$ and $0 \leq A$ acts as a bounded integral operator with kernel $A(x;y)$, namely

$$(Af)(x) = \int A(x;y) f(y) dy \ . \tag{3.96}$$

One says that A is pointwise strictly-positive if $0 < A(x;y)$ almost everywhere. We say that a vector f is positive if $0 \leq f(x)$ almost everywhere, and that f is strictly positive if $0 < f(x)$ almost everywhere.

If A is a positive, finite-dimensional matrix, then it is always the case that $\lambda = \|A\|$ is an eigenvalue of A. In the case that A acts on an infinite-dimensional Hilbert space $\mathcal{H}$, we need to assume that $\lambda = \|A\|$ is an eigenvalue. (The transformation A may also have continuous spectrum.)

Proposition 3.6.4 (Perron-Frobenius Theorem). *Assume that A is a positive, bounded transformation on $\mathcal{H}$, and that in a particular basis A is pointwise strictly-positive. Assume also that $\lambda = \|A\|$ be an eigenvalue of A. Then λ is a simple eigenvalue and the corresponding normalized eigenvector f can be chosen to be pointwise strictly-positive.*

Proof. Assume that f is an eigenvector of A corresponding to eigenvalue λ. In the basis for which A is pointwise strictly-positive, the eigenvalue equation becomes

$$\sum_j A_{ij} f_j = \lambda f_i \ , \qquad \text{or} \quad \int A(x;y) f(y) dy = \lambda f(x) \ . \tag{3.97}$$

Thus it is no loss of generality to take f to be pointwise-real. (One says that f is pointwise-real, if all the f_i are real, or if $f(x)$ is real for all x.) For if f is pointwise imaginary, we replace f by if. Write

$$f = f_+ - f_- \ , \qquad \text{and} \ |f| = f_+ + f_- \ . \tag{3.98}$$

where $f_\pm$ are pointwise-positive. Then

$$\begin{aligned} \langle f, Af \rangle &= \langle f_+, Af_+ \rangle + \langle f_-, Af_- \rangle - \langle f_+, Af_- \rangle - \langle f_-, Af_+ \rangle \\ &= \lambda \langle f, f \rangle = \lambda \langle f_+, f_+ \rangle + \lambda \langle f_-, f_- \rangle \ . \end{aligned} \tag{3.99}$$

But also $0 \leq \langle f_+, Af_- \rangle, \langle f_-, Af_+ \rangle$, so

$$\begin{aligned} \lambda \langle f, f \rangle &= \langle f, Af \rangle = \langle f_+, Af_+ \rangle + \langle f_-, Af_- \rangle - \langle f_+, Af_- \rangle - \langle f_-, Af_+ \rangle \\ &\leq \langle f_+, Af_+ \rangle + \langle f_-, Af_- \rangle + \langle f_+, Af_- \rangle + \langle f_-, Af_+ \rangle = \langle |f|, A|f| \rangle \\ &\leq \langle |f|, \|A\| |f| \rangle = \lambda \langle |f|, |f| \rangle \\ &= \lambda \langle f, f \rangle \ . \end{aligned} \tag{3.100}$$

It then follows that

$$\langle f_+, Af_- \rangle = \langle f_-, Af_+ \rangle = 0 \ . \tag{3.101}$$

But A is positivity-increasing, so if $f_+ \neq 0$ then Af_+ is pointwise strictly-positive; and in that case $f_- = 0$. On the other hand, if $f_- \neq 0$, then replace f by $-f$. In either case, we now have f pointwise-positive. But $A \neq 0$ ensures $\lambda \neq 0$, so

$$f = \lambda^{-1} A f \ . \tag{3.102}$$

Therefore as A is positivity increasing, we infer that f is pointwise strictly-positive.

3.7 Reflection Positivity Revisited

In this section we approach reflection positivity from the point of view of boundary conditions on the Laplacian Δ on $\mathbb{R}^d$. In particular, imposing Dirichlet or Neumann boundary conditions on surfaces in $\mathbb{R}^d$ leads to an alternative approach to understanding reflection positivity, and to a larger set of Green's functions that define reflection positive inner products.

3.7.1 Mirror Charges and Classical Green's Functions

One is also interested in self-adjoint Laplacians on $\mathbb{R}^d$ that have boundary conditions on certain surfaces in $\mathbb{R}^d$. Such operators are not translation invariant, by virtue of the surfaces on which one imposes boundary conditions, so they are certainly not Euclidean invariant. However, they play an important role. The simplest example is a Laplacian with Dirichlet and Neumann boundary conditions the surface time-zero surface $x_d = 0$. In the case of this elementary geometry, one can give a formula for the Dirichlet or Neumann Green's function based by the reflection principle.

The Dirichlet Green's function $C_D(x;y) = \left(-\Delta_D + m^2\right)^{-1}(x;y)$ arises from imposing the Dirichlet boundary condition on functions in the domain of Δ_D. This Green's function is no longer translation invariant (the boundary conditions are not translation invariant), so $C_D(x;y)$ depends on both $x+y$ as well as on $x-y$. However, the Green's function still can be interpreted as a potential function, so it obeys the reciprocity condition $C_D(x;y) = C_D(y;x)$.

In particular functions in the domain of Δ_D vanish on the time-zero plane, $f(\vec{x}, 0) = 0$. Since the range of C_D is the domain of Δ_D, this means that $C_D(x;y) = 0$ whenever $x_d = 0$, and by reciprocity $C_D(x;y) = 0$ also if $y_d = 0$. The Dirichlet boundary condition decouples the two sides of the time-zero plane, so one wants

$$C_D(x;y) = \begin{cases} C_D(x;y)\ , & \text{if } x_d y_d \geq 0 \\ 0\ , & \text{if } x_d y_d < 0 \end{cases} . \tag{3.103}$$

Placing a mirror charge of opposite sign at the reflected point in the $x_d = 0$ plane, one arrives at the formula

$$C_D(x;y) = \begin{cases} C(x-y) - C(x-\Theta y)\ , & \text{if } x_d y_d \geq 0 \\ 0\ , & \text{if } x_d y_d < 0 \end{cases} . \tag{3.104}$$

Since x and Θy lie on opposite sides of the time-zero plane, the Green's function $C(x-\Theta y)$ satisfies the homogeneous equation $(-\Delta_x + m^2)\,C(x-\Theta y) = 0$. One can add a solution of the homogeneous equation to the Green's function $C(x-y)$ to obtain a Green's function $C_D(x;y)$ satisfying different boundary conditions. The solution (3.104) satisfies the Green's function equation and also gives the Dirichlet boundary condition. So $C_D(x;y)$ satisfies the Green's function equation away from the boundary of the domain, and it vanishes on the boundary: the time-zero plane and at infinity.

Exercise 3.7.1. *Show that* $0 \leq C_D(x;y)$. Hint: *Use Proposition 3.2.1 and show that if* x *and* y *lie on opposite sides of the time-zero plane, then* $|x-y| < |x-\Theta y|$. *Note that the lengths are equal when* x *or* y *lies in the time zero plane.*

Similarly, one can give the Neumann Green's function C_N. This arises from the Neumann Laplacian, satisfying the boundary condition of vanishing normal derivative to the time-zero plane. An argument similar to that above shows that $0 \leq C_N$ and the Neumann Green's function arises by placing a source mirror charge of the same sign at the time-reflected point,

$$C_N(x;y) = \begin{cases} C(x-y) + C(x-\Theta y)\ , & \text{if } x_d y_d \geq 0 \\ 0\ , & \text{if } x_d y_d < 0 \end{cases} . \tag{3.105}$$

Exercise 3.7.2. *Verify that* (3.105) *satisfies* $C_N(x;y) = C_N(y;x)$. *Furthermore, show that* (3.105) *gives the usual Neumann boundary conditions: the normal component of the gradient* $\nabla_{\vec{x}} C_N(x;y)$ *vanishes at* $x_d = 0$.

From (3.104) and (3.105) and the interpretation of the Laplace operators Δ_D and Δ_N we immediately conclude,

Proposition 3.7.1. *The classical Green's functions satisfy the monotonicity relation*

$$0 \leq C_D(x;y) \leq C_N(x;y)\ . \tag{3.106}$$

Furthermore as operators,

$$0 \leq C_D\ , \qquad \text{and } 0 \leq C_N\ . \tag{3.107}$$

3.7.2 Reflection Positivity & Operator Monotonicity

We denote by Green's operators C, C_D, C_N the linear integral operators on $L^2(\mathbb{R}^d)$ defined by the Green's functions $C(x-y), C_D(x;y), C_N(x;y)$, etc. We see here that reflection positivity of C is equivalent to a monotonic relation between the Dirichlet and Neumann Green's operators, $C_D \leq C_N$. In other words, reflection positivity—which provides the relation between classical potential theory and quantum theory—turns on an intrinsic inequality between different boundary conditions on the Laplacian.

It happens to be the case that both $C_D(x;y) \leq C_N(x;y)$ and also $C_D \leq C_N$. In order to establish the relation to reflection positivity, we translate the last section into operator notation. Let $\mathfrak{p}_\pm$ denote the orthogonal projection of $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d_\pm)$, so $\mathfrak{p}_+ + \mathfrak{p}_- = I$.

Proposition 3.7.2. *The following statements are equivalent:*

1. *The operator C is reflection positive, namely* $0 \leq \mathfrak{p}_\pm \, \Theta C \, \mathfrak{p}_\pm$.

ii. $C_D \leq C_N$.

iii. $0 \leq \mathfrak{p}_\pm \, (C - C_D) \, \mathfrak{p}_\pm$.

iv. $0 \leq \mathfrak{p}_\pm \, (C_N - C) \, \mathfrak{p}_\pm$.

Corollary 3.7.3. *The operator monotonicity inequalities of Proposition 3.7.2.ii–iv hold.*

Proof. Note that $\Theta\mathfrak{p}_+ = \mathfrak{p}_+\Theta = \mathfrak{p}_-$, so $0 \leq \mathfrak{p}_+ \, \Theta C \, \mathfrak{p}_+$ is equivalent to $0 \leq \mathfrak{p}_- \, \Theta C \, \mathfrak{p}_-$. This is the operator statement of reflection positivity in Definition 3.3.1. The corollary follows from the proposition using the fact that we established reflection positivity of C in Proposition 3.3.2.

One can rewrite the identities (3.104) and (3.105) in operator form as

$$\begin{aligned} \mathfrak{p}_\pm C_D \mathfrak{p}_\pm &= \mathfrak{p}_\pm \, (C - C\Theta) \, \mathfrak{p}_\pm \, , & \mathfrak{p}_\pm C_D \mathfrak{p}_\mp = 0 \, , \\ \mathfrak{p}_\pm C_N \mathfrak{p}_\pm &= \mathfrak{p}_\pm \, (C + C\Theta) \, \mathfrak{p}_\pm \, , & \text{and } \mathfrak{p}_\pm C_N \mathfrak{p}_\mp = 0 \, . \end{aligned} \tag{3.108}$$

Recall also that $C\Theta = \Theta C$. Then from (3.108) we infer

$$\mathfrak{p}_\pm \, \Theta C \, \mathfrak{p}_\pm = \mathfrak{p}_\pm \, (C_N - C) \, \mathfrak{p}_\pm = \mathfrak{p}_\pm \, (C - C_D) \, \mathfrak{p}_\pm = \frac{1}{2} \mathfrak{p}_\pm \, (C_N - C_D) \, \mathfrak{p}_\pm \, , \tag{3.109}$$

and

$$\mathfrak{p}_\pm \, (C_N - C_D) \, \mathfrak{p}_\mp = 0 \, . \tag{3.110}$$

It follows from (3.109) that (i) ensures (ii–iv). Conversely (3.109)–(3.110) show that each of (ii–iv) ensure (i).

3.7.3 Reflection Invariance Ensures Monotonicity

It is of interest to investigate Green's operators of the form $C = (-\Delta_{\mathcal{B}} + m^2)^{-1}$, where $\mathcal{B}$ denotes some collection of Dirichlet or Neumann boundary data on various surfaces in $\mathbb{R}^d$. We also use $\mathcal{B}$ to denote these surfaces, as well as to denote the boundary conditions on the surfaces. The Green's operators studied in §3.7.1 correspond to the case of no boundary conditions, which we denote by $\mathcal{B} = \varnothing$ equal to the empty set.

In order to establish the propoerty of reflection positivity, one must first limit oneself to a geometric situation for which the two sides of the reflection plane are mirror images of each other. In more detail, the reflection operator Θ must map the surfaces on which one imposes boundary conditions into themselves; it must also map the specific boundary condition (in this case Dirichlet or Neumann) on one side of the reflection plane into the boundary conditions on the other side. We denote these two restrictions on the bounary conditions $\mathcal{B}$ by

$$\Theta\mathcal{B} = \mathcal{B} \, . \tag{3.111}$$

One can obtain a simple example of a reflection-invariant boundary condition on $\mathbb{R}^d$ by imposing Dirichlet boundary conditions on a dotted hyper-rectangle $\mathcal{B}$ placed symmetrically about the reflection plane $x_2 = 0$. We draw such a configuration in Figure 3.1, where we use the convention of taking time direction as the horizontal coordinate, increasing from left to right. More complicated examples could involve surfaces $\mathcal{B}$ with several components, and the imposition of different sorts of boundary conditions on the different components.

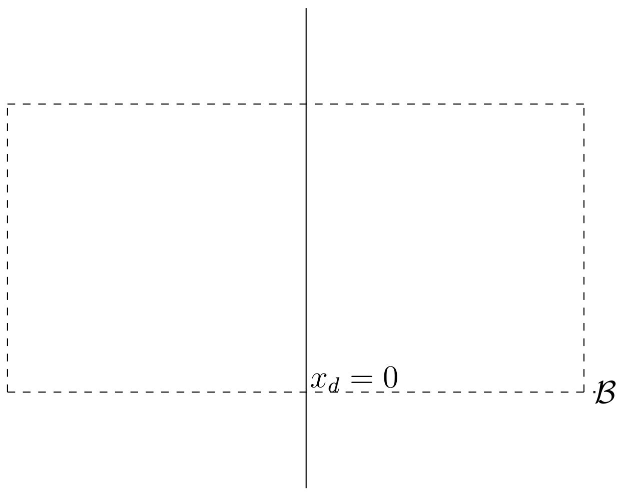

Figure 3.1 An example of reflection-invariant boundary conditions on $\mathcal{B}$.

In addition to the Green's operator $C = (-\Delta_{\mathcal{B}} + m^2)^{-1}$, we also require Green's operators C_D and C_N. These operators are both of the form $\left(-\Delta_{\widetilde{\mathcal{B}}} + m^2\right)^{-1}$, where $\widetilde{\mathcal{B}}$ denotes the same boundary conditions as specified by $\mathcal{B}$, but in addition Dirichlet or Neumann boundary conditions respectively on the time-zero ($x_d = 0$) plane.

Proposition 3.7.4 (Reflection Invariance Ensures Operator Monotonicity). *Assume that C, C_D, C_N have the form above with boundary conditions on $\mathcal{B}$ satisfying $\Theta\mathcal{B} = \mathcal{B}$. Then the Green's operators are monotonic in the operator sense that*

$$C_D \leq C \leq C_N \ . \tag{3.112}$$

Proof. The basic idea is to give a direct proof of the monotonicity inequality

$$C_N^{-1} \leq C^{-1} \leq C_D^{-1} \ , \tag{3.113}$$

or equivalently

$$-\Delta_{\mathcal{B},N} \leq -\Delta_{\mathcal{B}} \leq -\Delta_{\mathcal{B},D} \ . \tag{3.114}$$

and then to use the monotonicity-reversing Lemma 3.6.2.

In case that A and B are unbounded, positive, self-adjoint operators, the form domain $\mathcal{D}_F(A)$ is the closure of the operator domain $\mathcal{D}(A)$ in the norm $\langle f, Af\rangle$. The inequality $A \leq B$ means $\mathcal{D}_F(B) \subset \mathcal{D}_F(A)$ and

$$\langle f, Af\rangle \leq \langle f, Bf\rangle \ , \qquad \text{for all } f \in \mathcal{D}_F(B) \ . \tag{3.115}$$

One can specify the form domain for the Laplacian with Dirichlet or Neumann boundary conditions. Each Laplacian is a local operator, and the three Laplace operators only differ with respect to the boundary conditions on the hyperplane $t = 0$. Hence we need only compare how the three Laplacians act on functions in a neighborhood of the hyperplane $t = 0$. In fact, the Dirichlet Laplacian has fewest functions in its form domain, the free Laplacian is intermediate, and the Neumann Laplacian has the most functions,

$$\mathcal{D}_F(\Delta_{\mathcal{B},D}) \subset \mathcal{D}_F(\Delta_{\mathcal{B}}) \subset \mathcal{D}_F(\Delta_{\mathcal{B},N}) \ . \tag{3.116}$$

For simplicity of notation, let us ignore the boundary conditions on $\mathcal{B}$ and concentrate on the time-zero plane $\Gamma = \{x \colon x_d = 0\}$. The form domain for the Laplacian Δ are the functions

$$\mathcal{D}_F(\Delta) = \{f : f \in L^2, \nabla f \in L^2\} \ . \tag{3.117}$$

As long as $\nabla f \in L^2$, the restriction $f|\Gamma$ is defined, and is an $L^2(\Gamma)$-function. A similar analysis holds for one-sided derivatives. Let $\nabla_{\pm} f$ denote a gradient which is two-sided on the complement of Γ and one-sided in the normal direction to Γ. The Neumann Laplacian has a form domain

$$D_F(\Delta_N) = \{f : f \in L^2, \nabla_{\pm} f \in L^2\} \ . \tag{3.118}$$

These functions may have a jump continuity in the normal direction across Γ. Finally the Dirichlet Laplacian has a form domain

$$\mathcal{D}_F(\Delta_D) = \{f : f \in L^2, \nabla f \in L^2, f|\Gamma = 0\} \ . \tag{3.119}$$

These domains satisfy 3.116, and the forms satisfy 3.114.

The corresponding operators have domains

$$\mathcal{D}(\text{operator}) = \{f : f \in \mathcal{D}_F, |\langle \nabla f, \nabla g\rangle| \leq \text{const.} \|g\|_{L^2}\} \ . \tag{3.120}$$

Integration by parts shows that this ensures for example that that normal derivative of $f \in \mathcal{D}(\Delta_N)$ vanishes. Hence this definition coincides with the normal operator one.

3.7.4 Monotonicity & Reflection Positivity

The Green's operators $C = \left(-\Delta_{\mathcal{B}} + m^2\right)^{-1}$ of §3.7.3 are in general complicated, and explicit formulas exist only in very special cases. For example, we can write the Green's function for the boundary condition illustrated in Figure 3.1 using an infinite series of image charges, reflected through a lattice of hyperplanes converging to infinity. But in general, we do not attempt to give an exact formula for the Green's function arising from a complicated set of reflection-invariant boundaries. However, once we have obtained these Green's operators, claim that the corresponding Dirichlet Green's operator C_D and the Neumann Green's operator C_N with additional Dirichlet or Neumann boundary conditions on the $x_d = 0$ plane.

Proposition 3.7.5. *Let $C\left(-\Delta_{\mathcal{B}} + m^2\right)^{-1}$ be a Green's function for reflection invariant boundary conditions $\Theta\mathcal{B} = \mathcal{B}$. Then as for the elementary case, $\Theta C = C\Theta$, as well as*

$$\begin{aligned} \mathfrak{p}_\pm C_D \mathfrak{p}_\pm &= \mathfrak{p}_\pm \left(C - C\Theta\right) \mathfrak{p}_\pm \,, & \mathfrak{p}_\pm C_D \mathfrak{p}_\mp = 0 \,, \\ \mathfrak{p}_\pm C_N \mathfrak{p}_\pm &= \mathfrak{p}_\pm \left(C + C\Theta\right) \mathfrak{p}_\pm \,, & \text{and } \mathfrak{p}_\pm C_N \mathfrak{p}_\mp = 0 \,. \end{aligned} \tag{3.121}$$

Exercise 3.7.3. *Verify Proposition 3.7.5.*

As a consequence of the representations in Proposition 3.7.5, along with the operator monotonicity already proved in Proposition 3.7.4, we follow the proof of Proposition 3.7.2 to obtain the following:

Proposition 3.7.6. *Let $C = \left(-\Delta_{\mathcal{B}} + m^2\right)^{-1}$ be the Green's function for reflection invariant boundary conditions $\Theta\mathcal{B} = \mathcal{B}$. Then C is reflection positive with respect to Θ. More generally, Proposition 3.7.2 holds with the more general operators C, C_D, C_N considered here.*

3.8 Space-Time Compactification

Let us reinvestigate the ideas in the previous sections for a compact space-time. We consider here the simplest compactification of $\mathbb{R}^d$, replacing it by the torus T^d with periods

$$\ell = (\ell_1, \ell_2, \ldots, \ell_d) \ . \tag{3.122}$$

It is convenient to single out the time coordinate, which we denote by $t = x_d$, and the time period, which we denote by $\ell_d = \beta$. We parameterize the time by the interval $-\beta/2 \leq t \leq \beta/2$, with periodic boundary conditions on functions that are continuous in t, namely $f(\vec{x}, -\beta/2) = f(\vec{x}, \beta/2)$. We allow some periods $\ell_1, \ldots, \ell_{d-1}$ to be infinite, in which case those coordinate directions are not compactified. We write

$$\mathbb{T}^d = \mathbb{T}^{d-1} \times [-\beta/2, \beta/2] \ , \tag{3.123}$$

It is often useful to consider the imbedding of $L^2(T^d)$ as

$$L^2(\mathbb{T}^{d-1} \times [-\beta/2, \beta/2]) \subset L^2(\mathbb{T}^{d-1} \times \mathbb{R}) \ . \tag{3.124}$$

The symmetry group $\mathbb{T}^d$ includes translations (but we lose the analog of Euclidean rotations). There are many other possibilities, but we do not consider them here.[2] Let us introduce the (basis-dependent) commutative multiplication of vectors. We call this the pointwise product and use $*$ to denote this multiplication. The product $a * p$ of two vectors $a, p \in \mathbb{R}^d$ is a vector $a * p = p * a$ with components

$$(p * a)_i = p_i a_i \; . \tag{3.125}$$

The torus $\mathbb{T}^d$ is dual to a lattice $\mathfrak{K}^d$, which we also call the momentum space lattice. Explicitly

$$\mathfrak{K}^d = \{p : p * \ell \in 2\pi \mathbb{Z}^d\} \; . \tag{3.126}$$

This duality arises in a natural way between the Hilbert space $L^2(\mathbb{R}^d)$ of functions $f(x)$ on $\mathbb{T}^d$ with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \overline{f(x)} g(x) dx \; , \tag{3.127}$$

and the space $l^2(\mathfrak{K}^d)$ of sequences $\tilde{f}(p)$ on $\mathfrak{K}^d$ with the inner product

$$\left\langle \tilde{f}, \tilde{g} \right\rangle_{l^2(\mathfrak{K}^d)} = \sum_{p \in \mathfrak{K}^d} \overline{\tilde{f}(p)} \tilde{g}(p) \; . \tag{3.128}$$

Fourier transformation $\mathfrak{F}$ maps $L^2(\mathbb{T}^d)$-functions to $l^2(\mathfrak{K}^d)$-series. It has the explicit form

$$(\mathfrak{F} f)(p) = \tilde{f}(p) = \int_{\mathbb{T}^d} f(x) e^{-ipx} dx \; , \qquad \text{with } p \in \mathfrak{K}^d \; . \tag{3.129}$$

Then the inverse transformation is

$$\left(\mathfrak{F}^{-1} \tilde{f}\right)(x) = \frac{1}{\mathfrak{v}^{1/2}} \sum_{p \in \mathfrak{K}^d} \tilde{f}(p) e^{ipx} \; , \tag{3.130}$$

where $\mathfrak{v}$ denotes the volume of the d-torus, $\mathfrak{v} = \prod_{i=1}^{d} \ell_i$. With this normalization, $\mathfrak{F}$ is an isomorphism as a map between $L^2(\mathbb{T}^d)$ and $l^2(\mathfrak{K}^d)$, and

$$\langle f, g \rangle_{L^2(\mathbb{T}^d)} = \langle \mathfrak{F} f, \mathfrak{F} g \rangle_{l^2(\mathfrak{K}^d)} \; . \tag{3.131}$$

3.8.1 Periodic Green's Function

Here we define the Green's function and Green's operator for a Laplace operator that is periodic in space and time. In the limit that a period ℓ_j (either in a spatial direction or the time direction) becomes infinite, this Laplacian converges to the ordinary free Laplacian with respect to that coordinate direction.

[2] Even in $d = 2$ one might consider the multitude of space-times given by Riemann surfaces or other surfaces with singularities. The care of Riemann surfaces that are Shottky doubles, with identical components on each side of the time-zero plane has been studied in some detail.

The Laplacian $\Delta_{\mathbb{T}}$ on $\mathbb{T}^d$, and the corresponding Green's operator $C_{\mathbb{T}} = (-\Delta_{\mathbb{T}} + m^2)^{-1}$ which acts on $L^2(\mathbb{T}^d)$. Here

$$\Delta_{\mathbb{T}} = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} . \tag{3.132}$$

We also consider $\mathbb{T}^d$ as a d-dimensional rectangle $\mathcal{I}^d$ imbedded in $\mathbb{R}^d$ with the j^{th} side length ℓ_j, and with opposite faces identified. We can also regard functions $f \in L^2(\mathbb{T}^d)$ as functions on $\mathbb{R}^d$ satisfy periodic boundary conditions

$$f(x + n * \ell) = f(x) , \qquad \text{for } n \in \mathbb{Z}^d . \tag{3.133}$$

The $L^2(\mathbb{T}^d)$ inner product is obtained by restriction of $L^2(\mathbb{R}^d)$ to $L^2(\mathcal{I}^d)$. Using this representation we read off the formula for the Green's operator $C_{\mathbb{T}^d}$ and the Green's function $C_{\mathbb{T}^d}(x; y)$.

Proposition 3.8.1. *The Green's functions for the periodic Laplacian on $L^2(\mathbb{T}^d)$ is*

$$C_{\mathbb{T}}(x - y) = \sum_{n \in \mathbb{Z}^d} C(x - y + n * \ell) . \tag{3.134}$$

One can denote the corresponding Green's operator as

$$C_{\mathbb{T}} = \sum_{n \in \mathbb{Z}^d} T_{n*\ell}\, C , \qquad \textit{restricted to} \quad L^2(\mathcal{I}^d) , \tag{3.135}$$

Proof. Note that the sum (3.134) converges due to the exponential decay of the free Green's function C on $L^2(\mathbb{R}^d)$. This is a consequence of the assumption of positive mass, $m > 0$. Consequently, as any period ℓ_j tends to infinity, the sum in that coordinate direction tends to the single term without translation, yielding free boundary conditions in that coordinate direction. The resulting sum satisfies the Green's function equation and also has the appropriate periodics. Therefore it is the periodic Green's function. The representation of the Green's operator is just an interpretation of the formula for the Green's function.

3.8.2 Periodic Time Reflection

One would also like to introduce time-reflection on $\mathbb{T}^d$ or $\mathcal{I}^d$, and this requires the proper interpretation of time reflection. We regard the time coordinate in $\mathcal{I}^d$ as a periodic interval in the line, parameterized by $-\beta/2 \leq t \leq \beta/2$. When we take the time-reflection to occur relative to $t = 0$. The "positive-time" and "negative-time" time half-spaces by

$$\mathcal{I}^d_+ = \mathcal{I}^{d-1} \times [0, \beta/2] , \qquad \text{and } \mathcal{I}^d_- = \mathcal{I}^{d-1} \times [-\beta/2, 0] . \tag{3.136}$$

Then $\mathcal{I}^d = \mathcal{I}^d_- \cup \mathcal{I}^d_+$. The time reflection acts on the d-brane $\mathcal{I}^d$ as

$$\Theta(\vec{x}, t) = (\vec{x}, -t) , \tag{3.137}$$

so

$$\Theta \mathcal{I}^d_{\pm} = \mathcal{I}^d_{\mp} \,. \tag{3.138}$$

The Jacobian of the change of coordinates $x \to \Theta x$ on $\mathcal{I}^d$ has magnitude 1, so the isomorphism Θ of $\mathcal{I}^d$ defines a unitary transformation of $L^2(\mathcal{I}^d)$ into itself. We also use the symbol Θ to denote this unitary. Furthermore

$$L^2(\mathcal{I}^d) = L^2(\mathcal{I}^d_-) \oplus L^2(\mathcal{I}^d_+) \,, \tag{3.139}$$

the unitary Θ has the property that

$$\Theta L^2(\mathcal{I}^d_{\pm}) = L^2(\mathcal{I}^d_{\mp}) \,. \tag{3.140}$$

Exactly the same relations hold on the torus $\mathbb{T}^d$. We define positive-time and negative-time subspaces

$$\mathbb{T}^d_+ = \mathbb{T}^{d-1} \times [0, \beta/2] \,, \qquad \mathbb{T}^d_- = \mathbb{T}^{d-1} \times [-\beta/2, 0] \,, \qquad \text{and } \mathbb{T}^d = \mathbb{T}^d_- \cup \mathbb{T}^d_+ \,, \tag{3.141}$$

and a time-reflection operator Θ such that

$$\Theta \mathbb{T}^d_{\pm} = \mathbb{T}^d_{\mp} \,, \qquad \text{and } \Theta \mathfrak{S} = \mathfrak{S} \,, \tag{3.142}$$

where $\mathfrak{S} = \mathbb{T}^d_- \cap \mathbb{T}^d_+$ is the time-reflection invariant $(d-1)$-dimensional surface which one might also call a p-brane. Also

$$L^2(\mathbb{T}^d) = L^2(\mathbb{T}^d_-) \oplus L^2(\mathbb{T}^d_+) \,. \tag{3.143}$$

The operator Θ is unitary and idempotent on $L^2(\mathbb{T}^d)$ and has the property that

$$\Theta L^2(\mathbb{T}^d_{\pm}) = L^2(\mathbb{T}^d_{\mp}) \,. \tag{3.144}$$

One can picture this geometric configuration by imbedding the torus $\mathbb{T}^d$ in $\mathbb{R}^{2d}$. Represent $\mathbb{T}^d$ by $\mathcal{I}^d$ with opposite sides identified. Denote the coordinates on $\mathbb{T}^d$ by x and the coordinates on $\mathbb{R}^{2d}$ by y. Orient $\mathbb{T}^d$ in $\mathbb{R}^{2d}$ so that the first coordinate is parameterized by $x_1 \in [-\ell_1/2, \ell_1/2]$ with the endpoints identified; in the imbedding this coordinate increases counter-clockwise around a circle S^1 lying in the plane $y_1\, y_2$. The circle has circumference ℓ_1 and we place its center at the origin of $\mathbb{R}^{d+1}$. Represent the j^{th} coordinate x_j of $\mathbb{T}^d$ as a circle of circumference ℓ_j centered at the origin of the $y_{2j-1}\, y_{2j}$-plane. Parameterize the time coordinate $t = x_d$ by the interval $t \in [-\beta/2, \beta/2]$. Then Θ satisfies (3.137).

We draw this schematically Figure 3.2, in which we illustrate a cross-section of the plane $y_{2d-1}\, y_{2d}$. In the case $d = 1$, this gives the exact picture. For convenience, we depict the coordinate y_{2d} horizontally and increasing to the right, while we draw the coordinate y_{2d-1} in the vertical direction. We take the point $t = \mp\beta/2$ to lie on the positive y_{2d}-axis and the point $t = 0$ to lie on the negative y_d-axis. The coordinates $y_1, \ldots, y_{2d-2}$ are orthogonal to the cross-section that we illustrate.

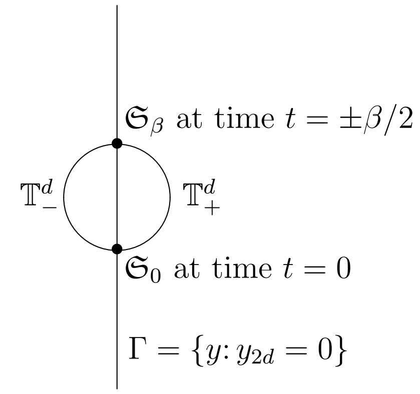

Figure 3.2 The imbedding of $\mathbb{T}^d$ in $\mathbb{R}^{2d}$ and the invariant p-brane $\mathfrak{S} = \mathbb{T}^d \cap \Gamma = \mathfrak{S}_0 \cup \mathfrak{S}_\beta$.

One might call the $(d-1)$-dimensional, time-reflection invariant set $\mathfrak{S}$ a $(d-1)$-dimensional p-brane. In our imbedding, the set $\mathfrak{S}$ arises as the intersection of T^d with the d-dimensional plane $\Gamma \subset \mathbb{R}^{2d}$,

$$\Gamma = \{y : y_{2d} = 0\} \; . \tag{3.145}$$

This plane Γ divides $\mathbb{T}^d$ into the two parts $\mathbb{T}^d_+$ and $\mathbb{T}^d_-$ that are the "positive-time" and "negative-time" subspaces—even though time is periodic. The plane Γ intersects $\mathbb{T}^d$ two times, at time $t = 0$ and at time $\pm\beta/2$. Thus $\mathfrak{S}$ has two disjoint components $\mathfrak{S} = \mathfrak{S}_0 \cup \mathfrak{S}_\beta$, a $(d-1)$-torus $\mathfrak{S}_0 = \mathbb{T}^{d-1}$ at time $t = 0$ and a second $(d-1)$-torus $\mathfrak{S}_\beta = \mathbb{T}^{d-1}$ at time $t = \mp\beta/2$. Each component is time-reflection invariant,

$$\Theta\mathfrak{S}_0 = \mathfrak{S}_0 \; , \qquad \text{and} \qquad \Theta\mathfrak{S}_\beta = \mathfrak{S}_\beta \; . \tag{3.146}$$

3.8.3 Reflection Positivity on $\mathbb{T}^d$

Now we use the time-reflection Θ of §3.8.2 defined on $\mathbb{T}^d$, which leaves $\mathfrak{S}$ invariant, and the periodic Green's function $C_\mathbb{T}$ of §3.8.1. Remark that $C_\mathbb{T}$ is time-reflection invariant,

$$[C_\mathbb{T}, \Theta] = 0 \; . \tag{3.147}$$

Define a form $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{T},1}}$ on the subspace of functions $L^2(\mathbb{T}^d_+) \subset L^2(\mathbb{T}^d)$ by

$$\langle f, f \rangle_{\mathcal{H}_{\mathbb{T},1}} = \langle f, \Theta C_\mathbb{T} f \rangle_{L^2(\mathbb{T}^d)} \; , \tag{3.148}$$

which one can also write as

$$\langle f, f \rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle f, \Theta \left(-\Delta_\mathbb{T} + m^2 \right)^{-1} f \right\rangle_{L^2(\mathbb{T}^d)} \; . \tag{3.149}$$

In other words,

Proposition 3.8.2. *On the Hilbert space $L^2(\mathbb{T}^d_+)$, the operator $C_{\mathbb{T}}$ is reflection positive with respect to Θ. In other words, the form $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{T},1}}$ is positive semi-definite, namely*

$$0 \le \langle f, f\rangle_{\mathcal{H}_{\mathbb{T},1}} = \langle f, \Theta C_{\mathbb{T}} f\rangle_{L^2(\mathbb{T}^d)} , \qquad \textit{for all } f \in L^2(\mathbb{T}^d_+) . \tag{3.150}$$

Proof. We show that reflection positivity for $C_{\mathbb{T}^d}$ on $L^2(\mathbb{T}^d_+)$ is a consequence of reflection positivity of C on $L^2(\mathbb{R}^d_+)$, established in Proposition 3.3.2. Use the representation of functions in $\mathbb{T}^d$ as functions on $\mathbb{T}^{d-1}\times\mathbb{R}$, periodic in time with period β and with $-\beta/2 \le t \le \beta/2$ being the domain for the $L^2(\mathbb{T}^d)$ inner product. Define the spatially-periodic Green's function

$$C_P(x-y) = \sum_{\vec{n}\in\mathbb{Z}^{d-1}} C(x-y+(\vec{n}*\vec{\ell},0)) , \qquad \text{so } C_{\mathbb{T}^d}(x-y) = \sum_{j\in\mathbb{Z}} C_P(x-y+(\vec{0},j\beta)) . \tag{3.151}$$

The operator C_P acts on $L^2(\mathbb{T}^{d-1}\times\mathbb{R})$. Since C is reflection positive, it C commutes with Θ and with spatial translations $T_{\vec{n}*\vec{\ell}}$. But $T_{\vec{n}*\vec{\ell}}$ also commutes with Θ. From this we infer that C_P is reflection positive with respect to Θ.

For shorthand, write

$$T_{\vec{x}} = T_{(\vec{x},0)} , \qquad \text{and } T_t = T_{(\vec{0},t)} . \tag{3.152}$$

Hence we need only analyze the sum in (3.135) over translation in the time direction. For $f \in L^2(\mathbb{T}^d_+)$, write

$$\langle f, f\rangle_{\mathcal{H}_{\mathbb{T},1}} = \sum_{j\in\mathbb{Z}} \langle f, \Theta T_{j\beta} C_P f\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} . \tag{3.153}$$

We claim that each individual term in the sum (3.153) is positive. For a term with $j \ge 0$, use the fact that $T_{j\beta}$ commutes with C_P and $T_{j\beta}\Theta = \Theta T_{-j\beta}$ to give

$$\langle f, \Theta T_{j\beta} C_P f\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} = \left\langle T_{j\beta/2} f, \Theta C_P T_{j\beta/2} f\right\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} \ge 0 . \tag{3.154}$$

Here we use $T_{j\beta/2} f \in L^2(\mathbb{R}^d_+)$ and that C_P is reflection positive with respect to Θ.

On the other hand, for terms with $j < 0$,

$$\begin{aligned}\langle f, \Theta C_P T_{j\beta} f\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} &= \left\langle T_{-j\beta/2}\Theta f, T_{j\beta/2} C_P f\right\rangle_{L^2(\mathbb{T}^d)} \\ &= \left\langle \left(T_{-j\beta/2}\Theta f\right), \Theta C_P \left(T_{-j\beta/2}\Theta f\right)\right\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} \ge 0 .\end{aligned} \tag{3.155}$$

In the final step we use the fact that $f \in \mathbb{T}^{d-1}\times\mathcal{I}_+$; therefore the translates $\left(T_{-j\beta/2}\Theta f\right)$ in question with $j \le -1$ lie in $L^2(\mathbb{T}^{d-1}\times\mathbb{R}_+)$. The positivity of (3.155) then follows from the reflection positivity of C_P with respect to Θ. Combining the results of (3.154)–(3.155), we conclude that each term in (3.153) is positive, and the proof is complete in this case. The general case follows by using $\omega_{\mathbb{T}}$ as the appropriate energy operator on $\mathbb{T}^{d-1}$.

We can extend this computation in a straight-forward way to give

Proposition 3.8.3. *For $f, g \in L^2(\mathbb{T}^d_+)$ and $0 \le t \le \beta$, we compute $\langle f, T_t g\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \hat{f}, \hat{T}_t \hat{g}\right\rangle_{\mathcal{H}_{\mathbb{T},1}}$ as*

$$\langle f, T_t g\rangle_{\mathcal{H}_{\mathbb{T},1}} = \sum_{j=0}^{\infty} \left(\langle f, \Theta C_P T_{t+j\beta}\, g\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} + \left\langle \left(T_{\beta/2}\Theta f\right), \Theta C_P T_{t+j\beta} \left(T_{\beta/2}\Theta f\right)\right\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} \right) \tag{3.156}$$

3.8.4 Quantization on $\mathbb{T}^d$ and the Role of $\mathfrak{S} = \Theta\mathfrak{S}$

We use the reflection-positive form $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{T},1}}$ on $L^2(\mathbb{T}^d_+)$ to define a Hilbert space of quantum-mechanical states and quantized operators acting on these states. This is the space

$$\mathcal{H}_{\mathbb{T},1} = L^2(\mathbb{T}^d_+)/\mathcal{N} = \{f + g : f \in L^2(\mathbb{T}^d_+) \ , \ \text{and} \ g \in \mathcal{N}\} \ , \tag{3.157}$$

where $\mathcal{N}$ is the null space of $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{T},1}}$.

Introduce the Sobolev space $\mathfrak{H}_{-1}(\mathcal{O};\mathbb{T}^d)$ in analogy with (3.158), as the Hilbert space of generalized functions f on $\mathbb{T}^d$ of the form

$$\mathfrak{H}_{-1}\left(\mathcal{O};\mathbb{T}^d\right) = \{f : C_{\mathbb{T}}^{1/2} f \in L^2(\mathbb{T}^d) \ , \ \text{and support} \ f \subset \mathcal{O}\} \ . \tag{3.158}$$

This space is a Hilbert space with the inner product

$$\langle f, g\rangle_{\mathfrak{H}_{-1}} = \left\langle C_{\mathbb{T}}^{1/2} f, C_{\mathbb{T}}^{1/2} g\right\rangle_{L^2(\mathbb{T}^d)} \ . \tag{3.159}$$

One can follow the construction of the quantization map $\wedge$ in §3.4, except we do not have as many symmetries of $\mathbb{T}^d_+$ as of $\mathbb{R}^d_+$. We obtain

$$\wedge\colon \mathfrak{H}_{-1}(\mathbb{T}^d_+) \mapsto \mathcal{H}_{\mathbb{T},1} \ , \qquad \text{or} \qquad \mathcal{H}_{\mathbb{T},1} = \left(\mathfrak{H}_{-1}(\mathbb{T}^d_+)\right)^{\wedge} \ . \tag{3.160}$$

We begin with the identification of the this Hilbert space as a Sobolev space. Let $\nabla^2_{\mathbb{T}^{d-1}}$ denote the Laplace operator on the $(d-1)$-dimensional torus $\mathbb{T}^{d-1}$, and let $\omega_{\mathbb{T}} = \left(\nabla^2_{\mathbb{T}^{d-1}} + m^2\right)^{1/2}$. Define the Sobolev space $\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})$ as the Hilbert space of generalized functions $\mathfrak{f}$ on $\mathbb{T}^{d-1}$ with the inner product

$$\langle \mathfrak{f}, \mathfrak{g}\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} = \left\langle (2\omega_{\mathbb{T}})^{-1/2}\,\mathfrak{f}, (2\omega_{\mathbb{T}})^{-1/2}\,\mathfrak{g}\right\rangle_{L^2(\mathbb{T}^{d-1})} \ . \tag{3.161}$$

Note that generalized functions $\mathfrak{f}\otimes\delta_s$ with $\mathfrak{f}\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})$ and localized at a sharp time $s \in [0, \beta/2]$ are elements of , $\mathfrak{H}_{-1}(\mathbb{T}^d_+)$.

Proposition 3.8.4 (Identification of the Inner Product). *The quantization* $\left(\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})\otimes\delta_t\right)^{\wedge}$ *of* $\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})\otimes\delta_t \subset \mathfrak{H}_{-1}(\mathbb{T}^d_+)$ *for any fixed* $t \in [0, \beta/2]$ *is dense in* $\mathcal{H}_{\mathbb{T},1}$*. Furthermore, the scalar product of the quantization of two such functions of the form* $\mathfrak{f}\otimes\delta_t$ *is*

$$\left\langle (\mathfrak{f}\otimes\delta_t)^{\wedge}, (\mathfrak{g}\otimes\delta_s)^{\wedge}\right\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \mathfrak{f}, e^{-(t+s)\omega_{\mathbb{T}}}\coth(\beta\omega_{\mathbb{T}}/2)\,\mathfrak{g}\right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} \ . \tag{3.162}$$

Remark 3.8.5. *The norm of the quantization of sharp-time functions is equivalent to the* $\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})$ *norm, in the sense that there is a constant* M *such that*

$$\|\mathfrak{f}\|_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} \le \left\|(\mathfrak{f}\otimes\delta_t)^{\wedge}\right\|_{\mathcal{H}_{\mathbb{T},1}} \le M\,\|\mathfrak{f}\|_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} \ . \tag{3.163}$$

In fact using (3.162) *we infer that the norm of sharp-time function is*

$$\left\|(\mathfrak{f}\otimes\delta_t)^\wedge\right\|_{\mathcal{H}_{\mathbb{T},1}} = \langle \mathfrak{f}, \coth(\beta\omega_{\mathbb{T}}/2)\,\mathfrak{f}\rangle^{1/2}_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} \ . \tag{3.164}$$

As $m \le \omega_{\mathfrak{T}}$*, the spectral theorem shows that*

$$I \le \coth(\beta\omega_{\mathbb{T}}/2) \le \coth(\beta m/2) = M^2\ , \tag{3.165}$$

from which the remark follows.

Remark 3.8.6. *This result shows that the quantization* $\hat{T}_t$ *of time translation* T_t *for* $0 \le t \le \beta/2$ *is the self-adjoint contraction* $e^{-t\omega_{\mathbb{T}}}$*. These operators extend to a semi-group for all* $t \ge 0$*.*

Remark 3.8.7. *A new feature of periodic time is the possibility that the Hilbert spaces generated by the quantization of functions localized on each of the disjoint components of* $\mathfrak{S}$ *are distinct. However, the two components of* $\mathfrak{S}$ *arise from localization at time* $t=0$ *and at time* $t=\beta/2$*. The proposition shows that quantization of functions* f *localized on each of the two disjoint components* $\mathfrak{S}_0$ *and* $\mathfrak{S}_\beta$ *of* $\mathfrak{S}$ *yield a dense set of the entire Hilbert space* $\mathcal{H}_{\mathbb{T},1} = \left(\mathfrak{H}_{-1}(\mathbb{T}^d_+)\right)^\wedge$*.*

Proof. First consider the case with all spatial periods infinite, so $\mathbb{T}^{d-1}$ becomes $\mathbb{R}^{d-1}$, and $\mathbb{T}^d = \mathbb{R}^{d-1}\times[-\beta/2,\beta/2]$. Suppose $t \ge 0$. We use result of Proposition 3.8.3 to compute the matrix element $\langle f, T_t g\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \hat{f}, \hat{T}_t\hat{g}\right\rangle_{\mathcal{H}_{\mathbb{T},1}}$. For $f, g\in\mathfrak{H}_{-1}(\mathbb{T}^d_+)$,

$$\begin{aligned}
\left\langle \hat{f}, \hat{T}_t\hat{g}\right\rangle_{\mathcal{H}_{\mathbb{T},1}} &= \langle f, T_t g\rangle_{\mathcal{H}_{\mathbb{T},1}} \\
&= \sum_{j=0}^{\infty}\left(\langle f, \Theta C_P T_{t+j\beta}\, g\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})} + \left\langle \left(T_{\beta/2}\Theta f\right), \Theta C_P T_{t+j\beta}\left(T_{\beta/2}\Theta g\right)\right\rangle_{L^2(\mathbb{T}^{d-1}\times\mathbb{R})}\right) \\
&= \sum_{j=0}^{\infty}\left\langle \hat{f}, e^{-(t+j\beta)\omega_{\mathbb{T}}}\hat{g}\right\rangle_{\mathcal{H}_{\mathbb{T},1}} + \sum_{j=0}^{\infty}\left\langle \left(T_{\beta/2}\Theta f\right)^\wedge, e^{-(t+j\beta)\omega_{\mathbb{T}}}\left(T_{\beta/2}\Theta g\right)^\wedge\right\rangle_{\mathcal{H}_{\mathbb{T},1}} \\
&= \left\langle \hat{f}, e^{-t\omega_{\mathbb{T}}}\left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1}\hat{g}\right\rangle_{\mathcal{H}_{\mathbb{T},1}} \\
&\quad + \left\langle \left(T_{\beta/2}\Theta f\right)^\wedge, e^{-t\omega_{\mathbb{T}}}\left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1}\left(T_{\beta/2}\Theta g\right)^\wedge\right\rangle_{\mathcal{H}_{\mathbb{T},1}}\ .
\end{aligned} \tag{3.166}$$

Note that $T_{\beta/2}\Theta\mathbb{T}^d_+ \subset \mathbb{T}^d_+$, so the function $T_{\beta/2}\Theta f$ has a quantization.

Now let us specialize to functions $f = \mathfrak{f}\otimes\delta$, $g = \mathfrak{g}\otimes\delta$ and t replaced by $t+s$, corresponding to localizing f at time t and g at time s. We simplify the two terms on the right of (3.166). As in the proof of Proposition 3.4.4, the first term is just

$$\left\langle \hat{\mathfrak{f}}, e^{-(t+s)\omega_{\mathbb{T}}}\left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1}\hat{\mathfrak{g}}\right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})}\ . \tag{3.167}$$

The second term simplifies as both f and g are localized at time zero, and thus time reflection invariant. This term then becomes

$$\left\langle \hat{\mathfrak{f}}, e^{-(t+s+\beta)\omega_{\mathbb{T}}} \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} . \tag{3.168}$$

Adding these two results yields the inner products (3.162).

We now establish that the states at $t = 0$ span $\mathcal{H}_{\mathbb{T},1} = \left(\mathfrak{H}_{-1}(\mathbb{T}^d_+)\right)^\wedge$. Write $f \in \mathfrak{H}_{-1}(\mathbb{T}^d_+)$ as a superposition of functions localized at time $s \in [0, \beta/2]$,

$$f = \int_0^{\beta/2} \mathfrak{f}_s \otimes \delta_s ds , \qquad \text{where } \mathfrak{f}_s(\vec{x}) = f(\vec{x}, s) . \tag{3.169}$$

Likewise,

$$T_{\beta/2}\Theta f = \int_0^{\beta/2} \mathfrak{f}_{\beta/2-s} \otimes \delta_s ds = \int_0^{\beta/2} \mathfrak{f}_s \otimes \delta_{\beta/2-s} ds . \tag{3.170}$$

Then we can use the contributions to the sharp-time inner product to write

$$\begin{aligned} \left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_{\mathbb{T},1}} &= \int_0^{\beta/2} \int_0^{\beta/2} \left\langle \hat{\mathfrak{f}}_s, e^{-(s+s')\omega_{\mathbb{T}}} \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_{s'} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} ds ds' \\ &\quad + \int_0^{\beta/2} \int_0^{\beta/2} \left\langle \hat{\mathfrak{f}}_s, e^{-(\beta-s-s')\omega_{\mathbb{T}}} \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_{s'} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} ds ds' . \end{aligned} \tag{3.171}$$

If we write

$$\hat{\mathfrak{f}}_+ = \int_0^{\beta/2} e^{-s\omega_{\mathbb{T}}} \hat{f}_s ds , \qquad \text{and } \hat{\mathfrak{f}}_- = \int_0^{\beta/2} e^{-(\beta/2-s)\omega_{\mathbb{T}}} \hat{f}_s ds , \tag{3.172}$$

then we have

$$\left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \hat{\mathfrak{f}}_+, \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_+ \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} + \left\langle \hat{\mathfrak{f}}_-, \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_- \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} . \tag{3.173}$$

Now we localize g to a fixed time $t \in [0, \beta/2]$, by taking it of the form $g_t = \mathfrak{g} \otimes \delta_t$ with $\mathfrak{g} \in \mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})$. The inner product of such a g_t with a general $f \in \mathfrak{H}_{-1}(\mathbb{T}^d_+)$ becomes

$$\begin{aligned} \left\langle \hat{f}, \hat{g}_t \right\rangle_{\mathcal{H}_{\mathbb{T},1}} &= \left\langle \hat{f}, (\mathfrak{g} \otimes \delta_t)^\wedge \right\rangle_{\mathcal{H}_{\mathbb{T},1}} \\ &= \left\langle \hat{\mathfrak{f}}_+, e^{-t\omega_{\mathbb{T}}} \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} \\ &\quad + \left\langle \hat{\mathfrak{f}}_-, e^{-(\beta/2-t)\omega_{\mathbb{T}}} \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} . \end{aligned} \tag{3.174}$$

Let us assume that functions of the form $g_t \in \left(\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1}) \otimes \delta_t\right)^{\wedge}$, for fixed $t \in [0, \beta/2]$, do not span $\mathcal{H}_{\mathbb{T},1}$. In this case there is a function $f \in \mathfrak{G}_{-1}(\mathbb{T}^d_+) \notin \mathcal{N}$ orthogonal to such g_t, so (3.174) vanishes for all g_t. In this case, choose

$$\hat{\mathfrak{g}} = e^{-t\omega_{\mathbb{T}}}\hat{\mathfrak{f}}_+ + e^{-(\beta/2-t)\omega_{\mathbb{T}}}\hat{\mathfrak{f}}_- \ . \tag{3.175}$$

For this particular $\mathfrak{g}$,

$$\left\langle \hat{f}, \hat{g}_t \right\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \hat{g}, \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{g} \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} = 0 \ . \tag{3.176}$$

But $\omega_{\mathbb{T}} \geq m$, so $S = \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1}$ satisfies

$$1 \leq S \leq (1 - e^{-\beta m})^{-1} \ . \tag{3.177}$$

In particular S is bounded and has a bounded inverse, so S has no null vectors. We therefore conclude that $\hat{f} = 0$, or $f \in \mathcal{N}$. This completes the proof.

We remark that we have in the course of this proof given a nice expression for the inner product $\left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_{\mathbb{T},1}}$ between two general functions in $L^2(\mathbb{T}^d_+)$.

Corollary 3.8.8. *For $f, g \in \mathcal{H}_{-1}(\mathbb{T}^d_+)$, one has*

$$\left\langle \hat{f}, \hat{g} \right\rangle_{\mathcal{H}_{\mathbb{T},1}} = \left\langle \hat{\mathfrak{f}}_+, \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_+ \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} + \left\langle \hat{\mathfrak{f}}_-, \left(I - e^{-\beta\omega_{\mathbb{T}}}\right)^{-1} \hat{\mathfrak{g}}_- \right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^{d-1})} , \tag{3.178}$$

with $\hat{\mathfrak{f}}_\pm$ defined in (3.172).

3.9 Mirror Space-Time Lattice

One way to treat ultra-violet regularization is to replace Euclidean space-time by a lattice space-time $\mathfrak{K}^d$. Discretization of space-time is dual to compactification studied in the previous section. However, now we introduce the lattice Laplacian as the fundamental operator and we consider symmetries of the lattice as fundamental symmetries of space-time.

In this case it is natural to take a cubic lattice with equal spacing δ in each coordinate direction. Denote a space-time coordinate by $x \in \mathfrak{K}^d$ and a function on space-time by a series $f(x)$ for $x \in \mathfrak{K}^d$. This space is dual to a momentum space torus $\mathbb{T}^d$ with equal periods $\ell_j = \frac{2\pi}{\delta}$.

3.9.1 Green's Functions

3.9.2 Time Reflection

3.9.3 Reflection Positivity

Part II

Fock Space

The Hilbert space of a free quantum field is called Fock space. We develop the elementary notions of this Hilbert space, as well as deriving some properties of standard operators on this space. For readers new to quantum field theory, this chapter is more or less self-contained. Others may wish to scan this part of the book for notation, and proceed quickly to the later parts. To begin, we review here some basic properties of Hilbert spaces and linear transformations (also called "operators") on Hilbert space.

In Schrödinger quantum theory, the wave function for a spinless boson with coordinates $\vec{x}_1$ is described by a wave functions $\mathfrak{f}_1(\vec{x}_1)$. The wave function for two particles at $\vec{x}_1$ and $\vec{x}_2$ is described by a composite wave function functions that is the product of the wave function for each particle, $\mathfrak{f}_1(\vec{x}_1)\mathfrak{f}_2(\vec{x}_2)$. This product is a tensor product wave function. So tensor products will play a key role in understanding the Hilbert space of n-particles.

In order to take into account the indistinguishability of bosons, one symmetrizes the wave function of two bosons to have the form $\mathfrak{f}_1(\vec{x}_1)\mathfrak{f}_2(\vec{x}_2) + \mathfrak{f}_2(\vec{x}_1)\mathfrak{f}_1(\vec{x}_2)$. Correspondingly fermionic two-particle states are anti-symmetric, having the form $\mathfrak{f}_1(\vec{x}_1)\mathfrak{f}_2(\vec{x}_2) - \mathfrak{f}_2(\vec{x}_1)\mathfrak{f}_1(\vec{x}_2)$. For example, if one chooses the Hilbert space $\mathcal{H}$ for single particle wave functions to be the appropriate space $\mathfrak{H}_{-1/2}(\mathbb{R}^s)$ for a massive, spinless boson, as introduced in (2.54), then a two-particle wave functions will lie in the tensor product $\mathfrak{H}_{-1/2}(\mathbb{R}^s) \otimes \mathfrak{H}_{-1/2}(\mathbb{R}^s)$.

In the case of some other type of particle, if the one-particle wave functions lie in $\mathcal{H}$, then the two particle wave function lies in $\mathcal{H} \otimes \mathcal{H}$. One also needs to take into account the type of particle, and to symmetrize or anti-symmetrize the tensor product. For example, in the case of two spinless bosons, one wants to restrice the space of two-particle states from the entire tensor product to the symmetric tensor product $\mathfrak{H}_{-1/2}(\mathbb{R}^s) \otimes_s \mathfrak{H}_{-1/2}(\mathbb{R}^s)$. On the other hand, in the case of fermions on the one-particle space $\mathcal{H}$, then one takes the anti-symmetric tensor product $\mathcal{H} \wedge \mathcal{H}$.

Fock space results from generalizing these ideas to describe an arbitrary number of particles. The two representations of the symmetric group corresponding to complete symmetry and complete anti-symmetry play a special role in physics: all particles observed to date in nature appear to be either bosons or fermions. The famous "spin and statistics" theorem of relativistic quantum field theory does not say that these are the only allowed possibilites; rather it makes the weaker statement that special relativity and positive energy are incompatible with anti-symmetrized bosons or symmetrized fermions.

In general, we consider a Hilbert space that describes an arbitrary number of particles, each of the form of a single particle given by states in $\mathcal{F}_1$. For a single type of particle, this Hilbert space is a sum

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}_n \; , \tag{3.179}$$

where $\mathcal{F}_n$ is the space of states for exactly n particles. We also write this as

$$\mathcal{F} = \exp \mathcal{H} \; , \tag{3.180}$$

where a natural exponential emerges relating the one-particle space $\mathcal{H}$ to the Fock space describing an arbitrary number of particles.

Chapter 4

Sums and Products

Tensor products arise early in the study of quantum theory. As soon as one analyzes a particle with spin in Schrödinger Theory, or several particles with or without spin, one meets a tensor product wave function. The Hilbert space for a particle with spin is the tensor product of the basic one-particle space $\mathcal{F}_1$ with a finite dimensional spin space $\mathbb{C}^N$. Thus the wave function of a single particle with spin has the form $\mathfrak{f}_i(\vec{x})$, where i indexes the finite-dimensional spin space. The spin space is one-dimensional for spin zero, corresponding to the case of $\mathcal{F}_1$.

Before delving into the details of the state space for a particular particle or field, we consider in this chapter some general properties of the tensor products that we will meet in later chapters. We construct a tensor algebra over a Hilbert space $\mathcal{H}$. Two particular subalgebras, one with symmetric multiplication and the other with anti-symmetric multiplication, correspond to the Fock spaces for bosons and fermions respectively.

4.1 The Direct Sum

The direct sum of two Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ is the Hilbert space

$$\mathcal{K}_1 \oplus \mathcal{K}_2 , \tag{4.1}$$

defined as pairs of vectors $\{f_1, f_2\}$ with $f_j \in \mathcal{K}_j$. The scalar product of two vectors in the direct sum is

$$\langle \{f_1, f_2\}, \{g_1, g_2\} \rangle_{\mathcal{K}_1 \oplus \mathcal{K}_2} = \langle f_1, g_1 \rangle_{\mathcal{K}_1} + \langle f_2, g_2 \rangle_{\mathcal{K}_2} . \tag{4.2}$$

There is a permutation transformation τ_{12} such that $\tau_{12}^2 = I$ with the property that

$$\tau_{12} : \mathcal{K}_1 \oplus \mathcal{K}_2 = \mathcal{K}_2 \oplus \mathcal{K}_1 . \tag{4.3}$$

Clearly the sum $\oplus$ is associative

$$(\mathcal{K}_1 \oplus \mathcal{K}_2) \oplus \mathcal{K}_3 = \mathcal{K}_1 \oplus (\mathcal{K}_2 \oplus \mathcal{K}_3) . \tag{4.4}$$

The direct sum often arises in considering several copies of the same or similar transformations (such as fields with components).

4.2 The Tensor Product

The tensor product of two Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ is a new Hilbert space $\mathcal{K}_1 \otimes \mathcal{K}_2$.

4.2.1 Definition of $\mathcal{K}_1 \otimes \mathcal{K}_2$

Define the elementary elements of the space $\mathcal{K}_1 \otimes \mathcal{K}_2$ as pairs of vectors $f_1 \in \mathcal{K}_1$ and $f_2 \in \mathcal{K}_2$, written as $f_1 \otimes f_2$. For $\lambda \in \mathbb{C}$, one identifies $\lambda(f_1 \otimes f_2) = (\lambda f_1) \otimes f_2 = f_1 \otimes (\lambda f_2)$, and considers formal sums of vectors of these elementary vectors. One takes the inner product of two elementary vectors in $\mathcal{K}_1 \otimes \mathcal{K}_2$ as the product of the corresponding inner products,

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \langle f_1, g_1 \rangle_{\mathcal{K}_1} \langle f_2, g_2 \rangle_{\mathcal{K}_2} \ . \tag{4.5}$$

One extends this definition by linearity to finite sums of $k_1 k_2$ elementary vectors

$$\Omega = \sum_{\alpha_1=1}^{k_1} \sum_{\alpha_2=1}^{k_2} c_{\alpha_1 \alpha_2} \, f_1^{\alpha_1} \otimes f_2^{\alpha_2} \ , \qquad \text{where } c_{\alpha_1 \alpha_2} \in \mathbb{C} \ . \tag{4.6}$$

These vectors comprise the *algebraic* tensor product.

The inner product of two such vectors Ω and Ω' must be linear in Ω' and conjugate linear in Ω. Thus for any choice of k_1 vectors $f_1^{\alpha_j} \in \mathcal{K}_1$ and k_2 vectors $f_2^{\alpha_j} \in \mathcal{K}_2$ the inner product must have the form

$$\begin{aligned} \langle \Omega, \Omega' \rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} &= \sum_{\alpha\alpha'} \overline{c_\alpha} \, c_{\alpha'} \left\langle f_1^{\alpha_1} \otimes f_2^{\alpha_2}, g_1^{\alpha'_1} \otimes g_2^{\alpha'_2} \right\rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} \\ &= \sum_{\alpha\alpha'} \overline{c_\alpha} \, c_{\alpha'} \left\langle f_1^{\alpha_1}, g_1^{\alpha'_1} \right\rangle_{\mathcal{K}_1} \left\langle f_2^{\alpha_2}, g_2^{\alpha'_2} \right\rangle_{\mathcal{K}_2} \ . \end{aligned} \tag{4.7}$$

Here one denotes $\alpha = (\alpha_1, \alpha_2)$ as a multi-index. The condition that this form make $\mathcal{K}_1 \otimes \mathcal{K}_2$ into a pre-Hilbert space is the statement that $0 \leq \langle \Omega, \Omega \rangle$, with vanishing only possible if $\Omega = 0$. In other words, the form (4.7) is positive definite on $(\mathcal{K}_1 \otimes \mathcal{K}_2) \times (\mathcal{K}_1 \otimes \mathcal{K}_2)$. In this case, the algebraic tensor product $\mathcal{K}_1 \otimes \mathcal{K}_2$ is a pre-Hilbert space that can be completed to a Hilbert space that we call $\mathcal{K}_1 \otimes \mathcal{K}_2$.

Proposition 4.2.1. *The form* (4.7) *is positive definite.*

Proof. Define M and N as $k_1 \times k_1$ and $k_2 \times k_2$ hermitian matrices with entries

$$M_{\alpha_1 \alpha'_1} = \left\langle f_1^{\alpha_1}, f_1^{\alpha'_1} \right\rangle \ , \qquad \text{and } N_{\alpha_2 \alpha'_2} = \left\langle f_2^{\alpha_2}, f_2^{\alpha'_2} \right\rangle \ . \tag{4.8}$$

Then

$$\langle \Omega, \Omega \rangle = \sum_{\alpha\alpha'} \overline{c_\alpha} \ M_{\alpha_1 \alpha'_1} N_{\alpha_2 \alpha'_2} \, , c_{\alpha'} \ , \tag{4.9}$$

In other words we need to study the positivity of $k_1 k_2$-dimensional square matrices with entries $M_{\alpha_1 \alpha'_1} N_{\alpha_2 \alpha'_2}$.

On the Hilbert space $\mathcal{K}_1$, the existence of the inner product is equivalent to the statement that all matrices of the form M are positive, and the statement that the vectors $f_1^{\alpha_1}$ are linearly independent, is equivalent to the matrix M being positive definite. The same is true for $\mathcal{K}_2$ and matrices of the form N. Therefore, the requirement that (4.9) is an inner product is that the matrix $k_1k_2 \times k_1k_2$ matrix $K = M \otimes N$ defined by the matrix elements

$$K_{\alpha\alpha'} = M_{\alpha_1\alpha_1'} N_{\alpha_2\alpha_2'} \tag{4.10}$$

is positive.

If we choose the $f_1^{\alpha_1}$ to be linearly independent in $\mathcal{K}_1$, and the $f_2^{\alpha_2}$ to be linearly independent in $\mathcal{K}_2$, one wants the $f_1^{\alpha_1} \otimes f_2^{\alpha_2}$ to be linearly independent in $\mathcal{K}_1 \otimes \mathcal{K}_2$. This is equivalent to the matrix K being positive definite when M and N are positive definite. If this is the case, then $\mathcal{K}_1 \otimes \mathcal{K}_2$ is a pre-Hilbert space; it can be turned into a Hilbert space by completing it in the given inner product. In other words, it is sufficient to establish the following:

Proposition 4.2.2. *Consider hermitian, positive matrices M and N with matrix elements $M_{\alpha_1\alpha_1'}$ and dimension $k_1 \times k_1$, and matrix elements $N_{\alpha_2\alpha_2'}$ and dimension $k_2 \times k_2$ respectively. Then the $k_1k_2 \times k_1k_2$ matrix $K = M \otimes N$ defined by the matrix elements* (4.10) *is positive. Furthermore if M and N are positive definite, then K is positive definite.*

Proof. Since $M_{\alpha_1\alpha_1'}$ is hermitian, there is a an orthonormal basis of eigenvectors $f^{(1)}, \ldots, f^{k_1} \in \mathbb{C}^{k_1}$ with components $f_{\alpha_1}^{(j)}$ and eigenvalues $0 \leq \lambda_j$. Similarly there is an orthonormal basis of eigenvectors $g^{(j')} \in \mathbb{C}^{k_2}$, with $1 \leq j \leq k_2$ for N, with eigenvalues $\mu_{j'}$. Note that the orthonormal vectors with components $f_{\alpha_1}^{(j)} g_{\alpha_2}^{(j')}$ are eigenvectors for K with eigenvalues $0 \leq \lambda_j \mu_{j'}$. There are exactly k_1k_2 such vectors, so they give an orthonormal basis for $\mathbb{C}^{k_1k_2}$. Therefore K is positive. Furthermore, if each λ_j and $\mu_{j'}$ is strictly positive, so is each eigenvalue $\lambda_j \mu_{j'}$.

SPECIFIC CASES **Example 1.** As we see from the proof above, the Hilbert space $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} = \mathbb{R}^{d_1d_2}$, and likewise $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} = \mathbb{C}^{d_1d_2}$. For finite dimensional Hilbert spaces,

$$\dim (\mathcal{K}_1 \otimes \mathcal{K}_2) = \dim \mathcal{K}_1 \times \dim \mathcal{K}_2 \; . \tag{4.11}$$

Example 2. In case we choose an orthonormal bases $e_1^{(\alpha_1)} \in \mathcal{K}_1$ and $e_2^{(\alpha_2)} \in \mathcal{K}_2$, then $e^{(\alpha_1)} \otimes e_2^{(\alpha_2)}$ is an orthonormal basis for $\mathcal{K}_1 \otimes \mathcal{K}_2$. The expansion

$$f = \sum_{\alpha_1\alpha_2} c_{\alpha_1\alpha_2} \, e_1^{(\alpha_1)} \otimes e_2^{(\alpha_2)} \; , \qquad \text{with } c_{\alpha_1\alpha_2} = \left\langle e_1^{(\alpha_1)} \otimes e_2^{(\alpha_2)}, f \right\rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} \; , \tag{4.12}$$

gives

$$\langle f, f' \rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \sum_{\alpha_1\alpha_2} \overline{c_{\alpha_1\alpha_2}} c'_{\alpha_1\alpha_2} \; . \tag{4.13}$$

Example 3. The tensor product of two $L^2(\mathbb{R}^d)$ spaces obeys the rule

$$L^2(\mathbb{R}^{d_1}) \otimes L^2(\mathbb{R}^{d_2}) = L^2(\mathbb{R}^{d_1+d_2}) \; . \tag{4.14}$$

Properties of the Hilbert space $L^2(\mathbb{R}^d)$ can be understood in terms of an orthonormal basis of eigenfunctions for the oscillator with d-components. These eigenfunctions factorize into (tensor) products of one-dimensional eigenfunctions, as does the measure of integration, dq, leading to the stated relation.

Example 4. The Sobolev Hilbert space $\mathfrak{H}_{-1}(\mathbb{R}^d)$ introduced in (3.51) displays another behavior, namely

$$\mathfrak{H}_{-1}(\mathbb{R}^{d_1}) \otimes \mathfrak{H}_{-1}(\mathbb{R}^{d_2}) \supsetneqq \mathfrak{H}_{-1}(\mathbb{R}^{d_1+d_2}) . \tag{4.15}$$

In Fourier space, the inner product in $\mathfrak{H}_{-1}(\mathbb{R}^d)$ is given by the measure

$$\frac{1}{p^2+m^2} dp . \tag{4.16}$$

The weight factor $(p^2+m^2)^{-1}$ in dimension d_1+d_2 does not factorize into the product of weight factors in dimension d_1 and d_2. Thus in the case $d_1 = d_2 = 1$, for example, the generalized function $\delta \in \mathfrak{H}_{-1}(\mathbb{R})$. Therefore $\delta \otimes \delta$ belongs to the two-particle space,

$$\delta \otimes \delta \in \mathfrak{H}_{-1}(\mathbb{R}) \otimes \mathfrak{H}_{-1}(\mathbb{R}) , \qquad \text{however} \qquad \delta \otimes \delta \notin \mathfrak{H}_{-1}(\mathbb{R}^2) . \tag{4.17}$$

Example 5. One can apply the tensor product construction to spaces with other topologies, such as the countably-normed Schwartz space $\mathcal{S}(\mathbb{R}^d)$, with norms (**??**). Giving the algebraic tensor product the topology induced from this countable set of norms, one has

$$\mathcal{S}(\mathbb{R}^{d_1}) \otimes \mathcal{S}(\mathbb{R}^{d_1}) = \mathcal{S}(\mathbb{R}^{d_1+d_2}) . \tag{4.18}$$

4.2.2 Tensor Products of Operators

Two linear transformation A_1 on $\mathcal{K}_1$ and A_2 on $\mathcal{K}_2$ respectively, yield a tensor product transformation $A_1 \otimes A_2$ that acts on $\mathcal{K}_1 \otimes \mathcal{K}_2$. Explicitly one defines

$$(A_1 \otimes A_2)(f \otimes g) = (A_1 f) \otimes (A_2 g) , \tag{4.19}$$

and extends this definition by linearity to all of $\mathcal{K}_1 \otimes \mathcal{K}_2$. As a consequence,

$$(A_1 \otimes A_2)(A_1' \otimes A_2') = (A_1 A_1') \otimes (A_2 A_2') , \tag{4.20}$$

and

$$(A_1 \otimes A_2)^* = A_1^* \otimes A_2^* . \tag{4.21}$$

Consequently,

$$(A_1 \otimes A_2)^* (A_1 \otimes A_2) = (A_1^* A_1) \otimes (A_2^* A_2) . \tag{4.22}$$

The matrix elements of $A_1 \otimes A_2$ in the basis $\{e_{i_1} \otimes f_{i_2}\}$ can be expressed in terms of the matrix elements of A_1 in the basis $\{e_{i_1}\}$ and A_2 in the basis $\{f_{i_2}\}$, namely

$$(A_1)_{i_1 j_1} = \langle e_{i_1}, A_1 e_{j_1} \rangle_{\mathcal{K}_1} , \qquad \text{and} \qquad (A_1)_{i_2 j_2} = \langle f_{i_2}, A_2 f_{j_2} \rangle_{\mathcal{K}_2} . \tag{4.23}$$

Then

$$(A_1 \otimes A_2)_{i_1 i_2, j_1 j_2} = \langle e_{i_1} \otimes f_{i_2}, (A_1 \otimes A_2)(e_{j_1} \otimes f_{j_2}) \rangle_{\mathcal{K}_1 \otimes \mathcal{K}_2} = (A_1)_{i_1 j_1} (A_2)_{i_2 j_2} \ . \tag{4.24}$$

On general vectors $g \in \mathcal{K}_1 \otimes \mathcal{K}_2$ of the form (4.6),

$$((A_1 \otimes A_2) g)_{i_1 i_2} = \sum_{j_1, j_2} (A_1)_{i_1 j_1} (A_2)_{i_2 j_2} g_{j_1 j_2} \ . \tag{4.25}$$

We establish two fundamental properties of the tensor product $A_1 \otimes A_2$ of two transformations. These two properties concern certain upper and lower bounds that we often encounter.

Proposition 4.2.3. *Let A_1, A_2 be bounded transformations on Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, and let $A_1 \otimes A_2$ be defined as above on $\mathcal{K}_1 \otimes \mathcal{K}_2$.*

i) The operator norm of $A_1 \otimes A_2$ on $\mathcal{K}_1 \otimes \mathcal{K}_2$ is given by the product of the norms of the factors,

$$\|A_1 \otimes A_2\|_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \|A_1\|_{\mathcal{K}_1} \|A_2\|_{\mathcal{K}_2} \ . \tag{4.26}$$

ii) If $0 \leq A_1$ and $0 \leq A_2$ each are positive, then the tensor product transformation is also positive,

$$0 \leq A_1 \otimes A_2 \ . \tag{4.27}$$

iii) The transformation $A_1 \otimes I$ has a bounded inverse on $\mathcal{K}_1 \otimes \mathcal{K}_2$ if and only if A_1 has a bounded inverse on $\mathcal{K}_1$. In that case,

$$(A_1 \otimes I)^{-1} = A_1^{-1} \otimes I \ . \tag{4.28}$$

iv) The spectrum of $A_1 \otimes I$ on $\mathcal{K}_1 \otimes \mathcal{K}_2$ coincides with the of A_1 on $\mathcal{K}_1$.

Proof. Any bounded transformation T on a Hilbert space $\mathcal{K}$ can be approximated strongly by a sequence T_n of finite rank transformations obtained by projecting T onto an n-dimensional subspace of $\mathcal{K}$, with $\|T_n\|_{\mathcal{K}} \to \|T\|\mathcal{K}$. Thus it is sufficient to establish the proposition for A_1 and A_2 equal to finite dimensional matrices. Furthermore, any transformation T on a Hilbert space satisfies the C^* property $\|T\| = \|T^*T\|^{1/2}$, so

$$\|A_1 \otimes A_2\|_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \|A_1^* A_1 \otimes A_2^* A_2\|_{\mathcal{K}_1 \otimes \mathcal{K}_2}^{1/2} \ . \tag{4.29}$$

As a consequence, it is sufficient to restrict attention to proving (4.26) for positive A_1 and A_2, as we assume also for (4.27). Thus it is convenient to take $\{e_{i_1}\}$ to be an orthonormal basis of eigenfunctions of A_1 with eigenvalues λ_{i_1} and to take $\{f_{i_2}\}$ to be an orthonormal basis of eigenfunctions of A_2 with eigenvalues μ_{i_2}. As a consequence, the vectors $e_{i_1} \otimes f_{i_2}$ are an orthonormal basis eigenvectors of $A_1 \otimes A_2$ with eigenvalues $\lambda_{i_1}\mu_{i_2}$. But this shows that for each finite dimensional approximation, all the eigenvalues of $A_1 \otimes A_2$ are positive, and also that

$$\|A_1 \otimes A_2\|_{\mathcal{K}_1 \otimes \mathcal{K}_2} = \sup_{i_1, i_2} \lambda i_1 \mu_{i_2} = \|A_1\|_{\mathcal{K}_1} \|A_2\|_{\mathcal{K}_2} \ . \tag{4.30}$$

This completes the proof of (ii). If the inverse of A_1 exists, then the multiplication law (4.20) shows that $A_1^{-1} \otimes I$ is the inverse of $A_1 \otimes I$. The spectrum of T is the complement of the set for which $T - \lambda I$ has a bounded inverse. But

$$(A_1 \otimes I) - \lambda\,(I \otimes I) = (A_1 - \lambda I) \otimes I\ , \tag{4.31}$$

so parts (iii) and (iv) also follow.

4.2.3 The Pointwise Operator Product

The pointwise operator product $A * B$ is a restriction of the tensor product $A \otimes B$ of operators on $\mathcal{H} \otimes \mathcal{H}$ to a subspace isomorphic to $\mathcal{H}$. When $\mathcal{H}$ is finite dimensional, the pointwise product of bounded transformations A, B always define a bounded transformation $A * B$. This may not be the case when $\mathcal{H}$ is infinite-dimensional case.

In an earlier discussion, we found it useful to introduce a basis-dependent, pointwise product of vectors $f * g$. For $f, g \in \mathbb{R}^d$, we introduced the vector $f * g$ with components $(f * g)_i = f_i g_i$, see (3.125). One could define an analogous notion on the Hilbert space ℓ^2 of square summable sequences, but in the infinite-dimensional case such a pointwise $*$-product of vectors is only defined on a subset of ℓ^2, namely for $f \in \ell^2 \cap \ell^p$ and $g \in \ell^2 \cap \ell^q$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. On the Hilbert space of L^2 functions, the pointwise product can also play a role, with $(f * g)(x) = f(x)g(x)$. But again such a multiplication is defined only on a subspace.

It is also useful to define the pointwise product of matrices (in a particular basis). Let A, B be $n \times m$ matrices with entries A_{ij} and B_{ij}. Define the pointwise product $A * B$ as the $n \times m$ matrix with entries

$$(A * B)_{ij} = A_{ij} B_{ij}\ . \tag{4.32}$$

The corresponding n^{th} $*$-power of A is

$$A^{\{*n\}} = \underbrace{A * A * A * \cdots * A}_{\text{n-factors}} \tag{4.33}$$

and the $*$-exponential of A is

$$\exp_*(A) = I + \sum_{n=1}^{\infty} \frac{1}{n!}\, A^{\{*n\}}\ . \tag{4.34}$$

We shortly use this notion in the case of finite-dimensional matrices. But one observes that the corresponding notion for linear transformations on an infinite-dimensional space arises for quantum fields. In particular, we have seen the central role played by the Green's operator $C = (-\Delta + m^2)^{-1}$ which is a bounded linear transformation on $L^2(\mathbb{R}^d)$, as long as $m > 0$. The integral kernel of this transformation is $C(x; y) = C(x - y)$ with the properties named in §3.2. In particular $C(x; y)$ is strictly positive and has a singularity on the diagonal in dimension $d \geq 3$ that is $\sim |x - y|^{-(d-2)}$. One can define the pointwise power $C^{\{*n\}}$ of C, as the integral operator with the kernel

$$\left(C^{\{*n\}}\right)(x; y) = C(x - y)^n\ . \tag{4.35}$$

Note that for

$$n < n_{\rm crit} = d/(d-2)\ . \tag{4.36}$$

this kernel is locally integrable. The operator norm of $C^{\{*n\}}$ on $L^2(\mathbb{R}^d)$ is

$$\left\| C^{\{*n\}} \right\| = \int_{\mathbb{R}^d} C(y)^n \, dy < \infty\ , \tag{4.37}$$

and $C^{\{*n\}}$ defines a bounded operator on $L^2(\mathbb{R}^d)$.

On the other hand for $n \geq n_c$, the kernel of C^{*n} is not locally integrable and $C^{\{*n\}}$ does not define a bounded operator on $L^2(\mathbb{R}^d)$. In case $n \geq n_{\rm crit}$, the pointwise power $C^{\{*n\}}$ is said to require *renormalization* in order to be defined.

4.2.4 Pointwise Products Preserve Positivity

We saw in Exercise 3.6.1 that the functions $h(t) = t^2$ and $h(t) = e^t$ do not preserve the monotonicity of matrices when we use the usual functional calculus. However we now see that these functions do preserve positivity, when we use the functional calculus defined by the $*$ product!

Proposition 4.2.4. *Let $0 \leq A, B$ be transformations on a given Hilbert space $\mathcal{K}$. Then when they are defined, the transformations $A * B$, $A^{\{*n\}}$, and $\exp_*(A)$ are all positive,*

$$0 \leq A * B\ , \qquad 0 \leq A^{\{*n\}}\ , \qquad \text{and} \qquad 0 \leq \exp_*(A)\ . \tag{4.38}$$

Remark 4.2.5. *This means that if A_{ij} and B_{ij} are the matrix elements of self-adjoint matrices A, B with non-negative eigenvalues, then also*

$$A_{ij} B_{ij}\ , \qquad (A_{ij})^n\ , \qquad \text{and } e^{A_{ij}}\ , \tag{4.39}$$

are elements of matrices with non-negative eigenvalues.

Proof. Consider the diagonal subspace $\mathfrak{K} \subset \mathcal{K} \otimes \mathcal{K}$ of vectors of the form (**??**) for which $\{g_{ii}\} \in \ell^2$. Also let $\mathfrak{K}$ denote the orthogonal projection of $\mathcal{K} \otimes \mathcal{K}$ onto $\mathfrak{K}$. There is a natural identification of $\mathfrak{K}$ with $\mathcal{K}$. Notice that $A * B$ is the restriction of $A \otimes B$ on $\mathcal{K} \otimes \mathcal{K}$ to act on $\mathfrak{K}$. In Proposition 4.2.3 we show that $0 \leq A \otimes B$. Hence restricted to the subspace $\mathfrak{K}$ we have $0 \leq A * B$. Continuing in this fashion, we infer $0 \leq A^{\{*n\}} = A * A^{\{*n-1\}}$. Furthermore the sum of positive transformations is a positive transformation, so $0 \leq \exp_*(A)$.

4.3 n-Fold Tensor Products

Define the n-fold tensor product of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ as the Hilbert space

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n\ , \tag{4.40}$$

constructed as follows. Write elementary vectors in this vector space as

$$\Omega^{(n)}_{f_1,\ldots,f_n} = f_1 \otimes f_2 \otimes \cdots \otimes f_n \;, \qquad \text{with } f_j \in \mathcal{H}_j \;, \tag{4.41}$$

with the inner product between two such elementary vectors (4.41) given by the product of the inner products of the individual vectors,

$$\left\langle \Omega^{(n)}_{f_1,\ldots,f_n}, \Omega^{(n)}_{g_1,\ldots,g_n} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} = \prod_{j=1}^{n} \langle f_j, g_j \rangle_{\mathcal{H}_j} \;. \tag{4.42}$$

Extend the definition of vectors to be linear over $\mathbb{C}$ in each factor space $\mathcal{H}_j$, so that finite formal sums

$$\Omega^{(n)} = \sum_{\alpha} c_\alpha \, \Omega^{(n)}_{f_1^{\alpha_1},\ldots,f_n^{\alpha_n}} \;, \qquad \text{with } c_\alpha \in \mathbb{C} \;, \text{ and } f_j^{\alpha_j} \in \mathcal{H}_j \;. \tag{4.43}$$

Here α is the multi-index $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ with $1 \le \alpha_j \le N$, for some $N < \infty$. The inner product of two such vectors $\Omega^{(n)}$ and $\Omega'^{(n)}$ is

$$\left\langle \Omega^{(n)}, \Omega'^{(n)} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} = \sum_{\alpha,\beta} \overline{c_\alpha}\, c'_\beta \left\langle \Omega^{(n)}_{f_{\alpha_1},\ldots,f_{\alpha_n}}, \Omega^{(n)}_{f'_{\beta_1},\ldots,f'_{\beta_n}} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} \;. \tag{4.44}$$

We see that this form (4.44) is positive definite for $\Omega^{(n)} = \Omega'^{(n)}$ as follows. Note that

$$\left\langle \Omega^{(n)}, \Omega^{(n)} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} = \sum_{\alpha,\beta} \overline{c_\alpha}\, c_\beta \prod_{j=1}^{n} \left\langle f_{\alpha_j}, f_{\beta_j} \right\rangle_{\mathcal{H}_j} \;. \tag{4.45}$$

Hence positivity of this putative inner product is equivalent to positivity of any $N^n \times N^n$ matrix $M^{(N,n)}$ with matrix elements

$$M^{(N,n)}_{\alpha,\beta} = \prod_{j=1}^{n} M^{(j)}_{\alpha_j\, \beta_j} \;, \tag{4.46}$$

where the individual matrices $M^{(j)}$ with matrix elements $M^{(j)}_{\alpha_j\, \beta_j}$ are positive, and strictly positive unless all the $f_{j\,\alpha_j}$ are zero.

$$\left\langle f_1^{(n)}, f_2^{(n)} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} = \sum_{\alpha_1,\ldots,\alpha_n} \overline{c(f_1^{(n)})_{\alpha_1\cdots\alpha_n}}\, c(f_2^{(n)})_{\alpha_1\cdots\alpha_n} \;, \tag{4.47}$$

and

$$c(f^{(n)})_{\alpha_1\cdots\alpha_n} = \left\langle f_{\alpha_1} \otimes f_{\alpha_2} \otimes \cdots \otimes f_{\alpha_n}, f^{(n)} \right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n} \;. \tag{4.48}$$

$$\| f_1 \otimes f_2 \otimes \cdots \otimes f_n \| = \prod_{j=1}^{n} \| f_j \|_{\mathcal{H}_j} \;. \tag{4.49}$$

A dense set of vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ have the form of finite linear combinations of such tensor products, with the vectors $\{f_j\}$ chosen from an ortho-normal basis for $\mathcal{H}$,

$$f^{(n)} \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \;. \tag{4.50}$$

With coefficients $c(f^{(n)})_\alpha$, such vectors have the form

A sequence of operators A_j acting on $\mathcal{H}_j$, for $1 \le j \le n$, defines a tensor product transformation $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ acting on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, namely

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)\,(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = A_1 f_1 \otimes A_2 f_2 \otimes \cdots \otimes A_n f_n \;. \tag{4.51}$$

It follows from the analysis of the case $n = 2$ in Proposition 4.2.3 that

$$\|A_1 \otimes A_2 \otimes \cdots \otimes A_n\|_{\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n} = \prod_{j=1}^{n} \|A_j\|_{\mathcal{H}_j} \;. \tag{4.52}$$

Also if $0 \le A_j$ for every j, then

$$0 \le A_1 \otimes A_2 \otimes \cdots \otimes A_n \;. \tag{4.53}$$

4.4 Tensor Powers

We often deal with a power of a given Hilbert space, namely the case of the tensor product for which all factors agree, $\mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}_n = \mathcal{H}$. In this case define the n^{th}-power of $\mathcal{H}$ to be

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n\ factors} \;. \tag{4.54}$$

Likewise, define the Fock space $\mathcal{F}$ as the direct sum of these spaces,

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots \oplus \mathcal{H}^{\otimes n} \oplus \cdots \;, \tag{4.55}$$

or

$$\begin{aligned} \mathcal{F} &= \oplus_{j=0}^{\infty} \mathcal{F}_j \\ &= \;, \end{aligned} \tag{4.56}$$

where $\mathcal{F}_j = \mathcal{H}^{\otimes n}$ and $\mathcal{F}_0 = \mathbb{C}$. Vectors in $\underline{f} \in \mathcal{F} = \exp_\otimes(\mathcal{H})$ are sequences of vectors of the form

$$\underline{f} = \{f^{(0)}, f^{(1)}, f^{(2)}, \ldots\} \;, \qquad \text{where } f^{(n)} \in \mathcal{F}_n \;. \tag{4.57}$$

Vectors with almost all the $f^{(n)} = 0$ are dense in $\mathcal{F}$. (These are called finite-particle vectors.) The inner product of two such vectors is

$$\left\langle \underline{f}, \underline{g} \right\rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{F}_n} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{H}^{\otimes n}} \;. \tag{4.58}$$

The Fock space $\mathcal{F}$ is the Hilbert space obtained by completing the space of finite-particle vectors in the norm given by the inner product (4.58).

4.4.1 The Map Γ

Let $\mathcal{B}(\mathcal{H})$ denote the bounded, linear transformations on $\mathcal{H}$. Given an contraction operator $T \in \mathcal{B}(\mathcal{H})$, namely a bounded operator with norm $\|T\|_{\mathcal{H}} \leq 1$, one associates a contraction operator $\Gamma(T) \in \mathcal{B}(\mathcal{F})$. This operator is

$$\Gamma : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{F}) . \tag{4.59}$$

In case T is not a contraction, then $\Gamma(T)$ is an unbounded operator whose domain includes all vectors with a finite number of particles. The operator $\Gamma(T)$ acts on $\mathcal{F}_n$ as the n-fold tensor product of T. In particular

$$\Gamma(T) = \oplus_{n=0}^{\infty} \Gamma(T)_n , \qquad \text{where } \Gamma(T)_n = T^{\otimes n} , \tag{4.60}$$

or

$$\Gamma(T)\underline{f} = \{f^{(0)},\ Tf^{(1)},\ (T \otimes T)\, f^{(2)},\ \ldots, T^{\otimes n} f^{(n)},\ \ldots\} . \tag{4.61}$$

It is also interesting to calculate the infinitesimal $\Gamma(T)$ when $T(\alpha) = e^{\alpha S}$. Then

$$\left(\frac{d}{d\alpha} \Gamma(T(\alpha)) \bigg|_{\alpha=0} \right)_n = \underbrace{\overbrace{S \otimes I \otimes \cdots I}^{n\ factors} + I \otimes S \otimes I + \cdots + I + \cdots + I \otimes \cdots \otimes S}_{n\ terms} \tag{4.62}$$

Exercise 4.4.1. *Check the following examples:*

i. With I denoting the identity both on $\mathcal{H}$ and on $\mathcal{F}$

$$\Gamma(I) = I , \tag{4.63}$$

ii. For $t \geq 0$,

$$\Gamma(e^{-t}) = e^{-tN} , \tag{4.64}$$

where the number operator N. Show that N is self-adjoint, with spectrum the non-negative integers. Also show that each $\mathcal{F}_n$ as an eigenspace of N for eigenvalue $n \in \mathbb{Z}_+$.

iii. Show that the transformation $-I$ on $\mathcal{H}$ gives the operator

$$\Gamma(-I) = (-I)^N . \tag{4.65}$$

4.5 Symmetric Powers

Bosonic multi-particle states are symmetrized. So we define the n^{th}-symmetric power $\mathcal{H}^{\otimes_s n}$ of $\mathcal{H}$ as a subspace of $\mathcal{H}^{\otimes n}$, denoted by

$$\mathcal{H}^{\otimes_s n} = \underbrace{\mathcal{H} \otimes_s \cdots \otimes_s \mathcal{H}}_{n\ factors} , \tag{4.66}$$

and where an element of $\mathcal{H}^{\otimes_s n}$ has the form,

$$f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n \ := \frac{1}{n!^{1/2}} \sum_{\{\text{perm } i\}} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n} . \tag{4.67}$$

We also write

$$\Omega^s_{f_1,f_2,\ldots,f_n} = f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n \ . \tag{4.68}$$

Note that we use $n!^{-1/2}$ and not $n!^{-1}$ as the normalizing factor in (4.67), so when $f_1 = \cdots = f_n$,

$$\underbrace{f \otimes_s f \otimes_s f \otimes_s \cdots \otimes_s f}_{n factors} = n!^{1/2}\, \underbrace{f \otimes f \otimes f \otimes \cdots \otimes f}_{n factors} \ . \tag{4.69}$$

The inner product on $\mathcal{H}^{\otimes_s n}$ is inherited from $\mathcal{H}^{\otimes n}$, so

$$\| f \otimes_s f \otimes_s f \otimes_s \cdots \otimes_s f \|_{\mathcal{H}^{\otimes_s n}} = n!^{1/2} \, \| f \|^n_{\mathcal{H}} \ . \tag{4.70}$$

On the other hand, when the f_j belong to an orthonormal basis for $\mathcal{H}$, then vectors of the form $f_{i_1} \otimes_s f_{i_2} \otimes_s \cdots \otimes_s f_{i_n}$, with indices $i_1 < i_2 < \cdots < i_n$ are orthonormal. Furthermore the vectors

$$f^{(n)}_{i_1^{n_{i_1}},\ldots,i_r^{n_{i_r}}} = \left(\prod_{j=1}^{r} \frac{1}{n_{i_1}!^{1/2}} \right) \underbrace{f_{i_1} \otimes_s \cdots \otimes_s f_{i_1}}_{n_{i_1} factors} \otimes_s \cdots \otimes_s \underbrace{f_{i_r} \otimes_s \cdots \otimes_s f_{i_r}}_{n_{i_r} factors} \ , \tag{4.71}$$

with $i_1 < i_2 < \cdots < i_n$ and with $n_{i_1} + \cdots + n_{i_r} = n$ yield an orthonormal basis for $\mathcal{H}^{\otimes_s n}$.

We easily compute the inner product on $\mathcal{H}^{\otimes_s n}$ between two pure tensor product vectors as

$$\begin{aligned} \langle f_1 \otimes_s \cdots \otimes_s f_n, \ g_1 \otimes_s \cdots \otimes_s g_n \rangle_{\mathcal{H}^{\otimes_s n}} &= \langle f_1 \otimes_s \cdots \otimes_s f_n, \ g_1 \otimes_s \cdots \otimes_s g_n \rangle_{\mathcal{H}^{\otimes n}} \\ &= \frac{1}{n!} \sum_{\{\text{perm } i,i'\}} \prod_{j=1}^{n} \left\langle f_{i_j}, g_{i'_j} \right\rangle_{\mathcal{H}} \\ &= \sum_{\{\text{perm } i\}} \prod_{j=1}^{n} \left\langle f_j, g_{i_j} \right\rangle_{\mathcal{H}} \ . \end{aligned} \tag{4.72}$$

Here $\sum_{\{\text{perm } i\}}$ denotes the sum over the $n!$ permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

The combinatoric expression of the form (4.78) is sometimes called the *permament* of the $n \times n$ matrix M with entries M_{ij}. The permanent is a homogeneous polynomial of degree n in the entries of the matrix, that one denotes $\text{Perm}\, M_{ij}$ or $\text{Perm}\, M$. In case there might be ambiguity in the dimension of M, one also writes $\text{Perm}_n\, M$, when M is an $n \times n$ matrix M. The permanent of an arbitrary $n \times n$ matrix M with elements M_{ij} has the form

$$\text{Perm}_n M = \sum_{\{\text{perm } i\}} M_{1\,i_i} M_{2\,i_2} \cdots M_{n\,i_n} \ . \tag{4.73}$$

This is exactly the type of homogeneous polynomial that enters the definition of the determinant of M,

$$\det{}_n M = \sum_{\{\text{perm } i\}} (-1)^{\text{sgn}\, i} M_{1 i_i} M_{2 i_2} \cdots M_{n i_n} \ , \tag{4.74}$$

where $\text{sgn}\, i$ denotes the sign of the permutation i. For the permanent, however, one removes all the minus signs. While the permanent does not have the same geometric interpretation as the determinant. However, the permanent satisfies a similar recursion relation. For an $(n+1)\times(n+1)$-matrix M,

$$\mathrm{Perm}_{n+1}M \;=\; \sum_{j=1}^{n+1} M_{ij}\ \mathrm{Perm}_n \hat{M}_{ij}\ , \tag{4.75}$$

where $\hat{M}_{ij}$ denotes the $n\times n$ minor of M obtained by removing the first $i^{\rm th}$ row and the $j^{\rm th}$ column.

Consider now the special case that M is a Gram matrix, namely when the components M_{ij} are equal to the inner products of a sequence of vectors $f_i \in \mathcal{H}$ with a second sequence of vectors $g_j \in \mathcal{H}$. In other words $M_{ij} = \langle f_i, g_j\rangle_{\mathcal{H}}$. Then one writes M in the compact form

$$M = \begin{pmatrix} f_1, f_2, \ldots, f_n \\ g_1, g_2, \ldots, g_n \end{pmatrix}\ . \tag{4.76}$$

Here the first row of vectors indexes the rows of the matrix M, while the second row of vectors indexes the columns. Then the matrix elements can be written,

$$M_{ij} = \mathrm{Perm}_1 \begin{pmatrix} f_i \\ g_j \end{pmatrix}\ , \tag{4.77}$$

and the general permanent equals

$$\begin{aligned} \mathrm{Perm}_n \begin{pmatrix} f_1, f_2, \ldots, f_n \\ g_1, g_2, \ldots, g_n \end{pmatrix} &= \sum_{\{\mathrm{perm\ i}\}} \prod_{j=1}^{n} \left\langle f_j, g_{i_j}\right\rangle_{\mathcal{H}} \\ &= \sum_{\{\mathrm{perm\ i}\}} \prod_{j=1}^{n} \mathrm{Perm}_1 \begin{pmatrix} f_j \\ g_{i_j} \end{pmatrix} \\ &= \langle f_1 \otimes_s \cdots \otimes_s f_n,\ g_1 \otimes_s \cdots \otimes_s g_n\rangle_{\mathcal{H}^{\otimes_s n}}\ . \end{aligned} \tag{4.78}$$

With this notation, the recursion relation for the permanent (4.75) also has a simple form. For any fixed $i = 1, 2, \ldots, n+1$, we can write the recursion relation in the form

$$\mathrm{Perm}_{n+1} \begin{pmatrix} f_1, f_2, \ldots, f_{n+1} \\ g_1, g_2, \ldots, g_{n+1} \end{pmatrix} = \sum_{j=1}^{n+1} \mathrm{Perm}_1 \begin{pmatrix} f_i \\ g_j \end{pmatrix} \mathrm{Perm}_n \begin{pmatrix} f_1, f_2, \ldots, \not{f}_i \cdots, f_{n+1} \\ g_1, g_2, \ldots, \not{g}_j \cdots, g_{n+1} \end{pmatrix}\ . \tag{4.79}$$

Iteration of this recursion relation yields the original permanent in (4.78). One can also write the recursion relation in terms of the inner product of vectors on $\mathcal{F}^s$. It has the form

$$\left\langle \Omega^s_{f_1,\ldots,f_{n+1}},\ \Omega^s_{g_1,\ldots,g_{n+1}} \right\rangle_{\mathcal{H}^{\otimes_s n+1}} = \sum_{j=1}^{n+1} \langle f_i,\ g_j\rangle_{\mathcal{H}} \left\langle \Omega^s_{f_1,\ldots,\not{f}_i,\ldots,f_{n+1}},\ \Omega^s_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \right\rangle_{\mathcal{H}^{\otimes_s n}} \tag{4.80}$$

4.5.1 Bosonic Fock Space

The symmetric (or bosonic) Fock space $\mathcal{F}^s = \mathcal{F}^s(\mathcal{H})$ over the (one-particle) Hilbert space $\mathcal{H}$ is

$$\mathcal{F}^s = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n} , \tag{4.81}$$

where we denote $\mathcal{H}^{\otimes_s 0} = \mathbb{C}$. Vectors $\Omega_{\mathbf{f}} \in \mathcal{F}^s$ are sequences on n-particle wave functions $\Omega_n^s \in \mathcal{F}_n^s = \mathcal{H}^{\otimes_s n}$,

$$\mathbf{f} = \{f^{(0)}, f^{(1)}, f^{(2)}, \ldots\} , \qquad \text{where } f^{(n)} \in \mathcal{F}_n^s . \tag{4.82}$$

and

$$\left\langle \underline{f}, \underline{g} \right\rangle_{\mathcal{F}^s} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{F}_n^s} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{H}^{\otimes_s n}} . \tag{4.83}$$

We sometimes use another notation for a vector in $f^{(n)} \in \mathcal{F}_n^s \subset \mathcal{F}^s$ that is a symmetric tensor product of vectors $f_1, \ldots, f_n \in \mathcal{H}$, that is when

$$f^{(n)} = f_1^{(n)} \otimes_s f_2^{(n)} \otimes_s \cdots \otimes_s f_n^{(n)} . \tag{4.84}$$

It is convenient to write

$$\Omega_{f^{(n)}}^s = f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n . \tag{4.85}$$

The inner product of two such vectors is

$$\left\langle \Omega_{f^{(n)}}^s, \Omega_{g^{(n')}}^s \right\rangle_{\mathcal{F}^s} = \delta_{nn'} \left\langle \Omega_{f^{(n)}}^s, \Omega_{g^{(n)}}^s \right\rangle_{\mathcal{F}_n^s} , \tag{4.86}$$

where

$$\left\langle \Omega_{f^{(n)}}^s, \Omega_{g^{(n)}}^s \right\rangle_{\mathcal{F}_n^s} = \mathrm{Perm}_n \begin{pmatrix} f_1^{(n)}, \ldots, f_1^{(n)} \\ g_1^{(n)}, \ldots, g_1^{(n)} \end{pmatrix} , \qquad \text{for } n \geq 1 . \tag{4.87}$$

The inner product of two such vectors is

$$\left\langle \Omega_{\mathbf{f}}^s, \Omega_{\mathbf{g}}^s \right\rangle_{\mathcal{F}^s} = \overline{\Omega_{f^{(0)}}^s} \, \Omega_{g^{(0)}}^s + \sum_{n=1}^{\infty} \mathrm{Perm}_n \begin{pmatrix} f_1^{(n)}, \ldots, f_1^{(n)} \\ g_1^{(n)}, \ldots, g_1^{(n)} \end{pmatrix} . \tag{4.88}$$

Proposition 4.5.1. *The map Γ transforming a contraction on $\mathcal{H}$ to a contraction on $\mathcal{F}$ restricts to a map Γ^s $\mathcal{F}^s$.*

Proof. The definition (4.61) of the transformation Γ is symmetric on each n-particle component $\mathcal{F}_n$. Thus it maps $\mathcal{F}^s \subset \mathcal{F}$ to itself.

Remark. One could leave the normalization factor $n!^{-1/2}$ out of the definition of the symmetrization of the tensor product, and rather put a factor $n!^{-1}$ into the definition of the scalar product on $\mathcal{H}^{\otimes_s n}$. With this notation, it is natural to write $\mathcal{F}^s$ as an exponential, $\mathcal{F}^s = \exp_{\otimes_s} \mathcal{H}$, where we interpret the exponential being applied to the scalar product.

4.5.2 Bosonic Creation and Annihilation Operators

Bosonic creation and annihilation operators are a representation of the up and down *shift* transformations between subspaces of Fock space with particle number n,

$$\mathcal{F}^s_n \leftrightarrow \mathcal{F}^s_{n+1} \ . \tag{4.89}$$

One defines the downward shift so $\mathcal{F}^s_0$ maps into 0. The the creation operator $\mathfrak{m}_s(f)$ adds a wave function f to a state, mapping

$$\mathfrak{m}_s(f) : \mathcal{F}^s_n \to \mathcal{F}^s_{n+1} \ , \tag{4.90}$$

by the rule

$$\mathfrak{m}_s(f)\, \Omega^s_{f_1,f_2,\ldots,f_n} = \Omega^s_{f,f_1,f_2,\ldots,f_n} \ . \tag{4.91}$$

The symmetry of the symmetric tensor product on $\mathcal{F}^s_n$ entails the commutativity of different creation operators,

$$\mathfrak{m}_s(f)\mathfrak{m}_s(g) = \mathfrak{m}_s(g)\mathfrak{m}_s(f) \ . \tag{4.92}$$

As the domain of $\mathfrak{m}_f$ includes the dense set of vectors $\mathcal{D}$ which are finite linear combinations of elementary tensor product states with a finite number of particles. Therefore $\mathfrak{m}_s(f)$ has a dense domain, and its adjoint it uniquely determines the adjoint $\mathfrak{m}_s(f)^*$, which we now identify. On vectors of the form $\Omega^s_{g_1,g_2,\ldots,g_{n+1}}$, the definition of $\mathfrak{m}_s(f)^*$ as the adjoint of $\mathfrak{m}_s(f)$ means that

$$\begin{aligned} \left\langle \Omega^s_{f_1,f_2,\ldots,f_n}, \mathfrak{m}_s(f)^*\, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^s_{n+1}} &= \left\langle \mathfrak{m}_s(f)\Omega^s_{f_1,f_2,\ldots,f_n}, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^s_{n+1}} \\ &= \left\langle \Omega^s_{f,f_1,f_2,\ldots,f_n}, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^s_{n+1}} \ . \end{aligned} \tag{4.93}$$

Linear combinations of the vectors $\Omega^s_{g_1,g_2,\ldots,g_{n+1}}$ span $\mathcal{F}^s_n$, so this procedure yields the matrix elements of the adjoint $\mathfrak{m}_s(f)^*$ in a basis, and hence determine it uniquely (at least when acting on vectors with a finite number of particles). All other matrix elements of $\Omega^s_{f,f_1,f_2,\ldots,f_n}$ vanish, so $\mathfrak{m}_s(f)^*$ is an annihilation map

$$\mathfrak{m}_s(f)^* : \mathcal{F}^s_{n+1} \to \mathcal{F}^s_n \ , \tag{4.94}$$

for $n \geq 0$ and $\mathfrak{m}_s(f)\mathcal{F}^s_0 = 0$. Using the recursion relation for the inner product given in (4.80), with the choice $i = 1$,

$$\left\langle \Omega^s_{f,f_1,f_2,\ldots,f_n},\ \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^s_{n+1}} = \sum_{j=1}^{n+1} \langle f, g_j \rangle_{\mathcal{H}} \left\langle \Omega^s_{f_1,f_2,\ldots,f_n},\ \Omega^s_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^s_n} \ . \tag{4.95}$$

From (4.93) and (4.95) one can read off that

$$\mathfrak{m}_s(f)^*\, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} \langle f, g_j \rangle_{\mathcal{H}}\, \Omega^s_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \ , \qquad \text{and } \mathfrak{m}_s(f)^*\Omega^s_0 = 0 \ . \tag{4.96}$$

Proposition 4.5.2. *Let $f, g \in \mathcal{H}$. On the domain $\mathcal{D}_0$ of vectors with a finite number of particles,*

$$[\mathfrak{m}_s(f)^*, \mathfrak{m}_s(g)] = \langle f, g\rangle_{\mathcal{H}} \ , \qquad \textit{while} \ \ [\mathfrak{m}_s(f), \mathfrak{m}_s(g)] = [\mathfrak{m}_s(f)^*, \mathfrak{m}_s(g)^*] = 0 \ . \tag{4.97}$$

Proof. To evaluate the commutation relation between $\mathfrak{m}_s(f)^*$ and $\mathfrak{m}_s(g)$, use the basis of tensor product vectors that we have been using. Then on an $(n+1)$-particle vector,

$$\mathfrak{m}_s(g)\mathfrak{m}_s(f)^*\Omega^s_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} \langle f, g_j\rangle_{\mathcal{H}}\, \Omega^s_{g,g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \ , \tag{4.98}$$

while

$$\mathfrak{m}_s(f)^*\mathfrak{m}_s(g)\Omega^s_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} \langle f, g_j\rangle_{\mathcal{H}}\, \Omega^s_{g,g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} + \langle f, g\rangle_{\mathcal{H}}\, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \ . \tag{4.99}$$

Thus on each subspace $\mathcal{F}^s_n$ one has (4.97), But the relations do not depend on n, and the full Fock space $\mathcal{F}^s$ is a direct sum of the $\mathcal{F}^s_n$. Therefore the commutation relations (4.97) hold on any subspace of $\mathcal{F}^s$ with a finite number of particles, completing the proof.

Let N^b denotes the bosonic number operator, namely the restriction of the number operator N, introduced on $\mathcal{F}$ in Exercise 4.4.1, to the subspace $\mathcal{F}^s \subset \mathcal{F}$. Each n-particle subspace $\mathcal{F}^s_n$ is an eigenspace of N^b with eigenvalue n.

Exercise 4.5.1. *Show that on vectors with a finite number of particles,*

$$\left[N^b, \mathfrak{m}_s(f)\right] = \mathfrak{m}_s(f) \ , \qquad \textit{and} \ \ \left[N^b, \mathfrak{m}_s(f)^*\right] = -\mathfrak{m}_s(f)^* \ . \tag{4.100}$$

Furthermore show that the self-adjoint unitary $\Gamma(-I)$ satisfies

$$\Gamma(-I)\mathfrak{m}_s(f)\Gamma(-I) = -\mathfrak{m}_s(f) \ . \tag{4.101}$$

4.6 Anti-Symmetric Powers

Anti-symmetric tensor powers of vectors correspond to fermion particle states. We can redo the analysis in §4.5 of $\mathcal{F}$ but this time focusing on the subspace of tensor-product vectors that are totally anti-symmetric.

Define the n^{th}-anti-symmetric power $\mathcal{H}^{\wedge n}$ of $\mathcal{H}$ as a subspace of $\mathcal{H}^{\otimes n}$,

$$\mathcal{H}^{\wedge n} = \underbrace{\mathcal{H} \wedge \cdots \wedge \mathcal{H}}_{n\ factors} \ , \tag{4.102}$$

and where an element of $\mathcal{H}^{\wedge n}$ has the form,

$$\Omega^a_{f_1,f_2,\ldots,f_n} = f_1 \wedge f_2 \wedge \cdots \wedge f_n \ := \frac{1}{n!^{1/2}} \sum_{\{\mathrm{perm}\ i\}} (-1)^{\mathrm{sgn}\, i} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n} \ , \tag{4.103}$$

and sgn i is the order of the permutation i. The inner product on $\mathcal{H}^{\wedge n}$ is inherited from $\mathcal{H}^{\otimes n}$, so

$$\begin{aligned}
\left\langle \Omega^a_{f_1,f_2,\ldots,f_n},\ \Omega^a_{g_1,g_2,\ldots,g_n} \right\rangle_{\mathcal{H}^{\wedge n}} &= \langle f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_n \rangle_{\mathcal{H}^{\otimes n}} \\
&= \frac{1}{n!} \sum_{\{\text{perm } i,i'\}} (-1)^{\operatorname{sgn} i + \operatorname{sgn} i'} \prod_{j=1}^{n} \left\langle f_{i_j}, g_{i'_j} \right\rangle_{\mathcal{H}} \\
&= \sum_{\{\text{perm } i\}} (-1)^{\operatorname{sgn} i} \prod_{j=1}^{n} \left\langle f_j, g_{i_j} \right\rangle_{\mathcal{H}} \\
&= \det_n \begin{pmatrix} f_1, f_2, \ldots, f_n \\ g_1, g_2, \ldots, g_n \end{pmatrix} . \qquad (4.104)
\end{aligned}$$

If $\{e_j\}$ are an orthonormal basis for $\mathcal{H}$. With the multi-index $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, vectors of the form

$$\Omega^a_{e^{(n)}_\alpha} = \Omega^a_{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}} , \qquad \text{with } \alpha_1 < \alpha_2 < \ldots < \alpha_n , \qquad (4.105)$$

are an orthonormal basis for $\mathcal{H}^{\wedge n}$. A general vector $f^{(n)} \in \mathcal{H}^{\wedge n}$ can be expanded in this basis. It then has the form

$$f^{(n)} = \frac{1}{n!} \sum_\alpha c_\alpha \, \Omega^a_{e^{(n)}_\alpha} , \qquad \text{where } c_\alpha = \left\langle \Omega^a_{e^{(n)}_\alpha},\ f^{(n)} \right\rangle_{\mathcal{H}^{\wedge n}} . \qquad (4.106)$$

One requires that the coefficients c_α are square summable,

$$\left\langle f^{(n)},\ f^{(n)} \right\rangle_{\mathcal{H}^{\wedge n}} = \frac{1}{n!} \sum_\alpha |c_\alpha|^2 < \infty . \qquad (4.107)$$

Finally we note that there is a recursion relation similar to (4.80) for the inner product of two tensor product vectors in $\mathcal{H}^{\wedge n}$. The derivation of this relation uses the recursion relation for determinants. For any i,

$$\det_{n+1} \begin{pmatrix} f_1, f_2, \ldots, f_{n+1} \\ g_1, g_2, \ldots, g_{n+1} \end{pmatrix} = \sum_{j=1}^{n+1} (-1)^{i+j} \det_1 \begin{pmatrix} f_i \\ g_j \end{pmatrix} \det_n \begin{pmatrix} f_1, f_2, \ldots, \not{f}_i \cdots, f_{n+1} \\ g_1, g_2, \ldots, \not{g}_j \cdots, g_{n+1} \end{pmatrix} . \qquad (4.108)$$

One can also write the recursion relation in terms of the inner product of vectors on $\mathcal{F}^s$. It has the form

$$\left\langle \Omega^a_{f_1,\ldots,f_{n+1}},\ \Omega^a_{g_1,\ldots,g_{n+1}} \right\rangle_{\mathcal{H}^{\wedge\, n+1}} = \sum_{j=1}^{n+1} (-1)^{i+j} \ \langle f_i,\ g_j \rangle_{\mathcal{H}} \left\langle \Omega^a_{f_1,\ldots,\not{f}_i,\ldots,f_{n+1}},\ \Omega^a_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \right\rangle_{\mathcal{H}^{\wedge\, n}} \qquad (4.109)$$

4.7 Fermionic Fock Space

Define the anti-symmetric or fermionic Fock space $\mathcal{F}^a$ (or $\mathcal{F}^f$) as

$$\mathcal{F}^a = \oplus_{n=0}^{\infty} \mathcal{F}^a_n , \qquad \text{where } \mathcal{F}^a_0 = \mathbb{C} , \ \text{ and } \mathcal{F}^a_j = \mathcal{H}^{\wedge j} , \ \text{ for } j \geq 1 . \qquad (4.110)$$

Vectors in $\mathcal{F}^a$ are sequences

$$\underline{f}^a = \{f^{(0)}, f^{(1)}, f^{(2)}, \ldots\} \;, \qquad \text{where } f^{(n)} \in \mathcal{F}^a_n \;. \tag{4.111}$$

and

$$\left\langle \underline{f}^a, \underline{g}^a \right\rangle_{\mathcal{F}^a} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{F}^a_n} = \sum_{n=0}^{\infty} \left\langle f^{(n)}, g^{(n)} \right\rangle_{\mathcal{H}^{\wedge n}} \;. \tag{4.112}$$

A fermionic Fock-space vector $\underline{f}^a$ for which each component $f^{(n)}$ is an anti-symmetric tensor product states has the form

$$\Omega^a_{\mathbf{f}} = \{\Omega^s_0, \Omega^a_{f_1^{(1)}},\; \Omega^a_{f_1^{(2)}, f_2^{(2)}},\; \Omega^a_{f_1^{(3)}, f_2^{(3)}, f_3^{(3)}},\; \ldots\} \in \mathcal{F}^a \;. \tag{4.113}$$

The inner product of two such vectors is

$$\left\langle \Omega^a_{\mathbf{f}}, \Omega^a_{\mathbf{g}} \right\rangle_{\mathcal{F}^a} = \sum_{n=0}^{\infty} \det{}^{(n)} \left\langle f_i^{(n)}, g_j^{(n)} \right\rangle_{\mathcal{H}} \;. \tag{4.114}$$

4.7.1 Fermionic Creation and Annihilation Operators

Fermionic creation and annihilation operators are a representation of the up and down *shift* transformations in $\mathcal{F}^a$,

$$\mathcal{F}^a_n \leftrightarrow \mathcal{F}^a_{n+1} \;, \tag{4.115}$$

where the downward shift maps $\mathcal{F}^a_0$ to 0. Denote by $\mathfrak{m}_a(f)$ the creation operator map,

$$\mathfrak{m}_a(f) : \mathcal{F}^a_n \to \mathcal{F}^a_{n+1} \;. \tag{4.116}$$

Its adjoint $\mathfrak{m}_a(f)^*$ is the annihilation map taking $\mathcal{F}^a_{n+1}$ to $\mathcal{F}^a_n$ (and it takes $\mathcal{F}^a_0$ to 0). The definition of $\mathfrak{m}$ on $\mathcal{F}^a$ is similar to the definition on $\mathcal{F}^s$, but it has different properties. Let

$$\mathfrak{m}_a(f)\Omega^a_{f_1,f_2,\ldots,f_n} = \Omega^a_{f,f_1,f_2,\ldots,f_n} \;. \tag{4.117}$$

By the anti-symmetry of the tensor product,

$$\mathfrak{m}_a(f)\mathfrak{m}_a(g) = -\mathfrak{m}_a(g)\mathfrak{m}_a(f) \;. \tag{4.118}$$

We now determine $\mathfrak{m}_a(f)^*$. On vectors of the form $\Omega^a_{g_1,g_2,\cdots,g_{n+1}}$, the definition of $\mathfrak{m}_a(f)^*$ as the adjoint of $\mathfrak{m}_a(f)$ is

$$\begin{aligned} \left\langle \Omega^a_{f_1,f_2,\ldots,f_n}, \mathfrak{m}_a(f)^*\, \Omega^a_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^a_{n+1}} &= \left\langle \mathfrak{m}_a(f)\Omega^a_{f_1,f_2,\ldots,f_n}, \Omega^a_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^a_{n+1}} \\ &= \left\langle \Omega^a_{f,f_1,f_2,\ldots,f_n}, \Omega^a_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^a_{n+1}} \;. \end{aligned} \tag{4.119}$$

Linear combinations of the vectors $\Omega^a_{g_1,g_2,\ldots,g_{n+1}}$ span $\mathcal{F}^a_n$, so this procedure yields the matrix elements of the adjoint $\mathfrak{m}_a(f)^*$ in a basis, and hence determine it uniquely. All other matrix elements of $\Omega^a_{f,f_1,f_2,\ldots,f_n}$ vanish, so $\mathfrak{m}_a(f)^*$ is an annihilation map

$$\mathfrak{m}_a(f)^* : \mathcal{F}^a_{n+1} \to \mathcal{F}^a_n \ , \tag{4.120}$$

for $n \geq 0$ and $\mathfrak{m}_a(f)\mathcal{F}^a_0 = 0$. Using the recursion relation for the inner product given in (4.109), with the choice $i = 1$,

$$\left\langle \Omega^a_{f,f_1,f_2,\ldots,f_n},\ \Omega^a_{g_1,g_2,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^a_{n+1}} = \sum_{j=1}^{n+1} (-1)^{j+1} \langle f, g_j \rangle_{\mathcal{H}} \left\langle \Omega^a_{f_1,f_2,\ldots,f_n},\ \Omega^a_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \right\rangle_{\mathcal{F}^a_n} . \tag{4.121}$$

From (4.119) and (4.121) one can read off that

$$\mathfrak{m}_a(f)^*\, \Omega^a_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} (-1)^{j+1} \langle f, g_j \rangle_{\mathcal{H}}\, \Omega^a_{g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \ , \qquad \text{and } \mathfrak{m}_a(f)^* \Omega^a_0 = 0 \ . \tag{4.122}$$

One can also compute the anti-commutation relation between $\mathfrak{m}_a(f)^*$ and $\mathfrak{m}_a(g)$. In fact,

$$\mathfrak{m}_a(g)\mathfrak{m}_a(f)^* \Omega^s_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} (-1)^{j+1} \langle f, g_j \rangle_{\mathcal{H}}\, \Omega^a_{g,g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} \ , \tag{4.123}$$

while

$$\mathfrak{m}_s(f)^* \mathfrak{m}_s(g) \Omega^s_{g_1,g_2,\ldots,g_{n+1}} = \sum_{j=1}^{n+1} (-1)^{j} \langle f, g_j \rangle_{\mathcal{H}}\, \Omega^s_{g,g_1,\ldots,\not{g}_j,\ldots,g_{n+1}} + \langle f, g \rangle_{\mathcal{H}}\, \Omega^s_{g_1,g_2,\ldots,g_{n+1}} \ . \tag{4.124}$$

With $\{A\,,\,B\} = AB + BA$, we have established the anti-commutation relations for the operators $\mathfrak{m}_s(f)$ and their adjoints.

Proposition 4.7.1. *Let $f, g \in \mathcal{H}$. On the domain $\mathcal{D}_0$ of vectors with a finite number of particles,*

$$\{\mathfrak{m}_s(f)^*\,,\,\mathfrak{m}_s(g)\} = \langle f, g \rangle_{\mathcal{H}} \ , \qquad \textit{while } \{\mathfrak{m}_s(f)\,,\,\mathfrak{m}_s(g)\} = \{\mathfrak{m}_s(f)^*\,,\,\mathfrak{m}_s(g)^*\} = 0 \ . \tag{4.125}$$

Exercise 4.7.1. *Show that $\left\{N^f\,,\,\mathfrak{m}_a(f)\right\} = \mathfrak{m}_a(f)$ on any vector in $\mathcal{F}^a$ with a finite number of particles.*

Chapter 5

Number Bounds

Many bounds on Fock space can be expressed in terms of a number operator, or some variation of the number operator. These Fock-space bounds give basic tools for comparing two transformations. We begin with an elementary example.

5.1 Estimates on $\mathfrak{m}(f)$

In this section we investigate the bosonic and fermionic creation operators $\mathfrak{m}_s(f)$ and $\mathfrak{m}_a(f)$, and their adjoints, introduced in §4.5.2 and §4.7.1. These transformations map $\mathcal{F}_n^s$ or $\mathcal{F}_n^a$ to $\mathcal{F}_{n\pm1}^s$ or $\mathcal{F}_{n\pm1}^a$ respectively. Recall that we use the domain of definition for $\mathfrak{m}_a(f)$ and its adjoint to be $\mathcal{D}_0^a \subset \mathcal{F}^a$. This is the set of vectors that are finite linear combinations of vectors in $\mathcal{F}^a$ with a finite number of particles. This domain $\mathcal{D}_0^a$ is dense in $\mathcal{F}^a$. (Similarly, we define and use the domain $\mathcal{D}_0^s \subset \mathcal{F}^s$, which is dense in $\mathcal{F}^s$, as the domain of definiton of $\mathfrak{m}_s(f)$ and its adjoint.) Let N^b denote the number operator on $\mathcal{F}^s$, namely the self adjoint operator that has $\mathcal{F}_n^s$ as an eigenspace with eigenvalue n.

Proposition 5.1.1. *Let $f \in \mathcal{H}$. Then in the fermionic case,*

$$\|\mathfrak{m}_a(f)\|_{\mathcal{F}^a} = \|\mathfrak{m}_a(f)^*\|_{\mathcal{F}^a} = \|f\|_{\mathcal{H}} \, . \tag{5.1}$$

In the bosonic case,

$$\left\|\mathfrak{m}_s(f)\left(N^b + I\right)^{-1/2}\right\|_{\mathcal{F}^s} = \|f\|_{\mathcal{H}} \, . \tag{5.2}$$

The null space of N^b equals $\mathcal{F}_0^s$ and is contained in the null space of $\mathfrak{m}_s(f)^*$. On the orthogonal complement $\mathcal{F}_{\geq1}^s = (\mathcal{F}_0^s)^\perp$, one has $0 \leq \left(N^b\right)^{-1/2} \leq 1$ and

$$\left\|\mathfrak{m}_s(f)^*\left(N^b\right)^{-1/2}\right\|_{\mathcal{F}_{\geq1}^s} = \|f\|_{\mathcal{H}} \, . \tag{5.3}$$

Proof. The transformations $\mathfrak{m}_s(f)$ and $\mathfrak{m}_a(f)$ are linear in f, and vanish for $f = 0$, in which case the claims hold. Thus without loss of generality assume $f \neq 0$, and by scaling assume $\|f\|_{\mathcal{H}} = 1$.

In the fermionic case, consider the positive, self-adjoint operators $N^a(f) = \mathfrak{m}_a(f)\mathfrak{m}_a(f)^*$ and $\tilde{N}^a(f) = \mathfrak{m}_a(f)^*\mathfrak{m}_a(f)$. The commutation relations (4.125) ensure for $g = f$ that

$$N^a(f) + \tilde{N}^a(f) = I \ . \tag{5.4}$$

Thus $0 \leq N^a(f) \leq I$, and similarly for $\tilde{N}^a(f)$. Also

$$\mathfrak{m}_a(f)^2 = (\mathfrak{m}_a(f)^*)^2 = 0 \ , \tag{5.5}$$

which entails that both $N^a(f)$ and $\tilde{N}^a(f)$ are projections. As they sum to I, they cannot both vanish. One checks the projection property for $N^a(f)$, for example, by writing

$$N^a(f)^2 = \mathfrak{m}_a(f) \{\mathfrak{m}_a(f) \, , \, \mathfrak{m}_a(f)^*\} \, \mathfrak{m}_a(f)^* = \|f\|^2_{\mathcal{H}} \, \mathfrak{m}_a(f)\mathfrak{m}_a(f)^* = N^a(f) \ . \tag{5.6}$$

As any linear transformation $\mathfrak{m}_a(f)$ on a Hilbert space and its adjoint have the same norm,

$$\|\mathfrak{m}_a(f)^*\|_{\mathcal{F}^a} = \|N^a(f)\|^{1/2}_{\mathcal{F}^a} = \|\mathfrak{m}_a(f)\|_{\mathcal{F}^a} = \|\tilde{N}^a(f)\|^{1/2}_{\mathcal{F}^a} \ . \tag{5.7}$$

so both $N^a(f)$ and $\tilde{N}^a(f)$ have norm 1.

In the bosonic case the commutation relation (4.97) in the case $g = f$ of norm one take the form

$$\mathfrak{m}_s(f)^*\mathfrak{m}_s(f) = \mathfrak{m}_s(f)\mathfrak{m}_s(f)^* + I = N^s(f) + I \ . \tag{5.8}$$

Again this allows us to diagonalize $N^s(f) = \mathfrak{m}_s(f)\mathfrak{m}_s(f)^*$. Choose an orthonormal basis e_j for $\mathcal{H}$, with $e_1 = f$, and let $\Omega^s_{f^{(n)}} = \Omega^s_{\ldots,0,f^{(n)},0,\ldots} \in \mathcal{F}^s \cap \mathcal{F}^s_n$ denote an element of the corresponding orthonormal basis for $\mathcal{F}^s$ obtained by choosing all possible $n = 0, 1, \ldots$ and all possible $f^{(n)}$ of the form (4.71). We show that these vectors are also eigenvectors for $N^s(f)$. The commutation relation (4.97) yields

$$[N^s(f), \mathfrak{m}_s(e_j)] = \delta_{1j} \, \mathfrak{m}_s(e_j) \ , \tag{5.9}$$

Thus any such vector $\Omega^s_{f^{(n)}}$ of the form (4.71), and with $i_1 > 1$, is a null vector for $N^s(f)$. Furthermore, the relation (5.9) shows that $\mathfrak{m}_s(f)$ acts on an eigenvector of $N^s(f)$ with eigenvalue λ, yields another eigenvector of $N^s(f)$ with the eigenvalue raised to $\lambda + 1$. We infer that an element $\Omega^s_{f^{(n)}}$ is an eigenvector of $N^s(b)$ with eigenvalue $\delta_{i_1 1} n_{i_1}$. Taken together with (5.8), we see observe that the $\Omega^s_{f^{(n)}}$ also give a basis of eigenvectors for $\mathfrak{m}_s(f)^*\mathfrak{m}_s(f)$. This completes the proof of the two claimed bounds.

Remark 5.1.2. *For $f \neq 0$, one sometimes calls*

$$N^a(f) = \|f\|^{-2}_{\mathcal{H}} \, \mathfrak{m}_a(f)\mathfrak{m}_a(f)^* \ , \tag{5.10}$$

the fermion number operator for mode f. Likewise

$$N^s(f) = \|f\|^{-2}_{\mathcal{H}} \, \mathfrak{m}_s(f)\mathfrak{m}_s(f)^* \ , \tag{5.11}$$

is called the boson number operator form mode f, and it has spectrum $\mathbb{Z}_+$.

5.2 Nice Vectors

We often need a suitable regular, dense subset of nice vectors $\mathcal{D} \subset \mathcal{F}$ of Fock space to use as the basis domain of definition of certain operators operators. Likewise we use $\mathcal{D}\times\mathcal{D}$ as the basis domain for certain forms. We also use the domain $\mathcal{D}$ to carry out computations, such as the verification of commutation relations for the densities of the fields, to discover the symmetry generated by some self-adjoint transformation, etc. We then extend these relations by continuity to some larger domain by continuity or by other means.

In fact we introduce two such domains: a nice domain $\mathcal{D}^b \subset \mathcal{F}^b = \mathcal{F}^s$ for bosons and also a nice domain $\mathcal{D}^f \subset \mathcal{F}^f = \mathcal{F}^a$ for fermions. Both spaces are defined in the same way. In each case, let $\mathcal{H}$ denote the one-particle space and let $\mathcal{H}_0 \subset \mathcal{H}$ denote a dense subspace of "nice" one-particle wave functions. We choose $\mathcal{D}$ to be vectors that are finite linear combinations of vectors in $\mathcal{D}_0$, where vectors $\Omega \in \mathcal{D}_0$ have the properties:

- Any $\Omega \in \mathcal{D}_0$ has a finite number of non-zero n-particle components Ω_n.
- Each Ω_n is the tensor products of n-nice, one-particle wave functions in $\mathcal{H}_0$.

In the bosonic case, the wave functions are symmetric tensor products of one-particle wave functions; in the fermionic case the wave functions are anti-symmetric tensor products of one-particle wave functions.

5.3 The Weyl Algebra

Define the self-adjiont part of the bosonic creation operator $\mathfrak{m}_s(f)$ of (4.91) as

$$X(f) = \mathfrak{m}_s(f) + \mathfrak{m}_s(f)^* \;, \tag{5.12}$$

with the domain $\mathcal{D}^b \subset \mathcal{F}^b$. This operator is a generator in the Weyl algebra as follows. Note also,

$$X(if) = i\left(\mathfrak{m}_s(f) - \mathfrak{m}_s(f)^*\right) \;. \tag{5.13}$$

Furthermore

$$[X(f), X(g)] = (f,g)_{\mathcal{H}} - (g,f)_{\mathcal{H}} = 2i\Im\,(f,g)_{\mathcal{H}} \;. \tag{5.14}$$

Proposition 5.3.1. *For $f \in \mathcal{H}$, the transformation $X(f)$ is essentially self adjoint on $\mathcal{D}^b$. We denote the closure also by $X(f)$. The unitary Weyl operators $W(f) = e^{iX(f)}$ satisfy*

$$W(f)W(g) = e^{i\Im\langle f,g\rangle_{\mathcal{H}}}\, W(f+g) \;, \tag{5.15}$$

which applied twice yields,

$$W(f)W(g) = e^{2i\Im\langle f,g\rangle_{\mathcal{H}}}\, W(g)W(f) \;. \tag{5.16}$$

Remark 5.3.2. *Two Weyl operators $W(f)$ and $W(g)$ commute, if and only if*

$$\Im \langle f, g\rangle_{\mathcal{H}} \in \pi\mathbb{Z} . \tag{5.17}$$

Proof. Let $\Omega \in \mathcal{D}_0$ be a vector with n_0 or fewer particles. Expand $X(f)^j$ into a linear combination of 2^j monomials of the form $Y(f) = \mathfrak{m}_s(f)^{\#_1}\mathfrak{m}_s(f)^{\#_2}\cdots\mathfrak{m}_s(f)^{\#_j}$, where each # denotes the choice of either $\mathfrak{m}_s(f)$ or $\mathfrak{m}_s(f)^*$, with $j = j_1 + j_2$ specifying $j_1 \geq 0$ choices of $\mathfrak{m}_s(f)$ and $j_2 \geq 0$ choices of $\mathfrak{m}_s(f)^*$. One verifies on each n particle eigenspace of N^b that

$$\mathfrak{m}_s(f)\left(N^b+1\right)^{-1/2} = \left(N^b\right)^{-1/2}\mathfrak{m}_s(f) . \tag{5.18}$$

It follows that

$$\begin{aligned} \|Y(f)\Omega\|_{\mathcal{F}} &= \left\|Y(f)\left(N^b+j+1\right)^{-j/2}\left(N^b+j+1\right)^{j/2}\Omega\right\|_{\mathcal{F}^b} \\ &\leq \left\|Y(f)\left(N^b+j+1\right)^{-j/2}\right\|_{\mathcal{F}^b}\left\|\left(N^b+j+1\right)^{j/2}\Omega\right\|_{\mathcal{F}^b} \\ &\leq (n_0+j+1)^{j/2}\left\|Y(f)\left(N^b+j+1\right)^{-j/2}\right\|_{\mathcal{F}^b}\|\Omega\|_{\mathcal{F}^b} . \end{aligned} \tag{5.19}$$

But (5.18) combined with Proposition 5.1.1 means that

$$\left\|Y(f)\left(N^b+j+1\right)^{-j/2}\right\|_{\mathcal{F}^b} \leq \|f\|_{\mathcal{H}}^j . \tag{5.20}$$

Therefore,

$$\left\|X(f)^j\Omega\right\|_{\mathcal{F}^b} \leq 2^j\left(n_0+j+1\right)^{j/2}\|f\|_{\mathcal{H}}^j\|\Omega\|_{\mathcal{F}^b} , \tag{5.21}$$

from which we infer that the power series

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\left\|X(f)^j\Omega\right\|_{\mathcal{F}^b} , \tag{5.22}$$

converges to a function of an entire function of $\lambda\|f\|_{\mathcal{H}}$ of exponential order 2. Thus $X(f)$ has a dense set of analytic vectors, and its closure is self-adjoint.

Next consider the self-adjoint operator

$$G(\lambda) = W(\lambda f)X(g)W(-\lambda f) , \tag{5.23}$$

with domain $\mathcal{D}$. On this domain, $G(\lambda)$ also has power series in λ with infinite radius of convergence. As all derivatives at the origin of order 2 or more vanish,

$$G(\lambda) = X(g) + 2i\lambda\Im\langle f, g\rangle . \tag{5.24}$$

This identity extends by continuity to the domain of $X(g)$.

Next with $\Omega \in \mathcal{D}_0$ consider the entire function

$$F(\lambda) = e^{-i\lambda^2 \Im\langle f,g\rangle}\, W(-\lambda(f+g))W(\lambda f)W(\lambda g)\Omega\ . \tag{5.25}$$

Note $F(0) = \Omega$. Use (5.24) to obtain

$$\begin{aligned}\frac{dF(\lambda)}{d\lambda} &= e^{-i\lambda^2 \Im\langle f,g\rangle}\, W(-\lambda(f+g))\left(X(-g) - 2i\lambda \Im\langle f,g\rangle\right) W(\lambda f)W(\lambda g)\Omega \\ &\quad + e^{-i\lambda^2 \Im\langle f,g\rangle}\, W(-\lambda(f+g))W(\lambda f)X(g)W(-\lambda f)W(\lambda f)W(\lambda g)\Omega = 0\ .\end{aligned} \tag{5.26}$$

Therefore, $F(\lambda) = \Omega$, which yields the desired identity (5.15) and completes the proof.

5.4 Some Additional Properties when $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$

In this section we assume that $\mathcal{H}$ is a function space. In particular for simplicity we choose $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. The important feature is that $\mathcal{H}$ includes a dense set of nice vectors, and we choose these vectors to be either

$$\mathfrak{d}_1 = \mathcal{S}(\mathbb{R}^{d-1})\ , \qquad \text{or} \qquad \mathfrak{d}_0 = e^{-\omega^2}\mathfrak{d}_1\ . \tag{5.27}$$

Part III

Quantum Fields

Quantum Fields describe an arbitrary number of particles. Thus the Hilbert space on which they live has a more complicated structure than the Hilbert spaces in Proposition I called We begin with the construction of a Fock space $\mathcal{H}$, the natural Hilbert space appropriate for a free (linear) quantum field. For simplicity we begin with a field that describes a single type of scalar (spin zero) particle with mass $m > 0$. One generally begins the study of non-linear or interacting field by perturbing such a linear field.

We take an appropriate Hilbert space $\mathcal{H}_1$ for a single particle. This space will be given an unitary, irreducible, positive energy representation $U(\Lambda, a)$ of the Poincaré group (the inhomogeneous Lorentz group). Such a representation is characterized by the spin and mass, so we have used up our freedom of choice. The full Fock space is the symmetric tensor product exponential of the one-particle space, $\mathcal{H} = \exp_{\otimes_s} \mathcal{H}_1$. The unitary representation of the Poincaré group on $\mathcal{H}_1$ determines a corresponding unitary representation $U(\Lambda, a)$. This representation is highly reducible, and has the interpretation of acting on each of the individual particles.

We begin with a discussion of tensor products that make up the Hilbert space to describe an arbitrary number of particles. This is the space of states for the free field.

Chapter 6

The Free Bosonic Field

In this chapter we define the mass-m, scalar, free quantum field $\varphi(\vec{x}, t)$, acting on Minkowski space-time $\mathbb{R}^{d-1} \times \mathbb{R}$. This field is an operator-valued distribution that satisfies the linear equation of motion

$$\left(\Box + m^2\right) \varphi(\vec{x}, t) = 0 \ , \tag{6.1}$$

along with the canonical constraints on the initial data

$$\varphi(\vec{x}) = \varphi(\vec{x}, 0) \ , \qquad \text{and } \ \pi(\vec{x}) = \left(\frac{\partial \varphi}{\partial t}\right)(\vec{x}, 0) \ , \tag{6.2}$$

that satisfy

$$[\pi(\vec{x}), \varphi(\vec{x}')] = -i\delta(\vec{x} - \vec{x}'), \qquad \text{and } \ [\varphi(\vec{x}), \varphi(\vec{x}')] = 0 = [\pi(\vec{x}), \pi(\vec{x}')] \ . \tag{6.3}$$

6.1 The Local Field

A local field $\varphi(x) = \varphi(\vec{x}, t)$ arises from giving the time-zero field $\varphi(\vec{x})$ the time-dependence generated by a local Hamiltonian H,

$$\varphi(x) = e^{itH} \varphi(\vec{x}) e^{-itH} \ . \tag{6.4}$$

A local Hamiltonian is one which propagates the field with finite speed, so that $\varphi(x)$ and $\varphi(x')$ commute when $x - x'$ is a space-like Minkowski vector. If $H = H_0$ is the free field Hamiltonian, then $\varphi(x)$ is hte free field. We first describe the space of state, introduce the initial field $\varphi(\vec{x})$, and then define the time-dependent free field $\varphi(\vec{x}, t)$.

6.1.1 The Hilbert Space

The Hilbert space of the massive, free scalar field is the bosonic Fock space $\mathcal{F}^b$ over the one particle space $\mathcal{H}$. In the notation of Chapter 4,

$$\mathcal{F}^b = \mathcal{F}^s = \exp_{\otimes_s} \mathcal{H} \ . \tag{6.5}$$

A single-component field has the one-particle space $\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$, namely the Sobolev space introduced (**??**), with inner product

$$\langle f, g\rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^s)} = \left\langle f, (2\omega)^{-1}\, g\right\rangle_{L^2(\mathbb{R}^s)} \ . \tag{6.6}$$

Here $\omega = \left(-\nabla^2 + m^2\right)^{1/2}$ is the relativistic energy for a mass-m particle. The action of ω on $\mathcal{F}^b$ is given by the free-field Hamiltonian H_0. In terms of the operator Γ^s defined in (4.60)–(4.61),

$$H_0 = -\left.\frac{d}{dt}\Gamma(e^{-t\omega})\right|_{t=0} \ . \tag{6.7}$$

The operator H_0 acts on states $\Omega^s_{f(n)} \in \mathcal{F}^s_n$ as

$$\underbrace{\overbrace{\omega \otimes I \otimes \cdots \otimes I}^{n\ factors} + \cdots + I \otimes \cdots \otimes I \otimes \omega}_{n\ terms} \ . \tag{6.8}$$

Exercise 6.1.1. *Show that the creation operators* $\mathfrak{m}_s(\mathfrak{f})$ *of* (4.91) *and their adjoints satisfy*

$$[H_0, \mathfrak{m}_s(\mathfrak{f})] = \mathfrak{m}_s(\omega\mathfrak{f}) \ , \qquad \text{and} \ \ [H_0, \mathfrak{m}_s(\mathfrak{f})^*] = -\mathfrak{m}_s(\omega\mathfrak{f})^* \ , \tag{6.9}$$

or in unitary form

$$e^{itH_0}\mathfrak{m}_s(f)e^{-itH_0} = \mathfrak{m}_s(e^{it\omega}f) \ , \qquad \text{and} \ \ e^{itH_0}\mathfrak{m}_s(f)^*e^{-itH_0} = \mathfrak{m}_s(e^{it\omega}f)^* \ . \tag{6.10}$$

The unitary representation $U(\Lambda, a)$ of the Poincaré group on $\mathcal{F}_1$ gives rise to a unitary representation of the Poincaré group on $\mathcal{F}$ equal to

$$\Gamma^s(U(\Lambda, a)) \ . \tag{6.11}$$

Thus we obtain a unitary representation of the Poincaré group on $\mathcal{F}^b$. By definition, This group leaves the vector $f = \{1, 0, 0, 0, \ldots\}$ invariant, and one calls this the no-particle vector the *vacuum-*vector

$$\Omega^s_0 = \{1, 0, 0, 0, \ldots\} \ . \tag{6.12}$$

The vacuum is an eigenvector of each generator of the Poincaré group with eigenvalue zero. These generators are identified with energy, momentum, angular momentum, and infinitesimal boosts. Thus

$$H_0\Omega^s_0 = P_j\Omega^s_0 = L_{ij}\Omega^s_0 = M_{ij}\Omega^s_0 = 0 \ . \tag{6.13}$$

6.1.2 Time-Zero Fields

We continue with the choice of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ as the space (6.49) of generalized functions $\mathfrak{H}_{-1/2}$. Then any $f \in \mathcal{H}$ can be decomposed into real and imaginary parts,

$$f = f_r + if_i \ , \ \text{ with } \ f_r, f_i \ \text{ real } \ . \tag{6.14}$$

Let us define

$$\mathfrak{H}_{-1/2 \text{ real}}(\mathbb{R}^s) = \{f : f \in \mathfrak{H}_{-1/2}(\mathbb{R}^s) \ , \ \text{ and } \ f = f_r\} \ . \tag{6.15}$$

The operator ω is real. This means that ω (as well as a real function of ω) transforms real function f to real functions f, so if $f \in \mathcal{D}(\omega)$,

$$\omega f = \omega f_r + \omega f_1 \ . \tag{6.16}$$

The self-adjoint part of the creation operator $\mathfrak{m}_s(f)$ for real and purely imaginary functions $f \in \mathcal{H}$ play a special role. Recall we already introduced the self adjoint part of $\mathfrak{m}_s(f)$ for general $f \in \mathcal{H}$ in (5.12); we called $X(f)$ the generator of a unitary Weyl operator $W(f) = e^{iX(f)}$.

Definition 6.1.1. *The time-zero boson field φ is*

$$\varphi(f) = X(f) = \mathfrak{m}_s(f) + \mathfrak{m}_s(f)^* \ , \qquad \textit{in case } \ f \in \mathfrak{H}_{-1/2 \text{ real}}(\mathbb{R}^s) \ . \tag{6.17}$$

The canonically conjugate time-zero field is π, given as

$$\pi(f) = X(i\omega f) = i\left(\mathfrak{m}_s(\omega f) - \mathfrak{m}_s(\omega f)^*\right) \ , \qquad \textit{in case } \ \omega f \in \mathfrak{H}_{-1/2 \text{ real}}(\mathbb{R}^s) \ . \tag{6.18}$$

With these choices, the fundamental commutation relation (5.14) for real f, g take their standard form.

Proposition 6.1.2. *The canonical fields satisfy*

$$[\pi(f), \varphi(g)] = -i\left\langle f, g\right\rangle_{L^2(\mathbb{R}^s)} \ , \tag{6.19}$$

and the Weyl relation (5.16) *becomes*

$$e^{i\pi(f)}\ e^{i\varphi(g)} = e^{i\langle f,g\rangle_{L^2(\mathbb{R}^s)}}\ e^{i\varphi(g)}\ e^{i\pi(f)} \ . \tag{6.20}$$

Proof. These relations follow from

$$[\pi(f), \varphi(g)] = 2i\Im\left\langle i\omega f, g\right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^s)} = -i\left\langle f, 2\omega g\right\rangle_{\mathfrak{H}_{-1/2}(\mathbb{R}^s)} = -i\left\langle f, g\right\rangle_{L^2(\mathbb{R}^s)} \ , \tag{6.21}$$

and the Weyl relation (5.16). We can extend $\varphi(f)$ and $\pi(f)$ in an linear manner to all functions $f = f_r + if_i$ in $\mathfrak{H}_{-1/2}(\mathbb{R}^s)$ or $\mathfrak{H}_{-3/2}(\mathbb{R}^s)$ respectively. In this way,

$$\begin{aligned} \varphi(f) &= \mathfrak{m}_s(f) + \mathfrak{m}_s(\overline{f})^* \\ \pi(f) &= i\left(\mathfrak{m}_s(\omega f) - \mathfrak{m}_s(\overline{\omega f})^*\right) \ . \end{aligned} \tag{6.22}$$

If one wanted, one could express this inelegantly in terms of the $X(f)$'s. For example,

$$\varphi(f) \;=\; \frac{1}{2}\left(X(f)+X(\overline{f})-iX(if)+iX(i\overline{f})\right)\ . \tag{6.23}$$

Once these fields have been defined, one can extend them (linearly) to complex test functions f. Thus with f decomposed as in (6.14) or (6.17), let

$$\varphi(f)=\varphi(f_r)+i\varphi(f_i)\ , \qquad \text{and } \pi(f)=\pi(f_r)+i\pi(f_i)\ , \tag{6.24}$$

6.2 The Free Field

The initial value for the field φ and its canonically conjugate field π do not determine the time evolution of the field. That is given by the Hamiltonian. For the free field, we already introduced the Hamiltonian H_0. We use the Hamiltonian to define the time translation. For the free field, take a real test function $\mathfrak{f}$ and define

$$\begin{aligned}\varphi(\mathfrak{f},t) &= e^{itH_0}\left(\mathfrak{m}_s(\mathfrak{f})+\mathfrak{m}_s(\bar{\mathfrak{f}})^*\right)e^{-itH_0}\\ &= \mathfrak{m}_s(e^{it\omega}\mathfrak{f})+\mathfrak{m}_s(\overline{e^{it\omega}\mathfrak{f}})^*\ .\end{aligned} \tag{6.25}$$

$$\varphi(\mathfrak{f})\Omega_0^s=\Omega_{\mathfrak{f}}^s\ . \tag{6.26}$$

One recovers the standard creation operators by introducing

$$a^*(\mathfrak{f})=\mathfrak{m}_s((2\omega)^{1/2}\mathfrak{f})\ , \qquad \text{and } a(\mathfrak{f})=a(\bar{\mathfrak{f}})^*=\mathfrak{m}_s(\bar{\mathfrak{f}})^*\ . \tag{6.27}$$

As a consequence of the commutation relations (4.97) for the operators $\mathfrak{m}_s$ and $\mathfrak{m}_s^*$, one sees that the transformations a, a^* satisfy

$$[a(\mathfrak{f}),a^*(\mathfrak{g})]=\left\langle\bar{\mathfrak{f}},g\right\rangle_{L^2(\mathbb{R}^s)}\ . \tag{6.28}$$

One can also write (6.9) as

$$[H_0,a^*(\mathfrak{f})]=a^*(\omega\mathfrak{f})\ . \tag{6.29}$$

Thus the time-zero field can be written in its usual form,

$$\varphi(\mathfrak{f})=\frac{1}{\sqrt{2}}\left(a^*(\omega^{-1/2}\mathfrak{f})+a(\omega^{-1/2}\mathfrak{f})\right)\ , \tag{6.30}$$

and the time-dependent field is

$$\varphi(\mathfrak{f},t)=\frac{1}{\sqrt{2}}\left(a^*(e^{it\omega}\omega^{-1/2}\mathfrak{f})+a(e^{-it\omega}\omega^{-1/2}\mathfrak{f})\right)\ . \tag{6.31}$$

This field satisfies the equation of motion. One has

$$\frac{\partial^2}{\partial t^2}\varphi(\mathfrak{f},t) = -a^*(e^{it\omega}(2\omega)^{-1/2}\omega^2\mathfrak{f}) - a(e^{-it\omega}(2\omega)^{-1/2}\omega^2\mathfrak{f}) \tag{6.32}$$

As $\omega^2 = -\nabla^2 + m^2$, one has the distribution equation (6.1) $\omega^2 = -\nabla^2 + m^2$. Furthermore, the initial data for the time derivative is determined by differentiating the representation (6.31) and defining

$$\begin{aligned}\pi(\mathfrak{f}) &= \left.\frac{\partial\varphi(\mathfrak{f},t)}{\partial t}\right|_{t=0} \\ &= \frac{i}{\sqrt{2}}\left(a^*(\omega^{1/2}\mathfrak{f}) - a(\omega^{1/2}\mathfrak{f})\right) .\end{aligned} \tag{6.33}$$

The commutation relations at fixed time are

$$[\pi(\mathfrak{f}),\varphi(\mathfrak{g})] = -i\left\langle\bar{\mathfrak{f}},\mathfrak{g}\right\rangle_{L^2(\mathbb{R}^s)} , \qquad \text{and } [\varphi(\mathfrak{f}),\varphi(\mathfrak{g})] = 0 = [\pi(\mathfrak{f}),\pi(\mathfrak{g})] . \tag{6.34}$$

6.2.1 Fields at a Point

We express $a(\mathfrak{f})$ as a density

$$a(\mathfrak{f}) = \int_{\mathbb{R}^s} a(\vec{x})\mathfrak{f}(\vec{x})d\vec{x} , \qquad \text{and } a^*(\mathfrak{f}) = \int_{\mathbb{R}^s} a^*(\vec{x})\mathfrak{f}(\vec{x})d\vec{x} . \tag{6.35}$$

The densities for a, a^* are forms, not operators, which we deal with shortly in §6.5. They satisfy

$$[a(\vec{x}),a^*(\vec{y})] = \delta(\vec{x}-\vec{y}) , \qquad \text{and } [a(\vec{x}),a(\vec{y})] = [a^*(\vec{x}),a^*(\vec{y})] = 0 . \tag{6.36}$$

Exercise 6.2.1. *Show that the bosonic number operator N^b and the bosonic Hamiltonian H_0 as defined above agree with the ususal definitions,*

$$N^b = \int_{\mathbb{R}^s} a^*(\vec{x})a(\vec{x})d\vec{x} , \qquad \textit{and } H_0 = \int_{\mathbb{R}^s} a^*(\vec{x})\,\omega\,a(\vec{x})d\vec{x} . \tag{6.37}$$

With these densities,

$$\begin{aligned}\varphi(\vec{x},t) &= e^{itH_0}\varphi(\vec{x})e^{-itH_0} \\ &= \frac{1}{\sqrt{2\omega}}\left(e^{it\omega}a^*(\vec{x}) + e^{-it\omega}a(\vec{x})\right) .\end{aligned} \tag{6.38}$$

6.2.2 Momentum Space Representation

We have a corresponding Fourier representation. Define

$$a(\vec{x}) = \frac{1}{(2\pi)^{s/2}}\int_{\mathbb{R}^s} e^{i\vec{k}\cdot\vec{x}}a(\vec{k})d\vec{k} , \qquad \text{or equivalently } a(\vec{k}) = \frac{1}{(2\pi)^{s/2}}\int_{\mathbb{R}^s} e^{-i\vec{k}\cdot\vec{x}}a(\vec{x})d\vec{x} . \tag{6.39}$$

Then

$$a^*(\vec{x}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{-i\vec{k}\cdot\vec{x}} a^*(\vec{k}) d\vec{k} \;, \qquad \text{or } a^*(\vec{k}) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{i\vec{k}\cdot\vec{x}} a^*(\vec{x}) d\vec{x} \;. \tag{6.40}$$

As a consequence of (6.36), these forms satisfy

$$\left[a(\vec{k}), a^*(\vec{k}')\right] = \delta(\vec{k} - \vec{k}') \;, \qquad \text{and } \left[a(\vec{k}), a(\vec{k}')\right] = \left[a^*(\vec{k}), a^*(\vec{k}')\right] = 0 \;. \tag{6.41}$$

In the momentum representation, ω acts as the multiplication operator $\omega(\vec{k})$. Then the representation for the field (6.38) takes the usual form

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} \frac{1}{\sqrt{2\omega(\vec{k})}} \left(e^{it\omega(\vec{k}) - i\vec{k}\cdot\vec{x}} a^*(\vec{k}) + e^{-it\omega(\vec{k}) + i\vec{k}\cdot\vec{x}} a(\vec{k}) \right) d\vec{k} \;. \tag{6.42}$$

6.2.3 Commutation Relation

The commutation relation for the free field is easy to commute, as the time dependence allows one to express the field as a linear combination of creation and annihilation operators, see (6.42) that satisfy the canonical commutation relations. Thus

$$\begin{aligned} [\varphi(\vec{x}, t), \varphi\vec{x}', t'] &= \langle \Omega_0, \varphi(\vec{x}, t)\varphi(\vec{x}', t')\Omega_0 \rangle - \langle \Omega_0, \varphi(\vec{x}', t')\varphi(\vec{x}, t)\Omega_0 \rangle \\ &= W(x - x') - W(x' - x) = \Delta(x - x') \;, \end{aligned} \tag{6.43}$$

where $W(x - x')$ is the Poincaré-invariant generalized function

$$W(x - x') = \frac{1}{(2\pi)^s} \int_{k_d > 0} e^{-ik(x - x')} \delta(k^2 - m^2) dk \;. \tag{6.44}$$

Note $\Delta(x)$ and $W(x)$ are are both Lorentz-invariant generalized functions that are solutions to the Klein-Gordon equation. (Do not confuse this Δ with the Laplace operator.) The solution $W(x)$ has the initial data

$$W(\vec{x}, 0) = G(\vec{x}) \;, \qquad \text{and} \qquad \left(\frac{\partial W}{\partial t} \right)(\vec{x}, 0) = -\frac{i}{2}\delta(\vec{x}) \;. \tag{6.45}$$

where $G(\vec{x})$ is the Green's function of $(2\omega)^{-1}(\vec{x})$ introduced in (2.53). The solution $\Delta(x)$ has the initial data

$$\Delta(\vec{x}, 0) = 0 \;, \qquad \text{and} \qquad \left(\frac{\partial \Delta}{\partial t} \right)(\vec{x}, 0) = -i\delta(\vec{x}) \;. \tag{6.46}$$

If $x - x'$ is a space-like vector, i.e. $(x - x')^2 < 0$, then there is a Lorentz transformation Λ such that $\Lambda(x - x') = x' - x$. Clearly $W(x) = W(\Lambda x)$ is invariant invariant (in the sense of generalized functions) so

$$[\varphi(\vec{x}, t), \varphi\vec{x}', t'] = \Delta(x - x') = 0 \;, \qquad \text{for } (x - x')^2 < 0 \;. \tag{6.47}$$

6.3 Imaginary Time Fields

The imaginary time field is defined by

$$\varphi_I(\vec{x}, t) = \varphi(\vec{x}, it) = e^{-tH_0}\varphi(\vec{x})e^{tH_0} \ . \tag{6.48}$$

6.4 Compact Space

On the other hand, in case we use the compactified space equal to a torus $\mathbb{T}^s$, then we choose

$$\mathcal{H} = \mathfrak{H}_{-1/2}(\mathbb{T}^s) \ , \tag{6.49}$$

where the one-particle inner product is

$$\langle f, g \rangle_{\mathfrak{H}_{-1/2}(\mathbb{T}^s)} = \left\langle (2\omega_T)^{-1/2} f, (2\omega_{\mathbb{T}})^{-1/2} g \right\rangle_{L^2(\mathbb{R}^s)} \ . \tag{6.50}$$

Here

$$\omega_T = \left(-\nabla_T^2 + m^2\right)^{1/2} \ . \tag{6.51}$$

Likewise, if we work on a different one-particle space such as a lattice space, a section of a Riemann surface, etc., we take the appropriate definition of $\mathcal{H}$.

6.5 Forms and Number Bounds

In case of ordinary quantum field theory defined on Euclidean space $\mathbb{R}^s$, we take $\mathcal{H}_0$ to be the Schwartz space

$$\mathcal{H}_0 = \mathcal{S}(\mathbb{R}^s) \ . \tag{6.52}$$

Then transformations such as (positive or negative fractional) powers of ω, or $e^{-t\omega}$ with $t \geq 0$, map $\mathcal{H}_0$ to $\mathcal{H}_0$. Such transformations are determined uniquely as self-adjiont transformations on $\mathcal{H}$ by defining them on this domain $\mathcal{H}_0$.

6.6 Poincaré Invariance

6.7 Locality

6.8 Wightman Functions

6.9 Reeh-Schlieder Property

Chapter 7

The Fundamental Bound for Fields

Perhaps the most fundamental operator in a field theory is the Hamiltonian H, which we assume is self-adjoint and positive. Stability of the Hamiltonian $0 \leq H$ is central to many aspects of physics. Having a Hamiltonian, we turn to the field itself. The property of the time-zero field $\varphi(g)$ that serves as a fundamental starting point is the *key bound*, comparing the the field $\varphi(g)$ with the Hamiltonian H. The key bound provides the input to a robust machine from which one can derive many desired properties of quantum field operators.

The key bound is not a consequence of the Wightman axioms for quantum field theory, nor of the Osterwalder-Schrader axioms for Euclidean Green's functions. The key bound is an additional property that we desire as a starting point for our quantum fields. It is a property that we return to in later chapters where we develop methods to ensure that the key bound holds.

In this chapter, we highlight why we want to establish the key bound, by showing that it ensures a number of fundamental properties of the fields. For example, the key bound lets us pass from fields that are forms to fields that are operators on Hilbert space. It also ensures that the field operators are self-adjoint. The key bound ensures also that expectation values of all products of field operators exist. In addition, the key bound entails that the fields are *local* in the sense that if the commutator of two fields $[\varphi(f), \varphi(g)] = 0$, then quantum mechanical observables depending on the field $\varphi(f)$ commute with those depending on the field $\varphi(g)$.

We introduce the time-zero field as a form.[1] This means that we initially study matrix elements of the field (or equivalently expectation values of the field) in states Ω that are "smooth vectors" for the Hamiltonian H. This means that Ω is in the domain of arbitrary powers H^n of the the Hamiltonian H, which we write $\Omega \in C^\infty(H)$.

[1]See §8.5 for the a full definition of forms, as well as for a discussion of certain properties of forms.

7.1 The Fundamental Bound

We assume that the time-zero field $\varphi(g)$ is a form with domain $C^\infty(H) \times C^\infty(H)$. The key bound says:

$$\boxed{\pm\varphi(g) \leq H + I}\ , \tag{7.1}$$

for all real $g \in C_0^\infty(\mathbb{R}^s)$ for which $\|g\|_{\mathcal{K}} \leq 1$. In more detail, g is a test function $g(\vec{x})$ with $\vec{x} \in \mathbb{R}^s = \mathbb{R}^{d-1}$. The norm $\|\cdot\|_{\mathcal{K}}$ is a norm that is bounded by some finite linear combination of Schwartz space norms on $\mathcal{S}(\mathbb{R}^s)$.[2] Define the normed space $\mathcal{K}$ as the completion of the set $\mathcal{S}(\mathbb{R}^s)$ in the $\mathcal{K}$-norm. As the field is a linear function of g, so once we have shown that it is true, we can substitute $g/\|g\|_{\mathcal{K}}$ for g. Thus the key bound (7.1) is also equivalent to the bound,

$$\boxed{\pm\varphi(g) \leq \|g\|_{\mathcal{K}}\ (H + I)}\ , \tag{7.2}$$

for all real $g \in C_0^\infty(\mathbb{R}^s)$.

The key bound is equivalent to a stability bound for perturbations of H, namely

$$\boxed{0 \leq H + I \pm \varphi(g)}\ , \tag{7.3}$$

for all real $g \in C_0^\infty$ for which $\|g\|_{\mathcal{K}} \leq 1$.

Example. The free time-zero, mass-m field and its Hamiltonian $H = H_0$ provide a useful guide. One can find the exact lower bound of $H_0 + \varphi(g)$, namely

$$0 \leq H_0 + \varphi(g) + \frac{1}{2}\left\|\omega^{-1}g\right\|^2_{L^2(\mathbb{R}^s)}\ , \tag{7.4}$$

where $\omega = (-\nabla^2 + m^2)^{1/2}$. Thus in the case of the free field one can take the norm $\|\cdot\|_{\mathcal{K}}$ to be

$$\|g\|_{\mathcal{K}} = \frac{1}{\sqrt{2}}\left\|\omega^{-1}g\right\|_{L^2(\mathbb{R}^s)}\ , \tag{7.5}$$

and $\mathcal{K}$ the corresponding Sobolev space $\mathfrak{H}_{-1}(\mathbb{R}^s)$. With our choice of normalization, $\varphi(g)$ satisfies the bound (7.1) for $\|g\|_{\mathcal{K}} \leq 1$.

In the particular case of spatial dimension $s = 1$, the Dirac measure $\delta_x \in \mathcal{K}$. Thus one can choose g to be a real multiple λ of δ_x, in which case the time-zero free field $\varphi(x)$ satisfies the key bound

$$\pm\lambda\,\varphi(x) \leq H_0 + I\ ,\ \text{ as long as }\ \lambda^2 \leq 4m\ \ (\text{when }\ s = 1)\ . \tag{7.6}$$

[2]For example, as explained in §9.2 one can define the Schwartz space $\mathcal{S}(\mathbb{R}^s)$ using the increasing family of norms $\|g\|_n = \|h^n g\|_{L^2(\mathbb{R}^s)}$, with h is the Hamiltonian of a homogeneous, unit-frequency harmonic oscillator on $\mathbb{R}^s$. In other words, take

$$h = \frac{1}{2}\left(-\Delta + x^2 - s\right)\ ,$$

where Δ denotes the Laplace operator on $\mathbb{R}^s$, and where $x \in \mathbb{R}^s$. In this case, $\mathcal{S}(\mathbb{R}^s) = C^\infty(h)$, and we assume that there exists some n for which $\|g\|_{\mathcal{K}} \leq \|g\|_n$. Then every function $g \in \mathcal{S}(\mathbb{R}^s)$ has finite norm $\|g\|_{\mathcal{K}}$.

7.1.1 The Fundamental Bound and Field Operators

The space-time field $\varphi(f)$ is

$$\varphi(f) = \int e^{itH}\varphi(\vec{x})e^{-itH}f(\vec{x},t)dx\ , \qquad \text{for } f \in C_0^\infty(\mathbb{R}^d)\ , \text{ with } \mathbb{R}^d = R^{s+1}\ . \tag{7.7}$$

Clearly our assumption that the time-zero field $\varphi(g)$ is a form on the domain $C^\infty(H)\times C^\infty(H)$, ensures that the space-time field $\varphi(f)$ is also such a form. In fact $C^\infty(H)$ is invariant under the unitary group e^{-itH}. Therefore define the sharp time field $\varphi(g^{(t)})$ for the test function $g^{(t)}$ equal to

$$g^{(t)}(\vec{x}) = f(\vec{x},t)\ . \tag{7.8}$$

This field is a form on the domain $C^\infty(H)\times C^\infty(H)$, and the resulting form is a C^∞ function of t. One can integrate this form over t to obtain the space-time field $\varphi(f) = \int \varphi(g^{(t)})\,dt$.

Define a norm $M(f)$ by

$$M(f) = \int \left\|g^{(t)}\right\|_{\mathcal{K}} dt = \int \|f(\cdot,t)\|_{\mathcal{K}}\, dt\ . \tag{7.9}$$

Then for real f the space-time field obeys the primitive bound

$$\boxed{\pm\varphi(f) \le M(f)\ (H+I)}\ . \tag{7.10}$$

The fundamental bound (7.1) allows us to pass from the form $\varphi(f)$ with domain $C^\infty(H)\times C^\infty(H)$ to an operator $\varphi(f)$ with the domain $\mathcal{D}(H)$. The requirement that $\varphi(f)$ determine an operator requires a slightly more restrictive norm on f that includes one time derivative $\partial_t f = \partial f/\partial t$. Define the norm

$$\boxed{|||f||| = M(f) + M(\partial_t f)}\ . \tag{7.11}$$

In the case of a product test function $f = g\otimes h$, namely $f(x) = g(\vec{x})h(t)$ with $f' = df/dt$, one has

$$|||g\otimes h||| = \|g\|_{\mathcal{K}}\left(\|f\|_{L^1(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}\right)\ . \tag{7.12}$$

Theorem 7.1.1 (Field Operators). *Assume that the time-zero field $\varphi(g)$ is a form on the domain $C^\infty(H)\times C^\infty(H)$, and that $\varphi(g)$ satisfies the key bound (7.1). Consider the space-time field $\varphi(f)$ as the form defined in (7.7). Then all the following hold:*

i. **Field Operators Exist.** *Let $f \in C_0^\infty$. The form $\varphi(f)$ determines a unique field operator $\varphi(f)$ with the domain $\mathcal{D}(H)$, whose matrix elements agree with those of the form $\varphi(f)$. The field satisfies*

$$\left\|\varphi(f)(H+I)^{-1}\right\| \le |||f|||\ . \tag{7.13}$$

The operator $\varphi(f)$ has a closure $\varphi(f)^-$. For real f the operator $\varphi(f)$ is symmetric.

ii. **Essential Self-Adjointness.** *For real f the closure $\varphi(f)^-$ of $\varphi(f)$ is self-adjoint.*

iii. **Locality.** *Let $\varphi(f)^-$ and $\varphi(g)^-$ be two such self-adjoint fields with real f, g. Suppose that*

$$[\varphi(f), \varphi(g)] = 0 \,, \qquad \text{as a form on } C^\infty(H) \times C^\infty(H) \,. \tag{7.14}$$

Then the unitary operators generated by $\varphi(f)^-$ and $\varphi(g)^-$ also commute: both $\varphi(f)^-\varphi(g)^-$ and $\varphi(g)^-\varphi(f)^-$ are defined and equal, and also

$$\left[e^{i\varphi(f)^-}, e^{i\varphi(g)^-}\right] = 0 \,. \tag{7.15}$$

iv. **Limiting Test Functions.** *Let f be any function with $|||f||| < \infty$ that can be approximated by a sequence $f_n \in C_0^\infty$ in the sense that $|||f_n - f||| \to 0$. Then there exists a field operator $\varphi(f)$ with the dense domain $\mathcal{D}(H)$ that satisfies the bound (7.13). The matrix elements of this operator agree with the matrix elements of the form $\varphi(f)$. The operator $\varphi(f)$ has a closure. If f is real, then $\varphi(f)$ is essentially self-adjoint. Such self-adjoint operators $\varphi(f)^-$ arising for real f are also local in the sense of (iii) if in addition $|||\partial_t f||| < \infty$.*

Denote the resolvent of $H + I$ by

$$R(\lambda) = (H + I + \lambda)^{-1} \,, \ \text{for } \lambda \geq 0 \,, \qquad \text{and} \qquad R = (H + I)^{-1} \,. \tag{7.16}$$

Then for $0 \leq \alpha$

$$\|R(\lambda)^\alpha\| \leq (1 + \lambda)^{-\alpha} \,, \qquad \text{and} \qquad \|(H + I)^\alpha R(\lambda)^\alpha\| \leq 1 \,. \tag{7.17}$$

We approximate $\varphi(f)$ by $\varphi_\lambda(f)$, defined as

$$\boxed{\varphi_\lambda(f) = \lambda R(\lambda)^{1/2} \varphi(f) R(\lambda)^{1/2}} \,. \tag{7.18}$$

With this notation, we collect some useful bounds:

Lemma 7.1.2. *Under the hypotheses of the theorem:*

a. For any $\lambda \geq 0$, the form $\varphi_\lambda(f)$ is bounded, and has with norm less than

$$\|\varphi_\lambda(f)\| \leq \lambda M(f) \,. \tag{7.19}$$

Thus $\varphi_\lambda(f)$ uniquely determines a bounded operator (which we also denote by $\varphi_\lambda(f)$) and for real f this operator is self-adjoint.

b. The bounded operators $\varphi_\lambda(f)$ and $R^{1/2}\varphi(f)R^{1/2}$ satisfy

$$\|R\left(\varphi_\lambda(f) - \varphi(f)\right) R\| \leq 2(1 + \lambda)^{-1/2} M(f) \,. \tag{7.20}$$

c. The commutator $[\varphi(f), R(\lambda)]$ is a bounded form on $C^\infty(H) \times C^\infty(H)$, with a norm that obeys

$$\| \left[\varphi(f), R(\lambda)\right] \| \leq (1 + \lambda)^{-1} M(\partial_t f) \,. \tag{7.21}$$

d. *The commutator* $\left[\varphi(f), R(\lambda)^{1/2}\right]$ *is a bounded form on* $C^\infty(H) \times C^\infty(H)$*, with a norm that obeys*

$$\left\|\left[\varphi(f), R(\lambda)^{1/2}\right]\right\| \leq (1+\lambda)^{-1/2} M(\partial_t f) \; . \tag{7.22}$$

e. *The operator* $T_\lambda(f) = (H+I)^{1/2} e^{i\varphi_\lambda(f)} R^{1/2}$ *is bounded. The norm of* $T_\lambda(f)$ *satisfies*

$$\|T_\lambda(f)\| \leq e^{\frac{1}{2}M(\partial_t f)} \; . \tag{7.23}$$

Proof. The bound (a) is an immediate consequence of the primitive bound (7.10) and (7.17) for $\alpha = 1/2$. On the domian $\mathcal{D}(H)$ we derive the identity

$$\begin{aligned} \delta_\lambda &= R^{1/2}\left(I - \lambda^{1/2} R(\lambda)^{1/2}\right) \\ &= \left(I + \lambda^{1/2} R(\lambda)^{1/2}\right)^{-1} R^{1/2}(I - \lambda R(\lambda)) \\ &= \left(I + \lambda^{1/2} R(\lambda)^{1/2}\right)^{-1} (H+I)^{1/2} R(\lambda) \; , \end{aligned} \tag{7.24}$$

leading to the bound $0 \leq \left(I + \lambda^{1/2} R(\lambda)^{1/2}\right)^{-1} \leq I$ and therefore,

$$\|\delta_\lambda\| \leq \left\|R(\lambda)^{1/2}\right\| \leq (1+\lambda)^{-1/2} \; . \tag{7.25}$$

It follows that

$$\begin{aligned} R\left(\varphi_\lambda(f) - \varphi(f)\right) R &= \left(R^{1/2}\lambda^{1/2}R(\lambda)^{1/2}\right)\left(R^{1/2}\varphi(f)R^{1/2}\right)\left(R^{1/2}\lambda^{1/2}R(\lambda)^{1/2}\right) - R\varphi(f)R \\ &= -\delta_\lambda\left(R^{1/2}\varphi(f)R^{1/2}\right)\left(\lambda^{1/2}R(\lambda)^{1/2}R^{1/2}\right) + R^{1/2}\left(R^{1/2}\varphi(f)R^{1/2}\right)\delta_\lambda \end{aligned} \tag{7.26}$$

Hence using (7.10) and (7.17),

$$\|R\left(\varphi_\lambda(f) - \varphi(f)\right) R\| \leq 2(1+\lambda)^{-1/2} M(f) \; , \tag{7.27}$$

as claimed.

In order to establish (c), use the fact that f is smooth and compactly supported. The domain $C^\infty(H)$ is left invariant by the unitary group e^{-itH}. Thus we can differentiate the matrix elements of the field on $C^\infty(H) \times C^\infty(H)$, and use integration by parts to establish the identity of forms,

$$[H, \varphi(f)] = i\varphi(\partial_t f) \; . \tag{7.28}$$

Furthermore

$$[\varphi(f), R(\lambda)] = R(\lambda)\,[H, \varphi(f)]\,R(\lambda) = iR(\lambda)\varphi(\partial_t f)R(\lambda) \; . \tag{7.29}$$

Therefore one can use the bound (7.19) and the bound (7.17) to obtain

$$\|\,[\varphi(f), R(\lambda)]\,\| \leq \left\|R(\lambda)^{1/2}\right\|\left\|R(\lambda)^{1/2}\,\varphi(\partial_t f)R(\lambda)^{1/2}\right\|\,\left\|R(\lambda)^{1/2}\right\| \leq (1+\lambda)^{-1}M(\partial_t f) \; , \tag{7.30}$$

as claimed.

In order to establish (d), use the representation

$$R(\lambda)^{1/2} = \frac{1}{\pi}\int_0^\infty \lambda'^{-1/2} R(\lambda+\lambda')d\lambda' \,, \tag{7.31}$$

which is the relation (3.89) for $\alpha = 1/2$. Hence one infers

$$\begin{aligned} \left[\varphi(f), R(\lambda)^{1/2}\right] &= \frac{1}{\pi}\int_0^\infty \lambda'^{-1/2}\left[\varphi(f), R(\lambda+\lambda')\right] d\lambda' \\ &= \frac{i}{\pi}\int_0^\infty \lambda'^{-1/2} R(\lambda+\lambda')\varphi(\partial_t f) R(\lambda+\lambda')d\lambda' \,. \end{aligned} \tag{7.32}$$

Its norm can be bounded by

$$\begin{aligned} \left\|\left[\varphi(f), R(\lambda)^{1/2}\right]\right\| \;\le\; & \frac{1}{\pi}\int_0^\infty \lambda'^{-1/2}\left\|R(\lambda+\lambda')^{1/2}\right\| \left\|R(\lambda+\lambda')^{1/2}\varphi(\partial_t f)R(\lambda+\lambda')^{1/2}\right\| \\ & \times\left\|R(\lambda+\lambda')^{1/2}\right\| d\lambda' \,. \end{aligned} \tag{7.33}$$

Using (7.17) and (7.19), one bounds (7.33) by

$$\left\|\left[\varphi(f), R(\lambda)^{1/2}\right]\right\| \le \left(\frac{1}{\pi}\int_0^\infty \lambda'^{-1/2}(1+\lambda+\lambda')^{-1}d\lambda'\right) M(\partial_t f) \,. \tag{7.34}$$

The identity (7.31) shows that the λ'-integral equals $(1+\lambda)^{-1/2}$. Therefore (7.22) follows.

Now we turn to the proof of (e). One observes that the bounded operator $\varphi_\lambda(f)$ maps $C^\infty(H)$ to $C^\infty(H)$, and as a consequence of part (d) of this lemma we infer that $S_\lambda(f) = (H+I)\varphi_\lambda(f)R$ is also bounded. Consequently $(H+I)e^{i\varphi_\lambda(f)}R$ equals the convergent exponential power series $e^{S_\lambda(f)}$, and

$$\begin{aligned} F(s) &= (H+I)^{1/2}\left(e^{S_\lambda(f)}\right)^* e^{S_\lambda(f)}(H+I)^{1/2} \\ &= R^{1/2}\, e^{-is\varphi_\lambda(f)}\,(H+I)\, e^{is\varphi_\lambda(f)}\, R^{1/2} \,, \qquad \text{for } 0\le s\le 1 \,. \end{aligned} \tag{7.35}$$

Thus setting $T_\lambda(f) = (H+I)^{1/2}e^{i\varphi_\lambda(f)}R^{1/2}$, this family interpolates between $F(0) = I$ and $F(1) = T_\lambda(f)^* T_\lambda(f)$, with the property that $\|F(1)\| = \|T_\lambda(f)\|^2$.

Compute the derivative of $F(s)$ as a form on the domain $C^\infty(H)\times C^\infty(H)$. This yields the differential inequality,

$$\begin{aligned} \frac{dF(s)}{ds} &= -iR^{1/2}\, e^{-is\varphi_\lambda(f)}\left[\varphi_\lambda(f), H\right] e^{is\varphi_\lambda(f)}\, R^{1/2} \\ &= -R^{1/2}\, e^{-is\varphi_\lambda(f)}\,\varphi_\lambda(\partial_t f)\, e^{is\varphi_\lambda(f)} R^{1/2} \\ &= -R^{1/2}\, e^{-is\varphi_\lambda(f)}\,(H+I)^{1/2}\, R^{1/2}\,\varphi_\lambda(\partial_t f)\, R^{1/2}\,(H+I)^{1/2}\, e^{is\varphi_\lambda(f)} R^{1/2} \\ &\le M(\partial_t f)\, R^{1/2}\, e^{-is\varphi_\lambda(f)}\,(H+I)\, e^{is\varphi_\lambda(f)} R^{1/2} \\ &= M(\partial_t f)\, F(s) \,. \end{aligned} \tag{7.36}$$

Here we used the identity (7.28) and the bound (7.19). We also used the property for self-adjoint B that

$$\pm A^*BA \leq \|B\| A^*A \;, \tag{7.37}$$

as a consequence of

$$\langle \chi, A^*BA\chi\rangle \leq \|B\| \|A\chi\|^2 = \|B\| \langle \chi, A^*A\chi\rangle \;. \tag{7.38}$$

Take $A = (H + I)^{1/2} e^{is\varphi_\lambda(f)} R^{1/2}$ and $B = -R^{1/2}\varphi(\partial_t f)R^{1/2}$ to obtain (7.36). Integrating (7.36) gives $\ln(F(s)/F(0)) \leq M(\partial_t f)s$, so

$$F(s) = e^{M(\partial_t f)s} \;, \tag{7.39}$$

and

$$\|T_\lambda(f)\| = \|F(1)\|^{1/2} \leq e^{\frac{1}{2}M(\partial_t f)} \;, \tag{7.40}$$

as claimed. This completes the proof of the lemma.

Proof of Theorem 7.1.1 (i): Field Operators Exist. The existence of the bounded operator $\varphi(f)R$ is equivalent to the existence of $\varphi(f)R$ as a bounded form. We obtain the desired bound on $\varphi(f)R$ using (7.17), (7.19), and (7.22) in the case $\lambda = 0$. In fact,

$$\varphi(f)R = R^{1/2}\varphi(f)R^{1/2} + \left[\varphi(f), R^{1/2}\right] R^{1/2} \;, \tag{7.41}$$

so

$$\begin{aligned} \|\varphi(f)R\| &\leq \left\|R^{1/2}\varphi(f)R^{1/2}\right\| + \left\|\left[\varphi(f), R^{1/2}\right] R^{1/2}\right\| \\ &\leq \left\|R^{1/2}\varphi(f)R^{1/2}\right\| + \left\|\left[\varphi(f), R^{1/2}\right]\right\| \\ &\leq M(f) + M(\partial_t f) = |||f||| \;. \end{aligned} \tag{7.42}$$

This completes the proof of the form bound (7.13).

The bounded form $\varphi(f)R$ yields a unique, bounded operator $\varphi(f)R$ whose matrix elements agree with those of the form, see Proposition 8.5.1. This existence of the bounded operator $\varphi(f)R$ is equivalent to the existence of the unbounded operator $\varphi(f)$ with domain[3] $D(H) = \mathcal{R}((H + I)^{-1}$.

The form $\varphi(f)$ has the adjoint form $\varphi(\bar{f})$. Applying the same argument to the adjoint form, it determines and operator $\varphi(\bar{f})$ with domain $\mathcal{D}(H)$, and this is a restriction of the adjoint operator $\varphi(f)^*$. Thus the adjoint operator is densely defined, and the original operator $\varphi(f)$ has a closure. If f is real, then $\varphi(f)^*$ is an extension of $\varphi(f)$ itself, so $\varphi(f)$ is symmetric.

Proof of (ii): Essential Self-Adjointness. From (i) we know that for real f the operator $\varphi(f)$ with domain $\mathcal{D}(H)$ is symmetric. Therefore its adjoint extends its closure, $\varphi(f)^- \subset \varphi(f)^*$. In order to show that $\varphi(f)$ is essentially self adjoint, we need to show the opposite inclusion, namely that

[3] Here $\mathcal{R}(T)$ denotes the range of the transformation T, namely the set of vectors $\mathcal{R}(T) = T\mathcal{D}(T)$. If $\mathcal{R}(T)$ is dense in $\mathcal{H}$, then the inverse of T exists, and T^{-1} has the domain $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$.

the closure extends the adjoint, $\varphi(f)^* \subset \varphi(f)^-$. Concretely, we prove that: if $\Omega \in \mathcal{D}(\varphi(f)^*)$, then Ω is in the domain of the closure of $\varphi(f)$,

$$\Omega \in \mathcal{D}(\varphi(f)^-) \;, \qquad \text{and also } \varphi(f)^-\Omega = \varphi(f)^*\Omega \;. \tag{7.43}$$

Let $\Omega \in \mathcal{D}(\varphi(f)^*)$. In order to prove that $\Omega \in \mathcal{D}(\varphi(f)^-)$, we need to approximate Ω by a sequence of vectors in the domain of $\varphi(f)$. We choose $\lambda R(\lambda)\Omega \in \mathcal{D}(H)$. We now show that

$$\lim_{\lambda\to\infty} \lambda R(\lambda)\Omega = \Omega \;, \qquad \text{and also} \qquad \lim_{\lambda\to\infty} \varphi(f)\lambda R(\lambda)\Omega = \varphi(f)^*\Omega \;. \tag{7.44}$$

These limits (7.44) mean that Ω does have the properties (7.43), so the proof of essential self-adjointness will be complete.

As a consequence of (7.17), we know $\|\lambda R(\lambda)\| \le 1$ for $0 \le \lambda$. Similarly as a consequence of (7.22), we know that $\|\,[\varphi(f), \lambda R(\lambda)]\,\| \le M(\partial_t f)$ for $0 \le \lambda$. Now we prove that both sequences of uniformly bounded operators converge strongly as $\lambda \to \infty$, with the limits

$$\operatorname*{st.lim}_{\lambda\to\infty} \lambda R(\lambda) = I \;, \qquad \operatorname{st.lim} \lambda^{1/2} R(\lambda)^{1/2} = I \;, \qquad \text{and} \qquad \operatorname*{st.lim}_{\lambda\to\infty} [\varphi(f), \lambda R(\lambda)] = 0 \;. \tag{7.45}$$

To establish the first limit, we use the uniform bound $\lambda\|R(\lambda)\| \le 1$ and show strong convergence $\|\lambda R(\lambda) f - f\| \to 0$ for f in the dense set $\mathcal{D}(H)$. Then strong convergence on all vectors follows by Proposition 8.7.1. On the domain $\mathcal{D}(H)$, write

$$I - \lambda R(\lambda) = (H+I)(H+I+\lambda)^{-1} \;, \tag{7.46}$$

so $0 \le I - \lambda R(\lambda) \le I$, and also

$$\|(I - \lambda R(\lambda))f\| \le (1+\lambda)^{-1}\|(H+I)f\| \to 0 \;. \tag{7.47}$$

Therefore we have established the first limit in (7.45). Likewise the second limit follows from the representation

$$I - \lambda^{1/2}R(\lambda)^{1/2} = \left(I + \lambda^{1/2}R(\lambda)^{1/2}\right)^{-1}(I - \lambda R(\lambda)) \;. \tag{7.48}$$

where $0 \le \lambda^{1/2}R(\lambda)^{1/2}$ and therefore $\left(I + \lambda^{1/2}R(\lambda)^{1/2}\right)^{-1} \le 1$. Consequently for $\chi \in \mathcal{H}$,

$$\left\|(I - \lambda^{1/2}R(\lambda)^{1/2})\chi\right\| \le \|(I - \lambda R(\lambda))\chi\| \to 0 \;, \tag{7.49}$$

as $\lambda \to \infty$.

To prove the third part of (7.45), consider vectors $\chi \in \mathcal{D}(H^{1/2})$, and use the representation (7.29). Then

$$[\varphi(f), \lambda R(\lambda)]\;\chi = i\lambda R(\lambda, \varphi(\partial_t f)R^{1/2}\,R(\lambda)\,(H+I)^{1/2}\chi \;. \tag{7.50}$$

Therefore,

$$\begin{aligned}\|\,[\varphi(f), \lambda R(\lambda)\,]\,\chi\| &\le \lambda\left\|R(\lambda)^{1/2}\right\|\,\left\|R(\lambda)^{1/2}\varphi(f_t)R^{1/2}\right\|\,\|R(\lambda)\|\,\left\|(H+I)^{1/2}\chi\right\| \\ &\le (1+\lambda)^{-1/2}\left\|(H+I)^{1/2}\chi\right\| \to 0 \;,\end{aligned} \tag{7.51}$$

from which we infer the third claimed limit of (7.45).

Now we return to the question of self-adjointness, and the proof of (7.44). The first limit in (7.45) includes the first desired limit in (7.44), so we need only study the second claim in (7.44). Assume $\Omega \in \mathcal{D}(\varphi(f)^*)$ and choose an an arbitrary vector $\chi \in \mathcal{D}(H)$. Then using the fact that $\varphi(f)$ is symmetric, the following computation is valid:

$$\begin{aligned} \langle \chi, \varphi(f)\lambda R(\lambda)\Omega\rangle &= \langle \lambda R(\lambda)\varphi(f)\chi, \Omega\rangle \\ &= \langle \varphi(f)R(\lambda)\chi, \Omega\rangle + \langle [\lambda R(\lambda), \varphi(f)]\, \chi, \Omega\rangle \\ &= \langle \chi, \lambda R(\lambda)\varphi(f)^*\Omega\rangle + \langle \chi, [\varphi(f), \lambda R(\lambda)]\,\Omega\rangle \ . \end{aligned} \tag{7.52}$$

Here we use the fact that the commutator $[\lambda R(\lambda), \varphi(f)]$ is a bounded operator and that it satisfies $[\lambda R(\lambda), \varphi(f)]^* = [\varphi(f), \lambda R(\lambda)]$. Since $\mathcal{D}(H)$ is dense, we have derived an identity for vectors,

$$\varphi(f)\lambda R(\lambda)\Omega = \lambda R(\lambda)\varphi(f)^*\Omega + [\varphi(f), \lambda R(\lambda)]\,\Omega \ . \tag{7.53}$$

Now we can take the limit $\lambda \to \infty$ in (7.53). Using the two statements (7.45), we infer that both sequences of vectors on the right converge, and that

$$\lim_{\lambda\to\infty} \varphi(f)\lambda R(\lambda)\Omega = \lim_{\lambda\to\infty} \lambda R(\lambda)\varphi(f)^*\Omega + \lim_{\lambda\to\infty} [\varphi(f), \lambda R(\lambda)]\,\Omega = \varphi(f)^*\Omega \ . \tag{7.54}$$

This is the second desired limit in (7.44), so we have completed the proof that $\varphi(f)$ is essentially self-adjoint on $\mathcal{D}(H)$.

Before proceeding to prove locality, we state separately some useful bounds.

Lemma 7.1.3. *Consider real f with both $|||f|||$ and $|||\partial_t f|||$ finite.*

a. *The operator $S_\lambda(f) = (H+I)e^{i\varphi_\lambda(f)}R$ is bounded. The norm of $S_\lambda(f)$ satisfies*

$$\boxed{\|S_\lambda(f)\| \le e^{|||\partial_t f|||}} \ . \tag{7.55}$$

b. *The approximating unitary operators $e^{i\varphi_\lambda(f)}$ generated by $\varphi_\lambda(f)$ satisfy*

$$\boxed{\left\| R\left(e^{i\varphi_\lambda(f)} - e^{i\varphi_{\lambda'}(f)}\right) R\right\| \le 2\left((1+\lambda)^{-1/2} + (1+\lambda')^{-1/2}\right) M(f)\, e^{|||\partial_t f|||}} \ . \tag{7.56}$$

c. *Assuming that $\varphi(f)$ and $\varphi(f)$ commute as forms on $C^\infty(H)\times C^\infty(H)$, the approximate fields approximately commute in the sense that*

$$\boxed{\|R\,[\varphi_\lambda(f), \varphi_\lambda(g)]\,R\| \le (1+\lambda)^{-1}\left(|||f|||\ |||\partial_t g||| + |||g|||\ |||\partial_t f|||\right)} \ . \tag{7.57}$$

d. *Assuming that $\varphi(f)$ and $\varphi(f)$ commute as forms on $C^\infty(H)\times C^\infty(H)$, the commutator of the unitary operators generated by the approximate fields converges zero in the following sense,*

$$\boxed{\left\|R\left[e^{i\varphi_\lambda(f)^-}, e^{i\varphi_\lambda(g)^-}\right]R\right\| \le (1+\lambda)^{-1}\left(|||f|||\ |||\partial_t g||| + |||g|||\ |||\partial_t f|||\right)\, e^{|||\partial_t f|||+|||\partial_t g|||}} \ . \tag{7.58}$$

Proof. In what follows we use the notation

$$U_\lambda^s = U_\lambda(f)^s = e^{is\varphi_\lambda(f)} \;, \qquad \text{for } \; s \in \mathbb{R} \;. \tag{7.59}$$

To establish (a) one performs a calculation similar to the proof of Lemma 7.1.2.e, with the interpolation function

$$G(s) = R\, U_\lambda(f)^{s*}\,(H+I)^2\, U_\lambda(f)^s\, R \;, \qquad \text{for } \; 0 \le s \le 1 \;, \tag{7.60}$$

for which

$$G(0) = I \;, \qquad \text{and } \; G(1) = S_\lambda(f)^* S_\lambda(f) \le \|S_\lambda(f)\|^2 \;. \tag{7.61}$$

Here $S_\lambda(f) = (H+I)U_\lambda(f)R$. Then

$$\begin{aligned} \frac{dG(s)}{ds} &= -iR\, U_\lambda(f)^{s*}\left([\varphi_\lambda(f), H](H+I) + (H+I)[\varphi_\lambda(f), H]\right) U_\lambda(f)^s\, R \\ &= -R\, U_\lambda(f)^{s*}\left(\varphi_\lambda(\partial_t f)(H+I) + (H+I)\varphi_\lambda(\partial_t f)\right) U_\lambda(f)^s\, R \\ &= -R\, U_\lambda(f)^{s*}(H+I)\left(R\varphi_\lambda(\partial_t f) + \varphi_\lambda(\partial_t f)R\right)(H+I)U_\lambda(f)^s\, R \\ &\le 2\, |||\partial_t f|||\, G(s) \;, \end{aligned} \tag{7.62}$$

where again we use (7.37), this time with $A = (H+I)U_\lambda(f)^s R$ and $B = \varphi(\partial_t f)R + R\varphi(\partial_t f)$. The estimate of Theorem 7.1.1.i, namely (7.13), assures $\|B\| \le 2|||\partial_t f|||$, and hence we obtain (7.62). Integrating this differential equality, we obtain the bound $\ln G(s) \le 2s|||\partial_t f|||$, from which the estimate (7.55) follows.

We prove (b) use the interpolation function $F(s) = RU_\lambda^s U_{\lambda'}^{1-s} R$ to show that

$$\begin{aligned} R\,(U_\lambda - U_{\lambda'})\, R &= \int_0^1 \frac{dF(s)}{ds} ds \\ &= i\int_0^1 RU_\lambda^s\,(\varphi_\lambda(f) - \varphi_{\lambda'}(f))\, U_{\lambda'}^{1-s} R\; ds \;. \end{aligned} \tag{7.63}$$

Part (a) of this lemma ensures that one can write

$$R\,(U_\lambda - U_{\lambda'})\, R = i\int_0^1 RU_\lambda^s(H+I)R\,(\varphi_\lambda(f) - \varphi_{\lambda'}(f))\, R(H+I)U_{\lambda'}^{1-s} R\; ds \;, \tag{7.64}$$

and (7.55) then gives

$$\begin{aligned} \|R\,(U_\lambda - U_{\lambda'})\, R\| &\le \int_0^1 \|RU_\lambda^s(H+I)\|\, \|R\,(\varphi_\lambda(f) - \varphi_{\lambda'}(f))\, R\| \left\|(H+I)U_{\lambda'}^{1-s} R\right\| ds \\ &\le \|R\,(\varphi_\lambda(f) - \varphi_{\lambda'}(f))\, R\|\, e^{|||\partial_t f|||} \;. \end{aligned} \tag{7.65}$$

Using the bound of Lemma 7.1.2.b, namely (7.20), we also have

$$\begin{aligned} \|R\,(\varphi_\lambda(f) - \varphi_{\lambda'}(f))\, R\| &\le \|R\,(\varphi_\lambda(f) - \varphi(f))\, R\| + \|R\,(\varphi(f) - \varphi_{\lambda'}(f))\, R\| \\ &\le 2\left((1+\lambda)^{-1/2} + (1+\lambda')^{-1/2}\right) M(f) \;, \end{aligned} \tag{7.66}$$

so we have established (7.56).

In order to establish (c), we use the first compute on the domain $C^\infty(H) \times C^\infty(H)$ and use the fact that the forms $\varphi(f)$ and $\varphi(g)$ commute on this domain. Then

$$\begin{aligned}
[\varphi_\lambda(f), \varphi_\lambda(g)] &= \lambda^2 R(\lambda)^{1/2}\varphi(f)R(\lambda)\varphi(g)R(\lambda)^{1/2} \\
&\quad -\lambda^2 R(\lambda)^{1/2}\varphi(g)R(\lambda)\varphi(f)R(\lambda)^{1/2} \\
&= \lambda^2 R(\lambda)^{1/2}\varphi(f)\left[R(\lambda), \varphi(g)\right] R(\lambda)^{1/2} \\
&\quad -\lambda^2 R(\lambda)^{1/2}\varphi(g)\left[R(\lambda), \varphi(f)\right] R(\lambda)^{1/2} \\
&= -i\lambda^2 R(\lambda)^{1/2}\varphi(f)R(\lambda)\varphi(\partial_t g)R(\lambda)^{3/2} \\
&\quad +i\lambda^2 R(\lambda)^{1/2}\varphi(g)R(\lambda)\varphi(\partial_t f)R(\lambda)^{3/2} \\
&= -i\left(\varphi_\lambda(f)\varphi_\lambda(\partial_t g) - \varphi_\lambda(g)\varphi_\lambda(\partial_t f)\right) R(\lambda)\,.
\end{aligned} \tag{7.67}$$

One then has

$$\begin{aligned}
\|R\left[\varphi_\lambda(f), \varphi_\lambda(g)\right] R\| &\le \|R\left(\varphi_\lambda(f)\varphi_\lambda(\partial_t g) - \varphi_\lambda(g)\varphi_\lambda(\partial_t f)\right) R\|\ \|R(\lambda)\| \\
&\le (1+\lambda)^{-1}\left(\|R\varphi_\lambda(f)\varphi_\lambda(\partial_t g)R\| + \|R\varphi_\lambda(g)\varphi_\lambda(\partial_t f)R\|\right)\,.
\end{aligned} \tag{7.68}$$

Use the bound on field operators (7.13) to obtain part (c) of the lemma, namely (7.57).

In order to establish (d), compute the regularized commutator using the interpolation function $F(s) = U_\lambda(f)^s U_\lambda(g) U_\lambda(f)^{1-s}$. This gives the representation

$$\begin{aligned}
[U_\lambda(f), U_\lambda(g)] &= \int_0^1 \frac{dF(s)}{ds}\, ds \\
&= i\int_0^1 U_\lambda(f)^s \left[\varphi_\lambda(f), U_\lambda(g)^{1-s}\right] ds \\
&= -\int_0^1\int_0^1 U_\lambda(f)^s U_\lambda(g)^t \left[\varphi_\lambda(f), \varphi_\lambda(g)\right] U_\lambda(g)^{1-t} U_\lambda(f)^{1-s}\, dsdt\,.
\end{aligned} \tag{7.69}$$

The estimate in part (f) of Lemma 7.1.2, namely (7.55), shows that $U_\lambda(f)^s$ maps $\mathcal{D}(H)$ to $\mathcal{D}(H)$. Thus write (7.69) as

$$\begin{aligned}
R\left[U_\lambda(f), U_\lambda(g)\right] R &= -\int_0^1\int_0^1 \left(RU_\lambda(f)^s(H+I)\right)\left(RU_\lambda(g)^t(H+I)\right)\left(R\left[\varphi_\lambda(f), \varphi_\lambda(g)\right] R\right) \\
&\qquad \times\left((H+I)U_\lambda(g)^{1-t}R\right)\left((H+I)U_\lambda(f)^{1-s}R\right) dsdt\,.
\end{aligned} \tag{7.70}$$

Estimate this product using (7.55) and part (c) of the present lemma. Thus

$$\begin{aligned}
\|R\left[U_\lambda(f), U_\lambda(g)\right] R\| &\le \int_0^1\int_0^1 \|RU_\lambda(f)^s(H+I)\|\ \left\|RU_\lambda(g)^t(H+I)\right\| \\
&\qquad \times\left\|(H+I)U_\lambda(g)^{1-t}R\right\|\ \left\|(H+I)U_\lambda(f)^{1-s}R\right\| \\
&\qquad \times\|R\left[\varphi_\lambda(f), \varphi_\lambda(g)\right] R\|\, dsdt \\
&\le e^{|||\partial_t f||| + |||\partial_t g|||}\ \|R\left[\varphi_\lambda(f), \varphi_\lambda(g)\right] R\| \\
&\le (1+\lambda)^{-1}\left(|||f|||\ |||\partial_t g||| + |||g|||\ |||\partial_t f|||\right)\, e^{|||\partial_t f||| + |||\partial_t g|||}\,,
\end{aligned} \tag{7.71}$$

which is (7.58) as claimed. This completes the proof of the lemma.

Proof of (iii): Locality. We begin by proving

$$\operatorname*{st.lim}_{\lambda\to\infty} U_\lambda(f) = U(f) = e^{i\varphi(f)^-} . \tag{7.72}$$

We have shown in Lemma 7.1.3.b that weak lim $\varphi_\lambda(f) = U$ exists on the domain $\mathcal{D}(H) \times \mathcal{D}(H)$. Since the U_λ are unitary, the strong limits also exist, see Proposition 8.7.3. Thus

$$\operatorname*{st.lim}_{\lambda\to\infty} U_\lambda = U . \tag{7.73}$$

To complete the proof of (7.72), we need to show that $U = e^{i\varphi(f)^-}$.

First note that the sequence of operators $(\varphi_\lambda(f) - \varphi(f))\, R$ converges strongly to zero. In fact,

$$\begin{aligned}(\varphi_\lambda(f) - \varphi(f))\, R &= \left(\left(\lambda^{1/2}R(\lambda)^{1/2}\right)\varphi(f)\left(\lambda^{1/2}R(\lambda)^{1/2}\right) - \varphi(f)\right) R \\ &= \left(\left(\lambda^{1/2}R(\lambda)^{1/2}\right) - I\right)(\varphi(f)R)\left(\lambda^{1/2}R(\lambda)^{1/2}\right) \\ &\qquad + (\varphi(f)R)\left(\left(\lambda^{1/2}R(\lambda)^{1/2}\right) - I\right)\end{aligned} \tag{7.74}$$

The bound (7.13) of Theorem 7.1.1.i shows that $\|\varphi(f)R\| \le |||f|||$. Using (7.45), the strong limits $I - \lambda^{1/2}R(\lambda)^{1/2} \to 0$ and $\lambda^{1/2}R(\lambda)^{1/2} \to I$ both exist. Thus both terms in (7.74) have a strong limit equal to zero. Hence

$$\operatorname*{st.lim}_{\lambda\to\infty} \varphi_\lambda(f)R = \varphi(f)R . \tag{7.75}$$

It follows that the bounded, self-adjoint generators $\varphi_\lambda(f)$ of the approximating unitary groups U^s_λ satisfy

$$\frac{d}{ds} U^s_\lambda R = iU^s_\lambda \varphi_\lambda(f)R . \tag{7.76}$$

Since the generator of U^s agrees with $\varphi(f)$ on $\mathcal{D}(H)$, and $\varphi(f)$ is essentially self-adjoint on this domain by Theorem 7.1.1.ii, we infer that $U^s = e^{is\varphi(f)^-}$ and (7.72) holds.

As the product of strongly convergent sequences is strongly convergent, also

$$U(f)U(g) = \operatorname*{st.lim}_{\lambda\to\infty} U_\lambda(f)U_\lambda(g) , \qquad \text{and } U(g)U(f) = \operatorname*{st.lim}_{\lambda\to\infty} U_\lambda(g)U_\lambda(f) . \tag{7.77}$$

We need to identify show that these two limits are the same. From (7.58), it follows that

$$\operatorname*{weak\,lim}_{\lambda\to\infty} U_\lambda(f)U_\lambda(g) = \operatorname*{weak\,lim}_{\lambda\to\infty} U_\lambda(g)U_\lambda(f) . \tag{7.78}$$

Since these operators are unitary, the strong limits also agree. Therefore

$$[U(f), U(g)] = \operatorname*{st.lim}_{\lambda\to\infty} [U_\lambda(f), U_\lambda(g)] = \operatorname*{weak\,lim}_{\lambda\to\infty} [U_\lambda(f), U_\lambda(g)] = 0 . \tag{7.79}$$

Proof of (iv). Having established the bound on the field (7.13) $f \in C_0^\infty$, we take limits. Since $\varphi(f)(H+I)^{-1}$ is linear in f, consider any sequence $f_n \in C_0^\infty$ that is a Cauchy sequence $|||f_n - f_{n'}||| \to 0$ converging to f. We obtain a Cauchy sequence of bounded operators $\|\varphi(f_n)(H+I)^{-1} - \varphi(f_{n'})(H+I)^{-1}\| \le |||f_n - f_{n'}||| \to 0$, converging to $\varphi(f)(H+I)^{-1}$ satisfying the same bound. This determines $\varphi(f)$ with domain $\mathcal{D}(H)$. Essential self-adjointness then also follows for real f. In analyzing the commutator of two field operators, we also required a finite norm $|||\partial_t f|||$.

7.1.2 The Fundamental Bound and Expectation Values

Proposition 7.1.4. *Let $f \in \mathcal{S}(\mathbb{R}^d)$. In this case the domain $C^\infty(H)$ plays a special role.*

i. **Invariant Domain.** *The operator $\varphi(f)$ maps $C^\infty(H)$ into $C^\infty(H)$.*

ii. **Regular Matrix Elements.** *The matrix elements of the field $\varphi(f)$ in vectors $\Omega_1, \Omega_2 \in C^\infty(H)$, can be written,*

$$W(f) = \langle \Omega_1, \varphi(f)\Omega_2 \rangle_{\mathcal{H}} = \int \langle \Omega_1, \varphi(x)\Omega_2 \rangle_{\mathcal{H}} f(x)dx \ , \tag{7.80}$$

where

$$W(x) = \langle \Omega_1, \varphi(x)\Omega_2 \rangle_{\mathcal{H}} \in C^\infty(\mathbb{R}^d) \ . \tag{7.81}$$

iii. **Distributions.** *If $\Omega \in C^\infty(H)$, the expectation values of products of the fields in Ω exist,*

$$W_n(f_1, \ldots, f_n) = \langle \Omega, \varphi(f_1) \cdots \varphi(f_n)\Omega \rangle_{\mathcal{H}} \ , \tag{7.82}$$

as tempered distributions in $\mathcal{S}'(\mathbb{R}^{nd})$. If Ω is a zero-energy eigenstate of H, then the W_n are called "Wightman functions."

iv. **Translation Invariance.** *Suppose that Ω is an eigenstate for the space-time translation group $U(I,a)$ (not only for the time-translation subgroup generated by H). Then the W_n are translation-invariant, namely*

$$W_n(x_1, \ldots, x_n) = \mathfrak{W}_n(\xi_1, \ldots, \xi_{n-1}) \ , \qquad \text{where } \xi_n = x_i - x_{i+1} \ , \tag{7.83}$$

where the $\mathfrak{W}_n$ are tempered distributions in $\mathcal{S}'(\mathbb{R}^{(n-1)d})$.

Proof. Let $\Omega \in C^\infty(H)$. We show that $\varphi(f)\Omega \in C^\infty(H)$. Since

$$H\varphi(f)\Omega = \varphi(f)H\Omega + i\varphi(\partial_t f)\Omega \ , \tag{7.84}$$

it is the case that $\varphi(f)\Omega \in \mathcal{D}(H)$. We argue by induction, assuming that $\varphi(f)\Omega \in \mathcal{D}(H^j)$ for $1 \le j \le n$. We show that $\varphi(f)\Omega \in \mathcal{D}(H^{n+1})$. In fact

$$\begin{aligned} H^{n+1}\varphi(f)\Omega &= H^n\varphi(f)H\Omega + H^n\left[H, \varphi(f)\right]\Omega \\ &= H^n\varphi(f)H\Omega + iH^n\varphi(\partial_t f)\Omega \ . \end{aligned} \tag{7.85}$$

Hence $\varphi(f)\Omega \in \mathcal{D}(H^{n+1})$.

Part IV

Euclidean Fields

Euclidean fields are classical fields, dual to the one-particle wave functions $f(x)$ on Euclidean space $\mathbb{R}^d$, that we studied in §3.4.1. For example, the Euclidean scalar field $\Phi(x)$ for $x \in \mathbb{R}^d$ is a classical field. It is "classical," in the sense that $\Phi(x)\Phi(x') = \Phi(x')\Phi(x)$ for all $x, x' \in \mathbb{R}^d$. We will quantize to obtain the quantum field $\varphi(\vec{x}, t)$ analytically continued to imaginary time, namely $\varphi_I(\vec{x}, t) = \varphi(\vec{x}, it)$.

So in this chapter we consider the bosonic field $\Phi(f)$, where f is a function in the one-particle space $\mathcal{H} = \mathfrak{H}_{-1}(\mathbb{R}^d)$—as opposed to the one particle space $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$ of Chapter 6. In order to distinguish the Hilbert space $\mathcal{F}^b(\mathcal{H})$ from the one in Chapter 6, we denote the Euclidean Hilbert space by

$$\mathcal{F}^{\mathcal{E},b} = \mathcal{F}^b(\mathfrak{H}_{-1}(\mathbb{R}^d)) \,. \tag{7.86}$$

We obtain the zero-particle state $\Omega_0^{\mathcal{E}} = \{1, 0, 0, \ldots\}$, and the Euclidean bosonic field

$$\Phi(f) = \mathfrak{m}_s(f) + \mathfrak{m}_s(\overline{f})^* \,, \qquad \text{for } f \in \mathfrak{H}_{-1}(\mathbb{R}^d) \,. \tag{7.87}$$

Part V

Some Analytic Tools

Chapter 8

Linear Transformations on Hilbert Space

A linear transformation on Hilbert space is the generalization to infinite dimensions of matrices on $\mathbb{C}^N$. However a number of subtleties arise in the infinite dimensional case, and we deal with a few of them here. The properties of certain linear transformations can be understood as limits of finite-dimensional matrices, but others cannot. The simplest property of linear transformations not encountered in finite dimensions is continuous spectrum of a self-adjoint transformation, which is an intrinsic property of infinite dimensional Hilbert space. Here we do not attempt to present a text on linear transformations. We only highlight some useful information.

8.1 Hilbert Space

A vector space $\mathcal{H}$ over the field of scalars $\mathfrak{k}$ (the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ in our examples) is a linear space with a scalar multiplication. In other of vectors $f, g \in \mathcal{H}$ then $f+g \in \mathcal{H}$ and if $\lambda \in \mathfrak{k}$ then $\lambda f \in \mathcal{H}$. A hermitian scalar product is a positive definite map $\langle f, g\rangle_{\mathcal{H}}$ from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$ that is linear in the second factor and conjugate linear in the first. Thus means that for all $f \in \mathcal{H}$,

$$0 \leq \langle f, f\rangle \ , \qquad \text{with} \ \ \langle f, f\rangle = 0 \ , \ \text{ if and only if } \ \ f = 0 \ , \tag{8.1}$$

and for all $f, g, h \in \mathcal{H}$ and $\lambda \in \mathfrak{k}$,

$$\langle f, g + \lambda h\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}} + \lambda \langle f, h\rangle_{\mathcal{H}} = \langle g + \lambda h, f\rangle^*_{\mathcal{H}} \ . \tag{8.2}$$

The scalar product determines a norm $\|f\|_{\mathcal{H}} = \langle f, f\rangle^{1/2}_{\mathcal{H}}$.

A Hilbert space is a vector space with a scalar product that is complete in the corresponding norm. In other words in a Hilbert space every Cauchy sequence converges. This means that if a sequence of vectors $f_n \in \mathcal{H}$ satisfies the Cauchy convergence criterium $\|f_n - f_m\| \to 0$, as $n, m \to \infty$, then there is a vector $f \in \mathcal{H}$ such that $\|f_n - f\| \to 0$.

Examples.

i. The spaces $\mathbb{R}^N$ and $\mathbb{C}^N$ are Hilbert spaces with the usual scalar product, $\langle f, g\rangle = \sum_{j=1}^N f_j^* g_j$.

ii. The space ℓ^2 of square summable sequences $f = \{f_j : j \in \mathbb{Z}\}$ is a Hilbert space with the inner product $\langle f, g\rangle = \sum_{j\in\mathbb{Z}} f_j^* g_j$.

iii. If $\rho = \{\rho_j > 0 : j \in \mathbb{Z}_+\}$ is a positive sequence, then sequences $f = \{f_j : j \in \mathbb{Z}_+\}$ is a Hilbert space $\ell^2(\rho)$ with inner product $\langle f, g\rangle = \sum_{j\in\mathbb{Z}_+} f_j^* g_j \rho_j$.

iv. The space of functions on $\mathbb{R}^N$ that are square-integrable with respect to the strictly positive measure $d\nu$ is a Hilbert space $L^2(\mathbb{R}^N; d\nu)$ with inner product,

$$\langle f, g\rangle_{L^2(\mathbb{R}^N;d\nu)} = \int_{\mathbb{R}^N} \overline{f(x)}\, g(x)\, d\nu(x)\ . \tag{8.3}$$

In case $d\nu(x) = dx$ is Lebesgue measure, one writes simply $L^2(\mathbb{R}^N)$.

A linear subspace $\mathcal{D}$ of $\mathcal{H}$ is dense in $\mathcal{H}$ if every element $f \in \mathcal{H}$ can be approximated by a sequence of elements $f_n \in V$. A subset $\mathcal{D}_0$ of $\mathcal{H}$ is said to be a *basis* for $\mathcal{H}$ if the linear subspace generated by elements in $\mathcal{D}_0$ is dense in $\mathcal{H}$. The dimension of $\mathcal{H}$ is the smallest number of vectors that comprise core for $\mathcal{H}$, which may be finite or infinite. A Hilbert space is said to be *separable* if it has a countable basis. The Hilbert spaces that occur in this work are all separable Hilbert spaces.

The Riesz representation theorem states that a Hilbert space $\mathcal{H}$ is isomorphic to its dual. In other words, every continuous linear function $F(f)$ from $\mathcal{H}$ to $\mathbb{C}$ can be represented by the scalar product with some vector $\chi(F) \in \mathcal{H}$, $F(f) = \langle \chi, f\rangle_{\mathcal{H}}$.

8.2 Operators

We use the word operator to denote a *linear* transformation on $\mathcal{H}$. On a finite dimensional Hilbert space, an operator maps all of $\mathcal{H}$ into $\mathcal{H}$; it is represented by a matrix. In infinite dimensions, a linear transformation may not be defined on every vector in $\mathcal{H}$. Thus specifying a linear transformation T means giving both

- the domain $\mathcal{D}(T) \subset \mathcal{H}$ which is the linear subspace on which T is defined, and
- the range $\mathcal{R}(T)$ composed of the values Tf for $f \in \mathcal{D}(T)$.

One says that T is densely defined if $\mathcal{D}(T)$ is dense in $\mathcal{H}$. Let T and S be operators. One says that S extends T if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Tf = Sf$ for all $f \in \mathcal{D}(T)$.

Operator Norm. The norm of an operator T on the Hilbert space $\mathcal{H}$ is

$$\|T\|_{\mathcal{H}} = \sup_{f\in\mathcal{D}(T)} \frac{\|Tf\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} \ . \tag{8.4}$$

An operator is called *bounded* on $\mathcal{H}$ if $\|T\|_{\mathcal{H}} < \infty$. A bounded operator is continuous, in the sense that if $f_n \in \mathcal{D}(T)$ converge, namely $\|f_n - f\|_{\mathcal{H}} \to 0$, then Tf_n also converges. Linearity of T shows that continuity of T as a transformation on $\mathcal{H}$ is equivalent to boundedness of T.

The Adjoint. In case that T is densely defined, T uniquely determines an adjoint transformation T^*. Suppose that for a vector $g \in \mathcal{H}$ there exists a vector $\chi \in \mathcal{H}$ such that for all $f \in \mathcal{D}(T)$,

$$\langle g, Tf\rangle_{\mathcal{H}} = \langle \chi, f\rangle_{\mathcal{H}} \ . \tag{8.5}$$

We want to say that

$$g \in \mathcal{D}(T^*) \ , \qquad \text{and } \ T^*g = \chi \ . \tag{8.6}$$

But this makes sense only if the identity (8.5) determines χ uniquely; otherwise could assign different values to T^*g. Suppose two vectors χ_1, χ_2 exist, both of which have the property (8.5). Then $\langle \chi_1 - \chi_2, f\rangle_{\mathcal{H}} = 0$, for all $f \in \mathcal{D}(T)$. We assumed that $\mathcal{D}(T)$ is dense. And every vector orthogonal to a dense set of vectors is zero. Therefore χ is unique, and the relation (8.6) does define T^*.

For a bounded operator T with an adjoint T^*,

$$\|T\|_{\mathcal{H}} = \|T^*\|_{\mathcal{H}} = \|T^*T\|^{1/2} \ . \tag{8.7}$$

Furthermore, the matrix elements of T can be used to calculate the norm of T. The expression (8.4) equals

$$\|T\|_{\mathcal{H}} = \sup_{f,g\in\mathcal{H}} \frac{|\langle g, Tf\rangle_{\mathcal{H}}|}{\|g\|_{\mathcal{H}}\|f\|_{\mathcal{H}}} \ . \tag{8.8}$$

A densely defined, bounded operator always has a closure, and $T^- = T^{**}$. A bounded operator on $L^2(\mathbb{R}^N; d\nu)$ can be represented as an integral operator,

$$(Tf)(x) = \int T(x,y)f(y)d\nu(y) \ . \tag{8.9}$$

The function $T(x;y)$ is called the integral kernel of T.

The adjoint T^* of T has the integral kernel

$$T^*(x;y) = \overline{T(y;x)} \ . \tag{8.10}$$

Integral kernels on $L^2(\mathbb{R}^N; d\nu)$ compose with the rules of matrix multiplication,

$$(TS)(x;y) = \int T(x;z)S(z;y)d\nu(z) \ . \tag{8.11}$$

If T is translation invariant, then $T(x;y) = T(x-y)$.

Symmetric Operators. An operator T is *symmetric* if T is densely defined, and T^* extends T. In other words, $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, and

$$\langle f, Tg\rangle_{\mathcal{H}} = \langle Tf, g\rangle_{\mathcal{H}} \ , \qquad \text{for all } \ f, g \in \mathcal{D}(T) \ . \tag{8.12}$$

Every symmetric transformation T has a densely defined adjoint, because the adjoint is an extension of T. Thus every symmetric operator has T uniquely determines its double adjoint T^{**}.

Exercise 8.2.1. *Suppose that T is a symmetric operator. Show that T^{**} extends T. In other words, show that $\mathcal{D}(T) \subset \mathcal{D}(T^{**})$, and that for all $f \in \mathcal{D}(T)$, one has $T^{**}f = Tf$. (Warning: it may not be the case that T^* is symmetric nor that T^{**} extends T^*.)*

Self-Adjoint Operators. A symmetric transformation T is *self-adjoint* if $T = T^*$. A symmetric transformation is *essentially self-adjoint* if T^* adjoint is self-adjoint, or $T^* = T^{**}$. If T is essentially self-adjoint, it uniquely determines the self-adjoint operator T^{**}.

A self adjoint transformation T is the infinitesimal generator of a unitary group e^{isT}, where s is a real parameter.

If T is a bounded, self-adjoint operator on $L^2(\mathbb{R}^N)$, one has a useful bound on the norm $\|T\|_{\mathcal{H}}$ in terms of its integral kernel. Let

$$\|T\|_{\infty,1} = \sup_x \int_{\mathbb{R}^N} |T(x;y)| d\nu(y) \ . \tag{8.13}$$

Proposition 8.2.1. *Let T be self-adjoint on $L^2(\mathbb{R}^N; d\nu)$ with integral kernel $T(x;y)$. Then*

$$\|T\|_{\mathcal{H}} \leq \|T\|_{\infty,1} \ . \tag{8.14}$$

Remark. If T is not self-adjoint, then a similar relation holds, see Proposition 8.4.1. This bound may be optimal: for example in the case that T is the approximate identity T_ϵ on $L^2(\mathbb{R}^N)$ defined by the integral kernel

$$T_\epsilon(x;y) = \frac{1}{(4\pi\epsilon)^{N/2}} e^{-(x-y)^2/4\epsilon} \ , \tag{8.15}$$

then

$$\|T_\epsilon\|_{L^2} = \|T_\epsilon\|_{\infty,1} = 1 \ . \tag{8.16}$$

On the other hand, consider the rank-one operator T with integral kernel $T(x;y) = \chi(x)\overline{\chi(y)}$, where $\chi \in L^2(\mathbb{R}^N)$. Then T has operator norm $\|T\|_{L^2} = \|\chi\|_{L^2}$. But in case $\chi \notin L^1$, the norm $\|T\|_{\infty,1}$ is infinite.

Proof. The expression $\langle f, Tg\rangle_{L^2(\mathbb{R}^N;d\nu)}$ can be bounded using the Schwarz inequality as

$$\begin{aligned}
\left|\langle f, Tg\rangle_{L^2(\mathbb{R}^N;d\nu)}\right| &= \left|\int \overline{f(x)}T(x;y)g(y)d\nu(x)d\nu(y)\right| \\
&\leq \int |f(x)||T(x;y)|^{1/2}|T(x;y)|^{1/2}|g(y)|d\nu(x)d\nu(y) \\
&\leq \left(\int |f(x)|^2|T(x;y)|d\nu(x)d\nu(y)\right)^{1/2}\left(\int |T(x;y)||g(y)|^2 d\nu(x)d\nu(y)\right)^{1/2} \\
&\leq \left(\int |f(x)|^2 d\nu(x)\right)^{1/2}\left(\sup_x \int |T(x;y)|d\nu(y)\right)^{1/2} \\
&\qquad \times \left(\int |g(y)|^2 d\nu(y)\right)^{1/2}\left(\sup_y \int |T(x;y)|d\nu(x)\right)^{1/2} \\
&= \|T\|_{\infty,1}\|f\|_{L^2(\mathbb{R}^N;d\nu)}\|g\|_{L^2(\mathbb{R}^N;d\nu)} . \qquad (8.17)
\end{aligned}$$

In the last equality we use (8.10) to identify $|T(x;y)| = |T(y;x)|$ for self-adjoint T, so

$$\|T\|_{\infty,1} = \left(\sup_x \int |T(x;y)|d\nu(y) \sup_y \int |T(x;y)|d\nu(x)\right)^{1/2} . \qquad (8.18)$$

If T were not self-adjoint, we would use (8.42) to define the norm $\|T\|_{\infty,1}$.

The Closure of an Operator. A symmetric operator T uniquely determines the operator T^{**} extending T. In general, an operator T may uniquely determine some extension of itself. This extension is called the closure T^- of T. In the case of a symmetric operator the closure is $T^- = T^{**}$.

The closure in general is defined as follows. Suppose that the two sequences $f_n \in \mathcal{D}(T)$ and $Tf_n \in \mathcal{R}(T)$ converge to $f \in \mathcal{K}$ and $g \in \mathcal{K}$ respectively. Then the domain $\mathcal{D}(T^-)$ of the closure T^- in includes f, and $T^-f = g$. However, it could be the case for an unbounded transformations T, that $f_n \to 0$, but $Tf_n \not\to 0$. Were that to happen, one cannot define the operator T^-. So we have the warning: *An arbitrary linear operator T may not have a closure.*

Exercise 8.2.2. *Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, containing the dense subspace $\mathcal{D}_0$ of C^∞ functions that are compactly supported. (The function f is compactly supported if $f = 0$ outside some bounded region.) Let $\Omega \in \mathcal{K}$ be a fixed unit vector. Define the linear operator T with dense domain $\mathcal{D}_0$ by*

$$Tf = f(0)\Omega . \qquad (8.19)$$

Show that T does not have a closure T^-.

The Graph of an Operator. The direct sum $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is a Hilbert space $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with scalar product

$$\langle f_1 \oplus f_2, g_1 \oplus g_2\rangle_{\mathcal{K}} = \langle f_1, g_1\rangle_{\mathcal{H}_1} + \langle f_2, g_2\rangle_{\mathcal{H}_2} . \qquad (8.20)$$

It is often helpful to think of the operator T in terms of its graph $G(T)$, namely the set of pairs $\{f, Tf\}$, where $f \in \mathcal{D}(T)$ and $Tf \in \mathcal{R}(T)$. The graph $G(T)$ can also be regarded as a subset of the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. In fact, it is a linear subspace of $\mathcal{K}$, for if $\{f, Tf\}, \{g, Tg\} \in G(T)$, then $\{f, Tf\} + \{g, Tg\} = \{f + g, T(f + g)\} \in G(T)$.

Any linear subspace V of a Hilbert space $\mathcal{K}$ has an orthogonal complement $V^\perp$. This is defined as the set of vectors $\chi \in \mathcal{K}$ such that $\langle \chi, f \rangle_{\mathcal{K}} = 0$ for all $f \in V$. The orthogonal complement $V^\perp$ of V is always a closed subspace. For if $\chi_n \in V^\perp$ and $\chi_n \to \chi$, then for any $f \in V$, one has $\langle \chi, f \rangle_{\mathcal{K}} = \lim_n \langle \chi_n, f \rangle_{\mathcal{K}} = 0$. The closure V^- of the linear subspace $V \in \mathcal{H}$ is $V^- = V^{\perp\perp}$. Therefore

$$G(T)^- = G(T)^{\perp\perp} . \tag{8.21}$$

Is $G(T)^-$ the graph of an operator? If so, this operator is the closure of T, and $G(T)^- = G(T^-)$, in agreement with the usual definition of the closure of T.

Equivalently, not every linear subspace of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ is the graph of a linear operator on $\mathcal{H}$. The closure $G(T)^- = G(T)^{\perp\perp} \subset \mathcal{K}$ always exists, but $G(T)^- =$ may not happen to be the graph of a linear operator.

Proposition 8.2.2. *If T and T^* are both densely defined, then T has a closure T^- and*

$$G(T^-) = G(T)^- = G(T)^{\perp\perp} = G(T^{**}) . \tag{8.22}$$

Proof. One can relate the properties of $G(T^-)$ to properties of $G(T)^\perp$. In fact,

$$G(T)^\perp = \{\{-T^*\chi, \chi\} : \text{ for } \chi \in \mathcal{D}(T^*)\} . \tag{8.23}$$

One see this because $G(T)^\perp = \{\chi_1, \chi_2\}$, where the vectors χ_1, χ_2 have the property

$$\langle \chi_2, Tf \rangle_{\mathcal{K}} = - \langle \chi_1, f \rangle_{\mathcal{K}} \ , \qquad \text{for all } f \in \mathcal{D}(T) . \tag{8.24}$$

This means $\chi_2 = \chi \in \mathcal{D}(T^*)$ and $\chi_1 = -T^*\chi$.

We ask whether $G(T)^\perp$ is the graph $G(S)$ of an operator S? If that is the case, then $\mathcal{D}(S) = \mathcal{R}(T^*)$ and

$$ST^*\chi = -\chi \ , \tag{8.25}$$

for all $\chi \in \mathcal{D}(T^*)$. In other words, if T^* is densely defined, then ST^* is densely defined, and in that case $S = (-T^*)^{-1}$.

We would like to apply the same reasoning to ask whether $G(T)^{\perp\perp}$ is the graph of an operator U? Then one needs to know that S^* is densely defined, which means we also need to know that S^* is defined at all, namely that $\mathcal{D}(S) = \mathcal{R}(T^*)$ is dense. In that case

$$US^*f = -f \ , \tag{8.26}$$

for $f \in \mathcal{D}(S^*)$. Then $U = (-S^*)^{-1} = T^{**}$. Thus we conclude that if $G(T)^{\perp\perp}$ is the graph of an operator, then

$$G(T)^{\perp\perp} = G(T^{**}) . \tag{8.27}$$

Since $G(T)$ is a subspace of $\mathcal{K}$, it is the case that $G(T)^{\perp\perp} = G(T)^-$. Furthermore we have checked above that if $G(T)^-$ is the graph of an operator, then it is the graph of T^-.

8.3 Self-Adjoint Operators

We often are interested to know whether a given operator is self-adjoint or essentially self adjoint.

Criteria for Self-Adjointness A closed, symmetric transformation T with dense domain $\mathcal{D}(T)$ is self-adjiont if any of the following hold:

- $\mathcal{D}(T)$ contains an orthonormal set of eigenvectors for T.
- The range of $T \pm i$ is $\mathcal{K}$, or $(T \pm i)\,\mathcal{D}(T) = \mathcal{K}$. (In other words, T^* has neither $-i$ nor i as an eigenvalue.)
- If $I \leq T$ on $\mathcal{D}(T) \times \mathcal{D}(T)$, and $T\mathcal{D}(T) = \mathcal{K}$.

Criteria for Essential Self-Adjointness A symmetric transformation T with dense domain $\mathcal{D}(T)$ is essentially self-adjiont if any of the following hold:

- $(T \pm i)\,\mathcal{D}(T)$ are both dense in $\mathcal{K}$.
- If $I \leq T$ on $\mathcal{D}(T) \times \mathcal{D}(T)$, and $T\mathcal{D}(T)$ is dense in $\mathcal{K}$.
- $D(T)$ contains a dense set of analytic vectors for T (and conversely).

We make two remarks about these criteria. If the range of $T + i$ is not dense in $\mathcal{K}$, then there is a vector χ orthogonal to this range. Then $\langle \chi, (T+i)f \rangle_{\mathcal{K}} = 0$ for all $f \in \mathcal{D}(T)$. In particular, according to (8.6), χ is in the domain of $(T+i)^*$ and $(T+i)^*\chi = 0$. Thus *chi* is an eigenvector of T^* with eigenvalue i. The dimension of the eigenspaces $\pm i$ of T^* are known as the deficiency indices of a symmetric operator T, and T is essentially self adjoint when both deficiency indices equal zero. In certain cases where T is a differential operator, this criterion can be studied directly by solving the differential equation $T^* f = if$ as an equation for a generalized function f which ultimately to be an eigenvalue must lie in $\mathcal{K}$.

Secondly, a useful criterion to show that a transformation T has a dense set of analytic vectors is to compare T with another operator for which this is known. For example, any eigenvector is an analytic vector. One might choose to compare T with an operator S which is known to be self-adjoint. The following criterion of Nelson is sufficient. There is a condition depending on the size of multiple commutators between T and S, sometimes written in terms of the operation Ad_T, where

$$\mathrm{Ad}_T(S) = [T, S] \ . \tag{8.28}$$

8.3.1 Analytic Vectors

Given T and S, the operator TS has a domain that consists of all vectors $f \in \mathcal{D}(S)$ such that $Sf \in \mathcal{D}(T)$. And in this case $(TS)f = T(Sf)$. Similarly, a vector f is said to be in $C^\infty(T)$, if $f \in \mathcal{D}(T^n)$ for all $n \in \mathbb{Z}_+$.

A vector $f \in C^\infty(T)$ is said to be an analytic vector for T, if there are finite constants a, b such that for all $n \in \mathbb{Z}_+$,

$$\|T^n f\|_{\mathcal{K}} \leq a b^n n! \; . \tag{8.29}$$

Proposition 8.3.1 (E. Nelson's analytic vector theorem). *Let S be self-adjoint, let $C^\infty(S) \subset C^\infty(T)$, and let $TC^\infty(S) \subset C^\infty(S)$. Suppose that*

$$\|Tf\| \leq c_0 \|Sf\| \; , \tag{8.30}$$

and also that there are constants c_n such that for all $f \in C^\infty(S)$,

$$\|\mathrm{Ad}_T^n(S) f\| \leq c_n \|Sf\| \; , \;\; \textit{for all} \;\;\; n \geq 1 \; , \tag{8.31}$$

with

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} t^n \; , \tag{8.32}$$

converging for $|t|$ sufficiently small. Then every analytic vector for S is an analytic vector for T.

8.4 Operators between Different Hilbert Spaces

Many of the concepts about operators extend to linear transformations T that map a domain in the Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$,

$$T : \mathcal{H}_1 \to \mathcal{H}_2 \; . \tag{8.33}$$

If $\mathcal{D}(T)$ is dense in $\mathcal{H}_1$, then the adjoint transformation

$$T^* : \mathcal{H}_2 \to \mathcal{H}_1 \; , \tag{8.34}$$

is defined with the domain $D(T^*)$ is the set of vectors $g \in \mathcal{H}_2$ for which there exists $\chi \in \mathcal{H}_1$ such that

$$\langle g, Tf \rangle_{\mathcal{H}_2} = \langle \chi, f \rangle_{\mathcal{H}_1} \; , \qquad \text{for all } \; f \in \mathcal{D}(T) \; . \tag{8.35}$$

In this case $T^* g = \chi$. The norm of T is

$$\|T\|_{\mathcal{H}_1 \to \mathcal{H}_2} = \sup_{\substack{f \in \mathcal{H}_1 \\ f \neq 0}} \frac{\|Tf\|_{\mathcal{H}_2}}{\|f\|_{\mathcal{H}_1}} \; . \tag{8.36}$$

It also equals

$$\|T\|_{\mathcal{H}_1 \to \mathcal{H}_2} = \sup_{\substack{f \in \mathcal{H}_1 \\ g \in \mathcal{H}_2 \\ f, g \neq 0}} \frac{\left| \langle g, Tf \rangle_{\mathcal{H}_2} \right|}{\|g\|_{\mathcal{H}_2} \|f\|_{\mathcal{H}_1}} \; . \tag{8.37}$$

Example. In case $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathbb{C}$, the operator T is a continuous linear functional, and by the Riesz representation theorem it is represented by $Tf = \langle \chi, f \rangle_{\mathcal{H}}$ with $\|T\|_{\mathcal{H}\to\mathbb{C}} = \|\chi\|_{\mathcal{H}}$. Furthermore T^* maps $\lambda \in \mathbb{C}$ into

$$T^*\lambda = \lambda\chi \in \mathcal{H} \ , \qquad \text{with norm } \|T^*\|_{\mathbb{C}\to\mathcal{H}} = \|T\|_{\mathcal{H}\to\mathbb{C}} = \|\chi\|_{\mathcal{H}} \ . \tag{8.38}$$

Furthermore, T^*T is a rank one operator on $\mathcal{H}$,

$$T^*Tf = \chi \langle \chi, f \rangle_{\mathcal{H}} \ , \qquad \text{while } TT^*\lambda = \langle \chi, \chi \rangle_{\mathcal{H}} \lambda \ . \tag{8.39}$$

In the further special case that $\mathcal{H} = L^2(\mathbb{R}^N; d\nu)$, then the operators T, T^*, T^*T, and $TT*$ are integral operators with integral kernels

$$T(x;y) = \overline{\chi(y)} \ , \quad T^*(x;y) = \chi(x) \ , \quad (T^*T)(x;y) = \chi(x)\overline{\chi(y)} \ , \qquad \text{and } (TT^*)(x;y) = \langle \chi, \chi \rangle_{\mathcal{H}} \ . \tag{8.40}$$

Since T^*T has rank one, $\|T^*T\| = T = \mathrm{Tr}(T^*T) = \langle \chi, \chi \rangle_{\mathcal{H}}$. In other words, when $T : \mathcal{H} \to \mathbb{C}$, the operator norm $\|T\|_{\mathcal{H}\to\mathbb{C}}$ equals the $L^2(\mathbb{R}^N, d\nu)$-norm (sometimes called the Hilbert-Schmidt norm) of its integral operator kernel.

One can establish the bound on the operator norm of an operator

$$T : \mathcal{H}_1 \to \mathcal{H}_2 \ , \qquad \text{where } \mathcal{H}_1 = L^2(\mathbb{R}^N; d\nu) \ , \ \text{ and } \mathcal{H}_2 = L^2(\mathbb{R}^{N'}; d\nu') \ , \tag{8.41}$$

generalizing the case $\mathcal{H}_1 = \mathcal{H}_2$ of Proposition 8.2.1. Let

$$|T|_{\infty,1} = \left(\sup_{x\in\mathbb{R}^{N'}} \int_{\mathbb{R}^N} |T(x;y)| d\nu(y) \right)^{1/2} \left(\sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^{N'}} |T(x;y)| d\nu'(x) \right)^{1/2} \ . \tag{8.42}$$

We follow the proof of Proposition 8.2.1 with minor modification to obtain,

Proposition 8.4.1. *An operator T mapping between L^2 spaces of the form* (8.41) *has a norm bounded by the norm* (8.42),

$$\|T\|_{\mathcal{H}_1\to\mathcal{H}_2} \le |T|_{\infty,1} \ . \tag{8.43}$$

8.5 Forms

We use the word *form* to mean a map from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathbb{C}$ that is linear in $\mathcal{H}_2$ and conjugate linear in $\mathcal{H}_1$. is a transformation T from $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathbb{C}$, which is linear on $\mathcal{H}_2$ and anti-linear on $\mathcal{H}_1$. If T is an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ with domain $\mathcal{D}(T)$, then the matrix elements of T, namely $\langle f_1, Tf_2 \rangle$, define a sesqui-linear form on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{H}_1 \otimes \mathcal{D}(T)$,

$$T(f \otimes g) = \langle f_1, Tf_2 \rangle \ . \tag{8.44}$$

However, there are many sesqui-linear forms on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that are not the matrix elements of operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For example, with $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R})$, the delta function δ_x is a sesquilinear form with domain $C_0^\infty \otimes C_0^\infty$, namely

$$\delta_x \left(f \otimes g \right) = \overline{f(x)} g(x) \ . \tag{8.45}$$

The norm $\|T\|_{\mathcal{H}_1\otimes\mathcal{H}_2}$ of the sesqui-linear form T is defined as

$$\|T\|_{\mathcal{H}_1\otimes\mathcal{H}_2} = \sup_{\substack{f\in\mathcal{H}_1\\ g\in\mathcal{H}_2\\ f,g\neq 0}} \frac{\left|\langle g, Tf\rangle_{\mathcal{H}_2}\right|}{\|g\|_{\mathcal{H}_2}\|f\|_{\mathcal{H}_1}} , \tag{8.46}$$

namely exactly the same expression as the norm of a bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ in (8.37). The form T is bounded, if $\|T\|_{\mathcal{H}_1\otimes\mathcal{H}_2} < \infty$. The following elementary result is a form of the Riesz representation theorem.

Proposition 8.5.1. *Let T be a bounded form on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then there exists a unique, bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that the values of the form T equal the matrix elements of the operator T given by* (8.44)*.*

8.5.1 The Graph of T

The graph $G(T)$ is a linear subspace of $\mathcal{H}_1 \oplus \mathcal{H}_2$,

$$G(T) = \{f \oplus Tf : \text{ where } f \in \mathcal{D}(T) \subset \mathcal{H}_1 \text{ and } Tf \in \mathcal{R}(T) \subset \mathcal{H}_2\} . \tag{8.47}$$

8.6 Trace

We consider the trace Tr of a positive, bounded operator T on $\mathcal{H}$. Let $\{e_i\}$, for $j \in \mathbb{Z}_+$, be an orthonormal basis for $\mathcal{H}$. Define

$$\mathrm{Tr}(T) = \sum_{i=0}^{\infty} \langle e_i, Te_i\rangle_{\mathcal{H}} , \tag{8.48}$$

If $\mathrm{Tr}(T) < \infty$, one says that T is *trace class.*

Proposition 8.6.1. *When the a positive operator T is trace class,* $\mathrm{Tr}(T)$ *is basis independent.*

Proof. In order to establish basis independence, suppose that $\{f_i\}$ is a second orthonormal basis. We show that $\mathrm{Tr}(T)$ also equals (8.48) computed in the f-basis. Since $0 \leq T$, the sum (8.48) is increasing and existence of the trace means that $\mathrm{Tr}(T) < \infty$ and

$$\mathrm{Tr}(T) = \lim_{N\to\infty} \sum_{i=0}^{N} \langle e_i, Te_i\rangle_{\mathcal{H}} . \tag{8.49}$$

Also as T is bounded, so

$$\begin{aligned} \langle e_i, Te_i\rangle_{\mathcal{H}} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \langle e_i, f_j\rangle_{\mathcal{H}} \langle f_j, Tf_k\rangle_{\mathcal{H}} \langle f_k, e_i\rangle_{\mathcal{H}} \\ &= \lim_{N\to\infty} \sum_{j=0}^{N}\sum_{k=0}^{N} \langle e_i, f_j\rangle_{\mathcal{H}} \langle f_j, Tf_k\rangle_{\mathcal{H}} \langle f_k, e_i\rangle_{\mathcal{H}} , \end{aligned} \tag{8.50}$$

and the truncated sum oven j, k is positive for each i. Thus one can sum (8.50) over i with N fixed, and the increasing series converges on the left to $\mathrm{Tr}(T)$ and on the right using the fact that the $\{e_i\}$'s and the $\{f_j\}$'s are both orthonormal bases, we obtain

$$\lim_{N\to\infty} \sum_{j=0}^{N} \langle f_j, T f_j \rangle_{\mathcal{H}} = \sum_{j=0}^{\infty} \langle f_j, T f_j \rangle_{\mathcal{H}} \ . \tag{8.51}$$

This equals the sum on the left of (8.50), so

$$\mathrm{Tr}(T) = \sum_{j=0}^{\infty} \langle f_j, T f_j \rangle_{\mathcal{H}} \ , \tag{8.52}$$

and the trace is basis independent.

In case the trace of positive T is computed in a basis of eigenvectors, then

$$\mathrm{Tr}(T) = \sum_j \lambda_j \ , \tag{8.53}$$

where λ_j are the eigenvalues of T. The converse is also true; if a positive T is trace class, then T has an orthonormal basis of eigenvectors and (8.53) holds.

If T is bounded but not self-adjoint or positive, then T has a polar decomposition $T = U|T|$, where $0 \leq |T|$ is the absolute value of T, defined as the positive square root of the self-adjoint (and therefore diagonalizable) operator T^*T. In other words, $|T| = (T^*T)^{1/2}$. The operator U has the property that U^*U and UU^* are projections onto subspaces of $\mathcal{H}$. Unlike the finite-dimensional case, the operator T^*, and hence U^*, may have null vectors even if T has none. If $|T|$ is trace class, then $\mathrm{Tr}(T)$ defined by (8.48) exists and is basis independent.

8.7 Convergence of Operators

If T_n is a sequence of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, one is often interested to check that the sequence has a limit T. But there are several different criteria for convergence; different criteria are useful in different contexts. They also have very different consequences. We mention several criteria, that just happen to be ordered in strength. We start from the strongest notion of convergence and progress to the weakest.

8.7.1 Convergence Based on Traces

Here we restrict attention to the case $T : \mathcal{H} \to \mathcal{H}$. Furthermore we suppose that T has pure discrete spectrum. (This is a big restriction. For example, it only applies to a Hamiltonian for a system in a finite spatial volume.) However this is often a useful approximation, or intermediate step. And in a finite volume, the trace enters in a natural way in the definition of a "finite temperature state" given by the normalized exponential distribution $\rho = \mathfrak{z}^{-1} e^{-\beta H}$, where $\mathfrak{z}$ is a normalization constant chosen so the distribution ρ has unit trace. The trace gives a basic norm, of which there are useful variations, the Schatten norms.

Schatten Norms. The Schatten norms of T are defined by the trace of powers of the absolute value $|T| = (T^*T)^{1/2}$. For $p \geq 1$, these norms are

$$\|T\|_{I_p} = \mathrm{Tr}(|T|^p)^{1/2} = \left(\sum_j \lambda_j^p \right)^{1/p} , \tag{8.54}$$

where $\{\lambda_j\}$ are the eigenvalues of $|T|$. For $T \neq 0$, one can write for $M = \|T\|_{\mathcal{H}}$, that

$$\|T\|_{I_p} = M \left\| \frac{T}{M} \right\|_{I_p} . \tag{8.55}$$

But $|T|/M$ has a finite number of eigenvalues equal to 1, and all the rest in the interval $[0, 1)$. This makes it clear that $\|T\|_{I_p}$ is a strictly decreasing function of $p \geq 1$,

$$\|T\|_{I_p} > \|T\|_{I_{p'}} > \|T\|_{\mathcal{H}} = M , \qquad \text{for } p < p' . \tag{8.56}$$

Hence if some I_p norm of T is finite, then

$$\|T\|_{\mathcal{H}} = \lim_{p\to\infty} \|T\|_{I_p} . \tag{8.57}$$

One says that $T_n \to T$ in I_p if $T_n, T \in I_p$ and

$$\lim_{n\to\infty} \|T_n - T\|_{I_p} = 0 . \tag{8.58}$$

Furthermore, all Cauchy sequences converge in I_p, so if $\|T_n - T_m\|_{I_p}$ is a Cauchy sequence, then there exists T such that $T_n \to T$ in I_p. This is a very restrictive type of convergence, the most restrictive for $p = 1$. However in finite, fixed volume situations it is often the case that sequences of partition functions $\mathfrak{Z}_n = \mathrm{Tr}(e^{-\beta H_n})$ do converge to $\mathfrak{Z} = \mathrm{Tr}(e^{-\beta H})$.

8.7.2 Uniform Convergence

The limiting (and weakest) case $p = \infty$ of convergence in the Schatten norms is convergence in the operator norm. This is also called uniform convergence,

$$\|T_n - T\|_{\mathcal{H}} \to 0 . \tag{8.59}$$

All Cauchy sequences converge in the operator norm. While less restrictive than convergence in a finite Schatten norm, uniform convergence is still quite restrictive.

Examples. Uniform limits of finite rank operators are compact: namely they have pure discrete spectrum and all eigenvalues have finite multiplicity. Uniform convergence of a unitary group e^{itH} to I as $t \to 0$ ensures that the self-adjoint generator of the group H is bounded. However, if A and B are bounded, then

$$\lim_{n\to\infty} \left\| e^{A+B} - \left(e^{A/n} e^{B/n} \right)^n \right\|_{\mathcal{H}} = 0 . \tag{8.60}$$

One can substitute $A \to itA$, $B \to itB$ with the new A, B both self adjoint. Then (8.60) gives a product of the one-parameter unitary groups e^{itA} and e^{itB} generated by A and by B, yielding a group $e^{it(A+B)}$ generated by $A + B$.

8.7.3 Strong Convergence

A sequence of bounded transformations $\{T_n\}$ converges strongly to T, if

$$\|T_n f - Tf\|_{\mathcal{H}} = 0 \ . \tag{8.61}$$

This is the operator analog of pointwise convergence of functions, as the T_n converge at each point in $\mathcal{H}$.

Proposition 8.7.1. *Let $\{T_n\}$ be a sequence of operators that is uniformly bounded, $\|T_n\|_{\mathcal{H}} \leq M$ with M independent of n, and for which*

$$T_n f \to Tf \ , \tag{8.62}$$

for every f in a dense subset $\mathcal{D} \subset \mathcal{H}$. Then T_n converges strongly to T.

Proof. This is a "3ϵ"-argument. Given $\epsilon > 0$ and $f \in \mathcal{H}$, one can choose and $g \in \mathcal{D}$ with $M\|g - f\|_{\mathcal{H}} < \epsilon$. Thus $\|T_n(g-f)\|_{\mathcal{H}} < \epsilon$, with the bound independent of n. Furthermore, our assumption is that $T_n g$ converges, namely $T_n g$ is a Cauchy sequence. Thus there exists n_0 such that when $n, n' > n_0$, then $\|(T_n - T_{n'})\, g\|_{\mathcal{H}} < \epsilon$. For $n, n' > n_0$, write

$$(T_n - T_{n'})\, f = T_n(f-g) + (T_n - T_{n'})\, g + T_{n'}(g-f) \ . \tag{8.63}$$

Then

$$\|(T_n - T_{n'})\, f\|_{\mathcal{H}} \leq \|T_n(f-g)\|_{\mathcal{H}} + \|(T_n - T_{n'})\, g\|_{\mathcal{H}} + \|T_{n'}(g-f)\|_{\mathcal{H}} \leq 3\epsilon \ . \tag{8.64}$$

Hence $T_n f$ is a Cauchy sequence for an arbitrary vector $f \in \mathcal{H}$.

Proposition 8.7.2. *Let T_n, T be a sequence of self-adjoint operators with a common dense domain $\mathcal{D}$, such that T_n, T are all essentially self-adjoint on $\mathcal{D}$. If*

$$\operatorname*{st.lim}_{n\to\infty} T_n \chi = T\chi \ , \ \textit{for all } \chi \in \mathcal{D} \ , \tag{8.65}$$

then

$$\operatorname*{st.lim}_{n\to\infty} e^{iT_n} \to e^{iT} \ . \tag{8.66}$$

Proof. Since T_n is self-adjoint, e^{isT_n} is unitary and strongly continuous in s for real s, and strongly differentiable on the domain $\mathcal{D}$. For $\chi \in \mathcal{D}$,

$$\frac{d}{ds} e^{isT_n}\chi = i e^{isT_n} T_n \chi \ , \tag{8.67}$$

and

$$e^{iT_n}\chi = \chi + i\int_0^1 e^{isT_n} T_n \chi \, ds \ . \tag{8.68}$$

Thus

$$\begin{aligned} e^{iT_n}\chi - e^{iT}\chi &= i\int_0^1 \left(e^{isT_n} T_n - e^{isT} T\right) \chi \, ds \\ &= i\int_0^1 e^{isT_n}(T_n - T)\chi + \left(e^{isT_n} - e^{isT}\right) T\chi \, ds \ . \end{aligned} \tag{8.69}$$

Trotter Product Formula

8.7.4 Weak Convergence

Proposition 8.7.3. Weak Convergence of Unitaries Ensures Strong Convergence *Let T_n be unitary operators on $\mathcal{H}$, and let $\mathcal{D} \subset \mathcal{H}$ be a dense subset. If the matrix elements $\langle \chi, T_n \chi \rangle$ are a convergent Cauchy sequence for all $\chi \in \mathcal{D}$, then then there is a unitary T such that*

$$\text{st.}\lim_{n\to\infty} T_n = T \ . \tag{8.70}$$

Proof. First note that the hypotheses ensure weak convergence of T_n. Let σ range over the fourth roots of unity, $\{\pm 1, \pm i\}$. The polarization identity

$$\langle \chi, S\psi \rangle = \frac{1}{4} \sum_{\sigma} \overline{\sigma} \, \langle \chi + \sigma\psi, S(\chi + \sigma\psi) \rangle \ , \tag{8.71}$$

shows that convergence of expectations ensures weak convergence. Furthermore, convergence of expectations of unitaries on a dense set ensures convergence of expectations. For if $\Omega \in \mathcal{H}$ then given $\epsilon > 0$, there is a vector $\chi \in \mathcal{D}$ such that $\|\chi - \Omega\| < \epsilon$.

Thus For unitaries, the norm $\|(T_n - T)\, f\|^2 = 2$

8.7.5 Graph Convergence

Chapter 9

Fourier Transformation

The Fourier inversion formula is central to quantum theory. Here we establish this formula. Define the Fourier operator $\mathfrak{F}$ on $\mathbb{R}^N$ by

$$(\mathfrak{F}f)(p) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x)\, e^{-ip\cdot x}\, dx\ . \tag{9.1}$$

With $(\Pi f)(x) = f(-x)$, the Fourier inversion theorem says $\mathfrak{F}^{-1} = \Pi\mathfrak{F}$, namely

$$\left(\mathfrak{F}^{-1} f\right)(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(p)\, e^{ip\cdot x}\, dp\ . \tag{9.2}$$

9.1 Fourier Transforms on L^2

Proposition 9.1.1. *The Fourier operator $\mathfrak{F}$ is unitary on $L^2(\mathbb{R}^N; dx)$. In other words,*

$$\boxed{\mathfrak{F}^*\mathfrak{F} = \mathfrak{F}\mathfrak{F}^* = I}\ . \tag{9.3}$$

Also,

$$\mathfrak{F}^* = \Pi\mathfrak{F}\ . \tag{9.4}$$

Remark 9.1.2. *We establish the Fourier inversion theorem in three steps.*

1. *We reduce inversion on N-dimensional Euclidean space $\mathbb{R}^N$ to inversion in the case $N = 1$.*

2. *We show that the correctness of the Fourier inversion theorem on $\mathbb{R}$ is equivalent to the statement that the normalized eigenfunctions of the "harmonic oscillator" Hamiltonian H are an orthonormal basis in $L^2(\mathbb{R})$.*

3. *We show that the normalized oscillator eigenfunctions are an orthonormal basis as desired.*

- *As a byproduct of this argument, one finds an elementary relation between Fourier transformation* $\mathfrak{F}$*, the reflection* Π*, and the oscillator Hamiltonian* $H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right)$*. Namely*

$$\boxed{\Pi = e^{-i\pi H}}\ , \qquad \textit{and} \qquad \boxed{\mathfrak{F} = e^{-i\pi H/2}}\ . \tag{9.5}$$

One could also write the second relation in the more provocative form $\mathfrak{F} = \Pi^{1/2}$*, but the convention of choosing* which *square root comes from* (9.5)*.*

Proof. Step 1: Reduction to One Dimension. Since the complex measure

$$e^{-ip\cdot x}dx = \prod_{j=1}^{N} e^{-ip_j x_j}dx_j\ , \tag{9.6}$$

the Hilbert space $L^2(\mathbb{R}^N)$ is the N-fold tensor product of $L^2(\mathbb{R})$. If we establish the $N = 1$ result, then the N-fold tensor product of operators $\mathfrak{F} \otimes \mathfrak{F} \otimes \cdots \otimes \mathfrak{F}$ is unitary on $L^2(\mathbb{R}^N)$ with the corresponding tensor-product inverse.

Step 2. Relation to the Oscillator. Consider the operator

$$a = \frac{1}{2^{1/2}}\left(x + \frac{d}{dx}\right)\ , \tag{9.7}$$

on $L^2(\mathbb{R})$ with the dense domain $\mathcal{D}(a)$ of C^∞, rapidly decreasing functions with rapidly decreasing derivatives. Then $\mathcal{D}(a) \subset \mathcal{D}(a^*)$ and on the domain $\mathcal{D}(a)$,

$$a^* = \frac{1}{2^{1/2}}\left(x - \frac{d}{dx}\right)\ , \qquad \text{and} \qquad H = a^*a = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right)\ . \tag{9.8}$$

Here H is the quantum mechanical "harmonic oscillator" Hamiltonian, and in terms of the momentum $p = -id/dx$, one writes $H = \frac{1}{2}\left(p^2 + x^2 - 1\right)$.

We claim that the set of eigenfunctions of H and of $\mathfrak{F}$ coincide. The elementary example is the function

$$\Omega_0 = \pi^{-1/4}e^{-x^2/2}\ , \tag{9.9}$$

in $\mathcal{D}(a)$. This vector is a normalized null vector for a, namely $a\Omega_0 = 0$. We infer that Ω_0 is a null vector for H. Furthermore Ω_0 is an invariant vector for $\mathfrak{F}$, namely

$$\left(\mathfrak{F}\Omega_0\right)(p) = \frac{1}{(2\pi)^{1/2}\pi^{1/4}}\int_{-\infty}^{\infty} e^{-x^2/2-ipx}dx = \left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-(x+ip)^2/2}dx\right)\Omega_0(p) = \Omega_0(p)\ . \tag{9.10}$$

We claim that the operator a^* has the following commutation relations on the domain $\mathcal{D}(a)$,

$$[H, a^*] = a^*\ , \qquad \text{and} \qquad \mathfrak{F}\, a^* = -ia^*\, \mathfrak{F}\ . \tag{9.11}$$

The first relation (9.11) is a consequence of

$$[a, a^*] = I \ . \tag{9.12}$$

For the second, one observed that the definition (9.1) and integration by parts ensures that $-\mathfrak{F} d/dx = -ip\mathfrak{F}$, where p denotes multiplication by the coordinate p. Likewise $\mathfrak{F}x = id/dp\mathfrak{F}$. Therefore we infer from (9.8) that

$$\mathfrak{F}a^* = 2^{-1/2}\mathfrak{F}\left(x - \frac{d}{dx}\right) = 2^{-1/2}\left(i\frac{d}{dp} - ip\right)\mathfrak{F} = -ia^*\mathfrak{F} \ . \tag{9.13}$$

As a consequence, for $n \in \mathbb{Z}_+$ the vectors

$$\Omega_n = \frac{1}{n!^{1/2}}\, a^{*n}\, \Omega_0 \ , \tag{9.14}$$

are orthogonal eigenvectors of both H and $\mathfrak{F}$ with eigenvalues n and $(-i)^n$ respectively. In fact, the commutation relation (9.12) ensures that these vectors are orthonormal.

Hence we conclude that if the set of eigenvectors $\{\Omega_n\}$ are an orthonormal basis for $L^2(\mathbb{R})$, then $\mathfrak{F}$ is a unitary operator with spectrum $\pm 1, \pm i$, that H is a self-adjoint operator with spectrum $\mathbb{Z}_+$, and that the relation (9.5) holds.

Furthermore the reflection operator Π satisfies $\Pi a^* = -a^*\Pi$. Therefore $\Pi\Omega_n = (-1)^n\Omega_n$, so on the eigenfunction Ω_n one has the identity

$$\Pi = e^{\pm i\pi H} \ . \tag{9.15}$$

Therefore, one also infers that if the functions $\{\Omega_n\}$ are a basis, then

$$\mathfrak{F}^* = e^{i\pi H/2} = e^{-i\pi H/2 + i\pi H} = \mathfrak{F}\Pi = \Pi\mathfrak{F} \ . \tag{9.16}$$

The functions $\Omega_n(x)$ are the normalized Hermite functions; they have the form,

$$\Omega_n(x) = 2^{-n/2} n!^{-1/2}\, H_n(x)\Omega_0(x) \ , \tag{9.17}$$

where $H_n(x)$ is the usual Hermite polynomial of degree n. From the relation (9.14) one sees

$$\Omega_n(x) \quad = \quad (-1)^n \pi^{-1/4} 2^{-n/2} n!^{-1/2} \left(\frac{d}{dx} - x\right)^n e^{-x^2/2} \ , \tag{9.18}$$

and therefore

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2} \ . \tag{9.19}$$

One can read off from the representation (9.19) that

$$H_{2n+1}(0) = 0 \ , \qquad \text{and } H_{2n}(0) = \frac{(2n)!^{1/2}}{2^n n!} \sim \frac{1}{n^{1/4}} \ , \ \text{ as } n \to \infty \ . \tag{9.20}$$

We also wish to introduce the generating function $G_z(x)$ for the Hermite polynomials. For a complex parameter z define

$$G_z(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{-z^2+2zx} . \tag{9.21}$$

We evaluate the series for $G_z(x)$ using (9.19) and the fact that e^{-x^2} extends to an entire function of x, yielding the right side of (9.21).

For fixed z, define the function

$$F_z(x) = G_z(x)\Omega_0(x) = \frac{1}{\pi^{1/4}} e^{-z^2+2zx-x^2/2} , \tag{9.22}$$

has a square-integrable dependence on the variable x, as a consequence of the Gaussian decrease of $\Omega_0(x)$. Thus for fixed z, the function $F_z(x)$ defines a vector $F_z \in L^2(\mathbb{R})$. This vector has a power series expansion in z, which one can interpret as a generating function for the eigenfunctions Ω_n. Using (9.17) and (9.21) one has,

$$F_z = \sum_{n=0}^{\infty} \frac{(\sqrt{2}z)^n}{\sqrt{n!}} \Omega_n \in L^2(\mathbb{R}) . \tag{9.23}$$

Since each Ω_n is a unit vector in $L^2(\mathbb{R})$, the sum (9.23) converges as a series of vectors in $L^2(\mathbb{R})$ for all $z \in \mathbb{C}$. In other words, F_z is an entire function from $\mathbb{C}$ to $L^2(\mathbb{R})$. In particular, for any vector $\chi \in L^2(\mathbb{R})$, the function

$$F_z(\chi) = \langle \chi, F_z \rangle_{L^2(\mathbb{R})} , \tag{9.24}$$

is an entire function of z in the ordinary sense.

Step 3. The Oscillator Eigenfunctions are a Basis. We complete the proof of the proposition by showing that the set of orthonormal oscillator eigenfunctions $\{\Omega_n\}$ are a basis for $L^2(\mathbb{R})$. This is equivalent to showing that any function $\chi \in L^2(\mathbb{R})$ orthogonal to all the Ω_n's must be zero.

Assume there is a such a function χ satisfying $\langle \chi, \Omega_n \rangle_{L^2(\mathbb{R})} = 0$, for all $n \in \mathbb{Z}_+$. In terms of the generating function F_z above, this means that every derivative of the inner product

$$\left. \frac{d^n}{dz^n} F_z(\chi) \right|_{z=0} = 2^{n/2} n!^{1/2} \langle \chi, \Omega_n \rangle_{L^2(\mathbb{R})} = 0 , \qquad \text{for all } n \in \mathbb{Z}_+ . \tag{9.25}$$

Since $F_z(\chi)$ is entire, if its derivatives all vanish at the $z = 0$, then the function $F_z(\chi)$ itself must be identically zero for all $z \in \mathbb{C}$. This means that

$$\int_{-\infty}^{\infty} \overline{\chi(x)} e^{-x^2/2} e^{2\bar{z}x} dx = 0 , \qquad \text{for all } z \in \mathbb{C} . \tag{9.26}$$

Set $z = ip/2$, choose $\epsilon > 0$, multiply (9.26) by $e^{-\epsilon p^2 + ipa}$ for real a, and integrate over all real p. Using the fact that the Fourier transform of a Gaussian is a Gaussian, we obtain

$$\frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \overline{\chi(x)} e^{-x^2/2} e^{-(x-a)^2/4\epsilon} dx = 0 , \tag{9.27}$$

for all real a and $\epsilon > 0$. For any C_0^∞ function f, we then find that

$$\left\langle \chi e^{-x^2/2}, T_\epsilon f \right\rangle_{L^2(\mathbb{R})} = \frac{1}{\sqrt{4\pi\epsilon}} \int_{\mathbb{R}^2} \overline{\chi(x)} e^{-x^2/2} e^{-(x-a)^2/4\epsilon} f(a) dx da = 0 \; . \tag{9.28}$$

Here T_ϵ is the integral operator

$$(T_\epsilon f)(x) = \int T_\epsilon(x-y) f(y) dy \; , \qquad \text{with integral kernel } \; T_\epsilon(x-y) = \frac{1}{\sqrt{4\pi\epsilon}} e^{-(x-y)^2/4\epsilon} \; . \tag{9.29}$$

This operator is useful in other contexts, so we state the following properties separately.

Lemma 9.1.3. *The operator T_ϵ on $L^2(\mathbb{R})$ defined for $\epsilon > 0$ by (9.29) has the properties:*

(i) *The operators T_ϵ are contractions,*

$$\|T_\epsilon\|_{L^2(\mathbb{R})} \leq 1 \; , \qquad \textit{for all } \; 0 < \epsilon \; . \tag{9.30}$$

(ii) *The T_ϵ converges strongly to I as $\epsilon \to 0$, namely*

$$\lim_{\epsilon \to 0} \|T_\epsilon f - f\|_{L^2(\mathbb{R})} = 0 \; , \qquad \textit{for all } \; f \in L^2(\mathbb{R}) \; . \tag{9.31}$$

Assume the lemma. As a consequence of (i), the vanishing scalar product (9.28) extends from $f \in C_0^\infty$ by continuity to all $f \in L^2$, namely

$$\left\langle \chi e^{-x^2/2}, T_\epsilon f \right\rangle_{L^2(\mathbb{R})} = 0 \; , \qquad \text{for all } \; f \in L^2(\mathbb{R}) \; . \tag{9.32}$$

And by (9.31) the vanishing extends further to the limit $\epsilon = 0$,

$$\lim_{\epsilon \to 0} \left\langle \chi e^{-x^2/2}, T_\epsilon f \right\rangle_{L^2(\mathbb{R})} = \left\langle \chi e^{-x^2/2}, f \right\rangle_{L^2(\mathbb{R})} = 0 \; , \qquad \text{for all } \; f \in L^2(\mathbb{R}) \; . \tag{9.33}$$

Therefore $\chi e^{-x^2/2}$ is orthogonal to all functions in $L^2(\mathbb{R})$; so it must vanish. Multiplying by $e^{x^2/2}$ we conclude that $\chi = 0$, and the proof of the proposition is complete.

Proof of Lemma 9.1.3. Both desired continuity statements (i–ii) are a consequence of elementary properties of the integral kernel $T_\epsilon(x-y)$. We begin by the observation

$$0 \leq T_\epsilon(x-y) \; , \qquad \text{and} \quad \int T_\epsilon(x-y) dy = 1 \; . \tag{9.34}$$

To prove bound (i) use Proposition 8.2.1, which in this case gives the claimed bound,

$$\|T_\epsilon\|_{L^2(\mathbb{R})} \leq \|T_\epsilon\|_{\infty,1} = \int T_\epsilon(x-y) dy = 1 \; . \tag{9.35}$$

The proof of property (ii) is slightly more involved. First we show that $\mathbb{T}_\epsilon$ is a contraction on $L^\infty(\mathbb{R})$. This means $\|T_\epsilon f\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}$, where the $L^\infty(\mathbb{R})$ norm is,

$$\|f\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}}|f(x)| \ . \tag{9.36}$$

In fact

$$\|T_\epsilon f\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} \left| \int T_\epsilon(x-y) f(y) dy \right| \leq \|f\|_{L^\infty(\mathbb{R})} \int T_\epsilon(x-y) dy = \|f\|_{L^\infty(\mathbb{R})} \ . \tag{9.37}$$

Two further elementary properties of T_ϵ follow from inspecting $T_\epsilon(x-y)$ in (9.29). First note that $T_\epsilon = T_\epsilon^*$ as $T_\epsilon(x-y)$ is real and symmetric. Also $T_\epsilon^2 = T_{2\epsilon}$, checked by computing a Gaussian integral. Therefore

$$\|T_\epsilon f - f\|^2_{L^2(\mathbb{R})} = \langle f, f - T_\epsilon f\rangle + \langle f, T_{2\epsilon} f - T_\epsilon f\rangle = 2\langle f, f - T_\epsilon f\rangle + \langle f, T_{2\epsilon} f - f\rangle \ . \tag{9.38}$$

Now we show (9.31). Using Proposition 8.7.1, we need only prove convergence for $f \in C_0^\infty \subset L^2(\mathbb{R})$, as C_0^∞ is a dense subspace of $L^2(\mathbb{R})$. We estimate the right side of (9.38) for such $f \in C_0^\infty$ using

$$\left|\langle f, g\rangle_{L^2(\mathbb{R})}\right| = \left| \int \overline{f(x)} g(x) dx \right| \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} \ , \tag{9.39}$$

where the $L^1(\mathbb{R})$ norm is

$$\|f\|_{L^1(\mathbb{R})} = \int |f(x)| dx < \infty \ . \tag{9.40}$$

Therefore (9.38) satisfies

$$\|T_\epsilon f - f\|^2_{L^2(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \left(2\|T_\epsilon f - f\|_{L^\infty(\mathbb{R})} + \|T_{2\epsilon} f - f\|_{L^\infty(\mathbb{R})} \right) \ . \tag{9.41}$$

Note that the $L^1(\mathbb{R})$ norm of any $f \in C_0^\infty$ function is finite, so the desired convergence (9.31) follows, if one can establish convergence in L^∞,

$$\lim_{\epsilon\to 0} \|T_\epsilon f - f\|_{L^\infty(\mathbb{R})} = 0 \ , \qquad \text{for each } f \in C_0^\infty \ . \tag{9.42}$$

We prove (9.42) using an elementary computation. From (9.34), one can write

$$T_\epsilon f(x) - f(x) = \int_{-\infty}^{\infty} T_\epsilon(x-y) \left(f(y) - f(x) \right) dy \ . \tag{9.43}$$

Divide the integration into two regions, according to whether $|x-y|\, \epsilon^{-1/4} \leq 1$. Hence

$$\begin{aligned} |T_\epsilon f(x) - f(x)| \ \leq \ & \left| \int_{|x-y|\leq \epsilon^{1/4}} T_\epsilon(x-y) \left(f(y) - f(x) \right) dy \right| \\ & + \left| \int_{|x-y| > \epsilon^{1/4}} T_\epsilon(x-y) \left(f(y) - f(x) \right) dy \right| \ . \end{aligned} \tag{9.44}$$

In the first term use $|f(y)-f(x)| \le |x-y| \|f'\|_{L^\infty(\mathbb{R})}$, so

$$\begin{aligned}\left|\int_{|x-y|\le\epsilon^{1/4}} T_\epsilon(x-y)\ \left(f(y)-f(x)\right) dy\right| &\le \epsilon^{1/4}\|f'\|_{L^\infty(\mathbb{R})}\int_{|x-y|\le\epsilon^{1/2}} T_\epsilon(x-y)dy \\ &\le \epsilon^{1/4}\|f'\|_{L^\infty(\mathbb{R})}\int T_\epsilon(x-y)dy \\ &= \epsilon^{1/4}\|f'\|_{L^\infty(\mathbb{R})}\ . \end{aligned} \tag{9.45}$$

Note $\|f'\|_{L^\infty(\mathbb{R})} < \infty$ for any $f \in C_0^\infty$, so this term vanishes as $\epsilon \to 0$.

The second term obeys the bound

$$\begin{aligned}\left|\int_{|x-y|>\epsilon^{1/4}} T_\epsilon(x-y)\ \left(f(y)-f(x)\right) dy\right| &\le 2\|f\|_{L^\infty(\mathbb{R})}\int_{|x-y|>\epsilon^{1/4}} T_\epsilon(x-y)dy \\ &\le \frac{1}{\sqrt{\pi}}\|f\|_{L^\infty(\mathbb{R})}\int_{|x|>\epsilon^{-1/4}} e^{-x^2/4}dx\ . \end{aligned} \tag{9.46}$$

The integral of the tail of the Gaussian in the last term vanishes faster than any power of ϵ as $\epsilon \to 0$. Note that with this method, the second term is small because the dimensionless variable of the Gaussian $T_\epsilon(x-y)$ satisfies $|x-y|\,\epsilon^{-1/2} \gg 1$. We have chosen $|x-y|\,\epsilon^{-1/2} > \epsilon^{-1/4}$, explaining the limit on the final integral.

Combining the bounds (9.45)–(9.46), we infer (9.42) as claimed, and hence we have established the stated convergence (ii) of Equation (9.31). This completes the proof.

Remark. Now that we know that $\mathfrak{F}$ is unitary, we can identify the smoothing operator T_ϵ in Fourier space. The operator $\mathfrak{F}T_\epsilon\mathfrak{F}^*$ is a multiplication operator,

$$\left(\mathfrak{F}T_\epsilon\mathfrak{F}^* f\right)(p) = e^{-\epsilon p^2} f(p)\ . \tag{9.47}$$

This displays the property of $\mathfrak{F}T_\epsilon\mathfrak{F}^*$ as a self-adjoint semi-group. Once we know that $\mathfrak{F}$ is unitary, it is apparant that strong convergence of $e^{-\epsilon p^2}-1 \to 0$ as $\epsilon \to 0$ on $\mathfrak{F}L^2(\mathbb{R})$ is equivalent to the condition (ii) of the lemma, namely strong convergence of $T_\epsilon - I \to 0$ on $L^2(\mathbb{R})$.

9.2 Schwartz Space