

COMPLEX CLASSICAL FIELDS: A FRAMEWORK FOR REFLECTION POSITIVITY

ARTHUR JAFFE, CHRISTIAN D. JÄKEL, AND ROBERTO E. MARTINEZ II

ABSTRACT. We explore a framework for complex classical fields, appropriate for describing quantum field theories. Our fields are linear transformations on a Hilbert space, so they are more general than random variables for a probability measure. Our method generalizes Osterwalder and Schrader’s construction of Euclidean fields. We allow complex-valued classical fields in the case of quantum field theories that describe neutral particles.

From an analytic point-of-view, the key to using our method is *reflection positivity*. We investigate conditions on the Fourier representation of the fields to ensure that reflection positivity holds. We also show how reflection positivity is preserved by various spacetime compactifications of \mathbb{R}^d in different coordinate directions.

CONTENTS

I. Classical Fields	2
I.1. Overview	2
I.2. More Details	2
I.3. This Work	3
I.4. The Problem with Measures	4
II. Classical Fields as Operators on Hilbert Space	6
II.1. Neutral, Classical Fields: the Case $\mathbf{X} = \mathbb{R}^d$	7
II.2. Schwinger Functions	8
II.3. Neutral Fields Φ as Operators	9
III. Time Reflection	11
III.1. Time-Reflection Positivity	12
IV. Quantization	14
IV.1. Spatial Reflection-Positivity for the Neutral Field	15
V. Charged Fields	16
V.1. Twist Symmetry of the Charged Field	17
V.2. Matrix Notation for the Charged Field	18
V.3. Charge Conjugation	20
V.4. Complex Conjugation	20
V.5. Time Reflection	21
V.6. Time-Reflection Positivity for the Charged Field	22
V.7. Spatial-Reflection Positivity for the Charged Field	23
VI. Compactification	23

VII. Summary of Positivity Conditions	27
VII.1. Neutral Fields	27
VII.2. Charged Fields	28
References	29

I. CLASSICAL FIELDS

I.1. Overview. We study a Fock-Hilbert space \mathcal{E} on which averaged classical fields act as linear transformations. These fields generate an abelian algebra of unbounded operators that are defined on a common, dense, invariant domain. We call a classical field *neutral/charged* if it arises in the description of neutral/charged particles. What is special in our framework is that our neutral fields can be either real or *complex*, so the usual distinction between real and complex fields does not coincide here with the distinction between neutral and charged fields. Real neutral fields reduce to the usual case; complex neutral fields allow for something new.¹

The two-point function of fields is the integral kernel of an operator D , which need not be hermitian. We require two things: First, the hermitian part of D should have strictly positive spectrum. Second, the transformation D should be reflection positive. As precise formulation of reflection positivity is provided in Definition III.2 (for the neutral case) and Definition V.3 (for the charged case).

We also deal with charge in a somewhat novel way. In the usual case, a charged field is represented as a complex-linear combination of two neutral fields. The usual charge conjugation arises as the complex conjugation of this field; this can be implemented as a unitary operator. However, in this paper we introduce distinct charged fields Φ_{\pm} —not related by complex conjugation—whose labels correspond to the fundamental charge carried by the field. Charge conjugation acts as a unitary transformation \mathbf{U}_c on \mathcal{E} such that $\mathbf{U}_c \Phi_{\pm} \mathbf{U}_c^{-1} = \Phi_{\mp}$.

I.2. More Details. Kurt Symanzik introduced the concept of a Euclidean-invariant Markoff field associated with an underlying probability distribution of classical fields [18]. Euclidean-covariant classical fields describing neutral scalar particles are typically real, as are time-zero quantum fields. With the standard distributions that occur for scalar quantum fields, the zero-particle expectations of products of fields are typically positive. This is also that case for the vacuum expectation values of time-ordered products of imaginary-time quantum fields (*i.e.*, the analytic continuation of quantum fields in Minkowski

¹ Parts of this work were carried out during authors' visits to the Institute for Theoretical Physics, ETH Zürich, the Erwin Schrödinger Institute (ESI), Vienna, and the mathematics and physics departments at Harvard University. The relevant authors thank these institutions both for hospitality and for support.

space). In fact, the analytic continuation of anti-time-ordered vacuum expectation values of quantum fields in Minkowski space should agree with the expectations of such classical fields.

Edward Nelson formulated a set of mathematical axioms interpreting Euclidean Markoff fields as random variables. A Markoff field satisfying these axioms yields a corresponding quantum field [12]. Although these axioms apply beautifully to the free scalar field [13], and to some other cases, verifying the global Markoff property for known examples of interacting scalar fields poses certain difficulties. Moreover, an analogous set of axioms has not been formulated for fermionic or gauge fields.

Konrad Osterwalder and Robert Schrader discovered an alternative and more-widely applicable approach based on a property that they called *reflection positivity* (RP), which today is often called Osterwalder-Schrader (OS) positivity. Every Euclidean-invariant, OS-positive and regular set of expectations yields a relativistic, local quantum field theory [15, 16]. Assuming certain growth conditions in both the quantum and classical framework, the OS axioms for a classical field theory were shown to be equivalent to the Wightman axioms for a corresponding quantum field. According to Zinoviev, these growth assumptions can be replaced by a *weak spectral condition* [20].

While RP allows one to give a rigorous meaning to *inverse Wick rotation*, RP has also found applications in various areas of mathematics outside of mathematical physics. For example, RP has had an impact on the theory of analytic continuation of group representations; recent results and extensive references can be found in [10]. Moreover, within mathematical physics, RP has had an enormous impact in statistical physics—especially in understanding properties of the spectrum of the transfer matrix, as well as in the theory of phase transitions. A contemporary review can be found in [2].

I.3. This Work. In this work we demonstrate that by no longer insisting that neutral fields are real, one gains a great deal of added flexibility. A number of relevant problems, which so far were not accessible from Euclidean quantum field theory, can now be formulated in terms of classical fields. The new framework allows us to consider Hamiltonians with complex-valued heat kernels such as, for example, $H = H_0 + \vec{v} \cdot \vec{P}$, which for $|\vec{v}| < 1$ equals (up to an overall multiplicative constant) the Hamiltonian H_0 (in a finite spatial volume) as seen from a Lorentz frame moving with velocity \vec{v} . In fact, this was our motivating example, and it arose in our attempt to understand the work of Heifets and Osipov [8]. We consider this example in detail in a separate publication [9].

We also study charged fields. We introduce fields Φ_{\pm} that differ from the usual ones in that charge conjugation $\Phi_{\pm} \rightarrow \Phi_{\mp}$ is given by a unitary transformation *different* from complex conjugation. We apply this framework to study the thermal equilibrium states of a charged field with a chemical potential at positive temperature in the forthcoming article [9]. Such states and fields occur in the study of the statistical mechanics of Bose-Einstein condensation; see [1].

The expectation values of our classical fields are defined on a “Euclidean Hilbert space.” In this paper we consider Euclidean Fock space, which is the simplest case. The classical fields are represented as unbounded operators and finiteness of their expectation values follows once certain operator domain questions have been resolved. In case the usual description in terms of functional integrals is available, the domain questions in our approach *can be* resolved, and the two descriptions are equivalent. Our method is similar to the construction of Euclidean fields given by Osterwalder and Schrader [17]. We require RP for the no-particle expectation.

We begin in §II.1 by studying what transformation properties of the field Φ under reflections are equivalent to RP. In §IV we briefly describe the associated quantization in the Gaussian case. We generalize this for charged fields in §V. In §VI we show that RP on \mathbb{R}^d gives RP on spacetimes \mathbf{X} that are compactified in one or more coordinate directions.

Certain non-Gaussian expectations can then be built as perturbations of the Gaussian case. These examples can be studied using a cut-off and a generalised Feynman-Kac formula [9]. We rely on RP to obtain a robust inner product that one can use to prove useful estimates. This inner product also provides the relation between the Euclidean Fock space \mathcal{E} and the Hilbert space \mathcal{H} of quantum theory.

I.4. The Problem with Measures. One might expect that the complex-valued Schwinger functions are moments of a complex measure. But even in the Gaussian examples we consider, such a countably-additive measure does not exist. Without measure theory one loses the possibility to use L_p estimates to study convergence of integrals, and thus one loses quantitative control of the theory.

The standard formulation of a real classical field $\Phi \in \mathcal{S}'_{\text{real}}(\mathbb{R}^d)$, or a complex field $\Phi \in \mathcal{S}'(\mathbb{R}^d)$, is to take it to be a random variable for a probability measure $d\mu(\Phi)$ on the space of tempered distributions $\mathcal{S}'_{\text{real}}(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$. In the real case the characteristic function of the measure $d\mu$ is given by

$$S(f) = \int e^{i\Phi(f)} d\mu(\Phi) . \quad (\text{I.1})$$

The exact criterion for the existence of a countably-additive probability measure $d\mu(\Phi)$ on a nuclear space (such as $\mathcal{S}_{\text{real}}(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$) is the following:

Proposition I.1 (Minlos' Theorem [11]). *A functional $S(f)$ on a nuclear space \mathcal{S} is the characteristic functional of a countably-additive, probability measure on the dual space \mathcal{S}' , if and only if $S(f)$ is continuous, of positive type, and normalized by $S(0) = 1$.*

Complex measures are more delicate mathematically. Borchers and Yngvason [3] studied possibilities for measures being associated with arbitrary Wightman field theories. In the Gaussian case, there is a clean result for the existence of a complex Gaussian measures. Suppose $d\mu$ is a complex-valued Gaussian with mean zero and covariance D , and the resulting characteristic function is given by

$$S_D(f) = e^{-\frac{1}{2} \langle \bar{f}, Df \rangle_{L_2}} . \quad (\text{I.2})$$

Then one has the existence criterion of Proposition 4.4 in [19]:

Proposition I.2 (Yngvason's Criterion). *Let $0 < K = K^*$ and $L = L^*$ be bounded transformations on $L_2(\mathbb{R}^d)$ and continuous transformations on $\mathcal{S}(\mathbb{R}^d)$. Then there exists a countably-additive, complex-valued, Gaussian measure $d\mu_D(\Phi)$ on $\mathcal{S}'_{\text{real}}(\mathbb{R}^d)$ with covariance $D = K + iL$ and with characteristic function $S_D(f)$ if and only if $K^{-1/2}LK^{-1/2}$ is Hilbert-Schmidt on $L_2(\mathbb{R}^d)$.*

This criterion is linked to the desire to write $d\mu_D$ as a phase times a probability measure $d\mu_G$ normalized by the constant \mathfrak{Z} , namely

$$d\mu_D(\Phi) = \frac{1}{\mathfrak{Z}} e^{i\frac{1}{2} \langle \Phi, Y\Phi \rangle} d\mu_G(\Phi) , \text{ so } |d\mu_D(\Phi)| \leq \frac{1}{|\mathfrak{Z}|} d\mu_G(\Phi) . \quad (\text{I.3})$$

To obtain insight into Yngvason's Criterion, let us assume for simplicity that K and L commute, and that the spectrum of $K^{-1/2}LK^{-1/2} = LK^{-1}$ is even. Then one finds

$$D^{-1} = |D|^{-2}K - i|D|^{-2}L , \quad G^{-1} = |D|^{-2}K , \quad Y = |D|^{-2}L$$

and

$$\mathfrak{Z} = (\det(I - iLK^{-1}))^{1/2} .$$

Denote the positive eigenvalues of LK^{-1} by λ_j . Then

$$\mathfrak{Z} = \left(\prod_j (1 + i\lambda_j)(1 - i\lambda_j) \right)^{1/2} = \left(\prod_j (1 + \lambda_j^2) \right)^{1/2} \geq 1 . \quad (\text{I.4})$$

In (I.4) the product defining \mathfrak{Z} converges if and only if LK^{-1} is Hilbert-Schmidt, which is Yngvason's Criterion.

In the examples we consider in [9], Yngvason's Criterion does *not* apply. In these examples not only is the spectrum of LK^{-1} continuous (and hence not Hilbert-Schmidt), but also an infrared cutoff would

yield eigenvalues λ_j that would not converge to zero. In this case, as well as in the continuous case without a cutoff, \mathfrak{Z} is infinite.²

II. CLASSICAL FIELDS AS OPERATORS ON HILBERT SPACE

One can define a neutral, random field by introducing the Fock-Hilbert space $\mathcal{E} = \mathcal{E}(\mathcal{K})$ over a one-particle space \mathcal{K} . This exponential Hilbert space has the form

$$\mathcal{E} = \bigoplus_{n=0}^{\infty} \mathcal{E}_n, \quad \text{where } \mathcal{E}_0 = \mathbb{C}, \quad \text{and } \mathcal{E}_n = \underbrace{\mathcal{K} \otimes_s \cdots \otimes_s \mathcal{K}}_{n \text{ factors}}, \quad (\text{II.1})$$

and where \otimes_s denotes the symmetric tensor product. Let the distinguished vector $\Omega_0^{\mathbb{E}} = 1 \in \mathcal{E}_0$ denote the zero-particle state.

In the following, we take $\mathcal{K} = L_2(\mathbf{X})$, but one could just as well take $\mathcal{K} = \bigoplus_{j=1}^N L_2(\mathbf{X})$ for an N -component field. Either \mathbf{X} denotes Euclidean spacetime \mathbb{R}^d , a toroidal spacetime $\mathbb{T}^d = S^1 \times \cdots \times S^1$, or more generally an intermediate case in which spacetime has the form $\mathbf{X} = X_1 \times \cdots \times X_d$, where each factor X_j either equals \mathbb{R} or a circle S^1 of length ℓ_j . We let $\mathcal{S}(\mathbf{X})$ denote the C^∞ functions on \mathbf{X} with the topology given by the usual family of seminorms $\|f\|_{r,s} = \sup_{x \in \mathbf{X}} |x^r D^s f(x)|$. Then $\Phi \in \mathcal{S}'(\mathbf{X})$, the dual space of continuous linear functionals on $\mathcal{S}(\mathbf{X})$, and it pairs linearly with test functions $f \in \mathcal{S}(\mathbf{X})$ yielding $\Phi(f) = \int \Phi(x) f(x) dx$.

We assume the neutral random field Φ is an operator-valued distribution on the Hilbert space \mathcal{E} with each $\Phi(f)$ for $f \in \mathcal{S}(\mathbf{X})$ defined on a common, dense, invariant domain $\mathcal{D} \subset \mathcal{E}$. Note that $\Phi(f)^* = \Phi^*(\bar{f})$ on the domain \mathcal{D} . This domain includes $\Omega_0^{\mathbb{E}}$ and is invariant under the action of the field, namely $\Phi(f)\mathcal{D} \subset \mathcal{D}$. The characteristic functional of a neutral field is defined as the exponential

$$S(f) = \langle \Omega_0^{\mathbb{E}}, e^{i\Phi(f)} \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Omega_0^{\mathbb{E}}, \Phi(f)^n \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}, \quad (\text{II.2})$$

and we assume this series converges for all $f \in \mathcal{S}(\mathbf{X})$. In the Gaussian case with mean zero, $\langle \Omega_0^{\mathbb{E}}, \Phi(f)^{2n} \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}} = (2n-1)!! \langle \Omega_0^{\mathbb{E}}, \Phi(f)^2 \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}^n$ demonstrating convergence explicitly. We reinterpret this for charged fields in §V.

For Gaussian fields with bounded two point function D , the exponential series for the vector $e^{i\Phi(f)} \Omega_0^{\mathbb{E}}$ converges strongly, and the characteristic function (I.1) is well-defined by its exponential series. In the case of real, neutral fields, non-Gaussian expectations, associated with an action \mathfrak{A} , approximated by cut-off actions \mathfrak{A}_n , have been constructed. One replaces the expectation $\langle \Omega_0^{\mathbb{E}}, \cdot \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}$ with a sequence of

²Related issues arise in Brydges and Imbrie's study of random walks; see Equation (7.2) of [4]. We thank John Imbrie for bringing this to our attention.

normalized expectations of the form

$$\omega_n(\cdot) = \frac{\langle \Omega_0^{\mathbb{E}}, \cdot e^{-\mathfrak{A}_n} \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}}{\langle \Omega_0^{\mathbb{E}}, e^{-\mathfrak{A}_n} \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}},$$

yield another translation invariant expectation in the limit $n \rightarrow \infty$. Some corresponding Euclidean Hilbert spaces have been given by *constructive quantum field theory*; see [6].

II.1. Neutral, Classical Fields: the Case $\mathbf{X} = \mathbb{R}^d$. In this section, we consider fields on Euclidean spacetime, $\mathbf{X} = \mathbb{R}^d$, with the one-particle space $\mathcal{K} = L_2(\mathbb{R}^d)$. The *annihilation operator* (which is actually a densely-defined bilinear form on $\mathcal{E} \times \mathcal{E}$) has non-vanishing matrix elements from \mathcal{E}_n to \mathcal{E}_{n-1} . In the Fourier representation it acts as

$$(A(k)f)_{n-1}(k_1, \dots, k_{n-1}) = \sqrt{n} f_n(k, k_1, \dots, k_{n-1}).$$

Then $[A(k), A(k')] = 0$. The adjoint creation form $A(k)^*$ satisfies the usual canonical relations, $[A(k), A(k')^*] = \delta(k - k')$. Define the complex coordinates

$$\tilde{Q}(k) = A(k)^* + A(-k).$$

These coordinates mutually commute, $[\tilde{Q}(k), \tilde{Q}(k')] = 0$, and also

$$\tilde{Q}^*(k) = \tilde{Q}(-k), \quad \text{and} \quad \left\langle \Omega_0^{\mathbb{E}}, \tilde{Q}(k) \tilde{Q}(k') \Omega_0^{\mathbb{E}} \right\rangle_{\mathcal{E}} = \delta(k + k'). \quad (\text{II.3})$$

The Gaussian coordinate field $Q(x)$ is the Fourier transform of $\tilde{Q}(k)$, namely

$$Q(x) = (2\pi)^{-d/2} \int \tilde{Q}(k) e^{ik \cdot x} dk. \quad (\text{II.4})$$

Take the general Gaussian, neutral, scalar, classical field $\Phi(x)$ to be a linear function of $Q(x)$. Assume that for some given function $\tilde{\sigma}(k)$,

$$\Phi(x) = (2\pi)^{-d/2} \int \tilde{Q}(k) \tilde{\sigma}(k) e^{ik \cdot x} dk. \quad (\text{II.5})$$

Whatever the choice of $\tilde{\sigma}(k)$, the fields $\Phi(x)$ and their adjoints mutually commute,

$$[\Phi(x), \Phi(x')] = [\Phi(x), \Phi^*(x')] = 0. \quad (\text{II.6})$$

The field $\Phi(x) = \Phi^*(x)$ is hermitian in case that $\tilde{\sigma}(k) = \overline{\tilde{\sigma}(-k)}$, or, equivalently, when $\sigma = \bar{\sigma}$ is real³. In the standard free, Euclidean-field example, one takes $\tilde{\sigma}(k) = (k^2 + m^2)^{-1/2}$.

³For a transformation S on L_2 with integral kernel $S(x; x')$, the integral kernel of the transpose S^{T} is $S(x'; x)$, the kernel of the complex-conjugate \bar{S} is $\overline{S(x; x')}$, and the kernel of the hermitian-adjoint S^* is $\overline{S(x'; x)}$. The operator S is defined to be *symmetric* if $S = S^{\text{T}}$, is defined to be *real* if $S = \bar{S}$, and is defined to be *hermitian* if $S = S^*$. The kernel of a translation-invariant operator has the form $S(x; x') = S(x - x')$. In Fourier space:

$$\widetilde{S^{\text{T}}}(k) = \widetilde{S}(-k), \quad \widetilde{\bar{S}}(k) = \overline{\widetilde{S}(-k)}, \quad \text{and} \quad \widetilde{S^*}(k) = \overline{\widetilde{S}(k)}.$$

In configuration space, the definition (II.5) amounts to the relation

$$\Phi(x) = (2\pi)^{-d/2} (\sigma Q)(x) . \quad (\text{II.7})$$

Here, one defines the convolution operator σ by

$$(\sigma f)(x) = (2\pi)^{-d/2} \int \sigma(x - x') f(x') dx'$$

where

$$\sigma(x) = (2\pi)^{-d/2} \int \tilde{\sigma}(k) e^{ik \cdot x} dk .$$

The expectation of two fields defines an operator $D: \mathcal{K} \mapsto \mathcal{K}$ with integral kernel

$$D(x, x') = \langle \Omega_0^{\mathbb{E}}, \Phi(x) \Phi(x') \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}} .$$

One can introduce commuting, canonically-conjugate coordinates (which generally do not enter the functional integrals), namely

$$\tilde{P}(k) = \frac{i}{2} (A(k)^* - A(-k)) = \tilde{P}(-k)^* ,$$

so

$$[\tilde{P}(k), \tilde{Q}(k')] = -i\delta(k + k') .$$

In case that σ^{T} is invertible, the conjugate field is

$$\Pi(x) = (2\pi)^{d/2} (\sigma^{\text{T}})^{-1} P(x) = (2\pi)^{-d/2} \int \tilde{\sigma}(-k)^{-1} \tilde{P}(k) e^{ik \cdot x} dk .$$

With these conventions, $[\Pi(x), \Phi(x')] = -i\delta(x - x')$.

II.2. Schwinger Functions. The zero-particle expectations

$$S_n(x_1, \dots, x_n) = \langle \Omega_0^{\mathbb{E}}, \Phi(x_1) \cdots \Phi(x_n) \Omega_0^{\mathbb{E}} \rangle_{\mathcal{E}}$$

satisfy a Gaussian recursion relation,

$$S_n(x_1, \dots, x_n) = \sum_{j=2}^n S_2(x_1, x_j) S_{n-2}(x_2, \dots, \not{x}_j, \dots, x_n) , \quad (\text{II.8})$$

where \not{x}_j denotes the omission of x_j and

$$S_2(x, x') = D(x - x') = (2\pi)^{-d} \int \tilde{\sigma}(k) \tilde{\sigma}(-k) e^{ik \cdot (x - x')} dk . \quad (\text{II.9})$$

Commutativity of the fields assures that $D(x)$ is an even function, $D(x - x') = D(x' - x)$. So

$$D = \sigma \sigma^{\text{T}} = \sigma^{\text{T}} \sigma = D^{\text{T}} , \quad \text{and} \quad (2\pi)^{d/2} \tilde{D}(k) = \tilde{\sigma}(k) \tilde{\sigma}(-k) . \quad (\text{II.10})$$

Given $\tilde{D}(k)$, the general solution for $\tilde{\sigma}(k)$ is

$$\tilde{\sigma}(k) = (2\pi)^{d/4} \tilde{D}(k)^{1/2} e^{h(k)} ,$$

with $h(k)$ an odd function $h(k) = -h(-k)$. Here, we take the straightforward choice $h(k) = 0$, so

$$\tilde{\sigma}(k) = (2\pi)^{d/4} \tilde{D}(k)^{1/2} = \tilde{\sigma}(-k) ,$$

and thus (II.4) equals

$$\Phi(x) = (2\pi)^{-d/4} \int \tilde{Q}(k) \tilde{D}(k)^{1/2} e^{ik \cdot x} dk . \quad (\text{II.11})$$

Note that the recursion relation (II.8) ensures that

$$\langle \Omega_0^E, \Phi(f)^{2n} \Omega_0^E \rangle_{\mathcal{E}} = (2n-1)!! \langle \bar{f}, Df \rangle_{L_2}^n .$$

We want to ensure that multiplication by $\tilde{D}(k)^{1/2}$ defines a continuous transformation of Schwartz space $\mathcal{S}(X)$ into itself. This requires choosing an appropriate square root. Consider the case $\tilde{D}(k) = \tilde{K}(k) + i\tilde{L}(k)$, with $0 < \tilde{K}$, and \tilde{L} real. One can use the positive square root of $\tilde{K}(k)$ to write

$$(2\pi)^{-d/4} \tilde{\sigma}(k) = \tilde{D}(k)^{1/2} = \tilde{K}(k)^{1/2} \left(1 + i\tilde{L}(k)\tilde{K}(k)^{-1} \right)^{1/2} .$$

Here, one chooses the square root $\tilde{D}(k)^{1/2}$ so that $\tilde{K}(k)^{1/2}$ is positive, and the real part of the term $(1 + i\tilde{L}(k)\tilde{K}(k)^{-1})$ lies in the complex half-plane with real part greater than 1. Since we assume that multiplication by $\tilde{D}(k)$ provides a continuous transformation of $\mathcal{S}(\mathbb{R}^d)$ into itself, it is a smooth function all of whose derivatives are polynomially bounded. Since the square root we choose is unambiguous and non-vanishing, the function $\tilde{D}(k)^{1/2}$ also has these properties.

II.3. Neutral Fields Φ as Operators. Define the field $\Phi(f)$, $f \in C_0^\infty$, as an operator on \mathcal{E} with the domain \mathcal{D} consisting of vectors with a finite number of particles (namely vectors in $\bigoplus_{j=0}^n \mathcal{E}_j$) and with C_0^∞ wave functions in each \mathcal{E}_j . Note that with our choice of σ , the field is hermitian as a sesquilinear form, $\Phi(x) = \Phi^*(x)$, if and only if $\sigma = \bar{\sigma}$ is real. This is the case if and only if the operator $D = \bar{D}$ is real. In Fourier space, taking the symmetry of σ into account, this is equivalent to $\overline{\tilde{\sigma}(k)} = \tilde{\sigma}(k)$, or $\overline{\tilde{D}(k)} = \tilde{D}(k)$. In any case, we assume that σ (or D) is a bounded transformation on $L_2(\mathbb{R}^d)$,

$$\|D\| = \||D|\| = \|D^*D\|^{1/2} = \|\sigma\|^2 < \infty .$$

In fact, it is convenient to assume a stronger bound to ensure good properties for the time-zero fields. Let us decompose $D = K + iL$ into its real and imaginary parts

$$K = \frac{1}{2}(D + D^*) , \quad \text{and} \quad L = \frac{i}{2}(-D + D^*) .$$

We assume that there are strictly positive constants M_1, M_2, M_3 such that for $C = (-\Delta + m^2)^{-1}$,

$$M_1 C \leq K \leq M_2 C , \quad \text{and} \quad \pm K^{-1/2} L K^{-1/2} \leq M_3 . \quad (\text{II.12})$$

Now decompose the field into its real and imaginary parts,

$$\Phi(f) = c(f) + id(f) ,$$

where $c(f) = \frac{1}{2}(\Phi(f) + \Phi^*(f))$ and $d(f) = -\frac{i}{2}(\Phi(f) - \Phi^*(f))$. Let $\mathfrak{H}_{-\frac{1}{2}}$ denote the Hilbert space of functions on \mathbb{R}^{d-1} with inner product

$$\langle f, g \rangle_{\mathfrak{H}_{-\frac{1}{2}}} = \langle f, (2\mu)^{-1}g \rangle_{L_2(\mathbb{R}^{d-1})}, \quad (\text{II.13})$$

where $\mu = (-\nabla^2 + m^2)^{1/2}$.

Proposition II.1. *Assume the bounds (II.12). Then for $f, g \in L_2$ the closure of $\Phi(f)$ is normal and commutes with the closure of $\Phi(g)$. The fields $c(f)$ and $d(f)$ are essentially self-adjoint on \mathcal{D} , and their closures commute. These results extend to $f = \delta \otimes h$ for $h \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$.*

Proof. The operator $\sigma = (K + iL)^{1/2}$ acting on $L_2(\mathbb{R}^d)$ is invertible as long as K is invertible. The latter follows from the first bound in the assumption (II.12). As a consequence, each vector Ψ in the dense subset $\mathcal{D} \subset \mathcal{E}$ can be represented as $\Psi = F_\Psi \Omega_0^E$, where F_Ψ is a polynomial in fields. The degree of this polynomial equals the maximum number of particles in the vector. Thus, for $f \in C_0^\infty$,

$$\begin{aligned} \|\Phi(f)^n \Psi\|_{\mathcal{E}} &= \langle \Phi(f)^{*n} \Phi(f)^n \Omega_0^E, F_\Psi^* F_\Psi \Omega_0^E \rangle_{\mathcal{E}}^{1/2} \\ &\leq \|\Phi(f)^{*n} \Phi(f)^n \Omega_0^E\|_{\mathcal{E}}^{1/2} \|F_\Psi^* F_\Psi \Omega_0^E\|_{\mathcal{E}}^{1/2} \\ &\leq (4n-1)!!^{1/4} \| |D|^{1/2} |f| \|^n \|F_\Psi^* F_\Psi \Omega_0^E\|_{\mathcal{E}}^{1/2} \\ &\leq M 2^n n!^{1/2} \|D\|_{L_2}^{n/2} \|f\|_{L_2}^n \|F_\Psi^* F_\Psi \Omega_0^E\|_{\mathcal{E}}^{1/2}. \end{aligned} \quad (\text{II.14})$$

Here, M is a constant. The same bound holds for $\|\Phi(f)^{*n} \Psi\|_{\mathcal{E}}$, and as $\Phi(f)$ commutes with $\Phi^*(f)$ it shows that Ψ is an analytic vector for $c(f)$ and for $d(f)$. One can extend this bound to all $f \in L_2(\mathbb{R}^d)$ by limits, so $\Phi(f)$ is also defined in this case. The essential self-adjointness follows from Nelson's Lemma 5.1 of [14]. The commutativity of the closures of $c(f)$ and $d(g)$ follows from a similar estimate for $c(f)^n d(g)^{n'} \Psi$, showing that the sums

$$\sum_{n, n'=1}^N \frac{i^{n+n'}}{n!n'} c(f)^n d(g)^{n'} \Psi = \sum_{n, n'=1}^N \frac{i^{n+n'}}{n!n'} d(g)^{n'} c(f)^n \Psi$$

converge to $e^{ic(f)} e^{id(g)} \Psi = e^{id(g)} e^{ic(f)} \Psi$ as $N \rightarrow \infty$. The vectors $\Psi \in \mathcal{D}$ are dense, so the exponentials commute as unitaries. The same is true with f and g interchanged.

In general, use (II.12) to show that

$$\begin{aligned} |D| &= (K^2 + L^2)^{1/2} = K^{1/2} (I + K^{-1} L^2 K^{-1})^{1/2} K^{1/2} \\ &\leq (1 + M_3^2)^{1/2} K \leq (1 + M_3^2)^{1/2} M_2 C. \end{aligned}$$

This ensures

$$\|\Phi(f) \Omega_0^E\|_{\mathcal{E}}^2 = \langle f, |D| f \rangle_{L_2} \leq (1 + M_3^2)^{1/2} M_2 \langle f, C f \rangle_{L_2}.$$

We use this to restrict to time zero. If $f = \delta \otimes h$, then

$$\begin{aligned} \langle f, |D|f \rangle_{L_2} &\leq (1 + M_3^2)^{1/2} M_2 \langle f, Cf \rangle_{L_2} \\ &= (1 + M_3^2)^{1/2} M_2 \langle h, h \rangle_{\mathfrak{H}_{-1/2}} . \end{aligned}$$

The argument then proceeds for the time-zero fields in the same manner as for the spacetime averaged field. \square

III. TIME REFLECTION

Let ϑ denote time reflection on \mathbb{R}^d , namely $\vartheta : (t, \vec{x}) \mapsto (-t, \vec{x})$, and also let ϑ denote the implementation of time reflection as a real, self-adjoint, unitary transformation on $L_2(\mathbb{R}^d)$. Let $\Theta = \Theta^* = \Theta^{-1}$ denote the push-forward of ϑ to a corresponding time-reflection unitary on \mathcal{E} . The operator Θ acts on each tensor product \mathcal{E}_n as the n -fold tensor product of ϑ , and $\Theta \Omega_0^E = \Omega_0^E$.

Proposition III.1. *Time-reflection transforms the neutral field $\Phi(x)$ defined by (II.11) according to*

$$\Theta \Phi(x) \Theta = \Phi^*(\vartheta x) \quad \text{if and only if} \quad \vartheta \sigma^T \vartheta = \sigma^* . \quad (\text{III.1})$$

In case that (III.1) holds, then both ϑD and $D\vartheta$ are self-adjoint operators on $L_2(\mathbb{R}^d)$. In fact,

$$\vartheta D = \sigma^* \vartheta \sigma \quad \text{and} \quad D\vartheta = \sigma \vartheta \sigma^* .$$

Proof. Since $\vartheta(E, \vec{k}) = (-E, \vec{k})$, the unitary ϑ acts as $(\vartheta f)(k) = f(\vartheta k)$ on \mathcal{E}_1 . Thus, Θ has the property $\Theta A(k) \Theta = A(\vartheta k)$, and as a consequence $\Theta \tilde{Q}(k) \Theta = \tilde{Q}(\vartheta k)$. Using (II.5), one has

$$\begin{aligned} \Theta \Phi(x) \Theta &= (2\pi)^{-d/2} \int \Theta \tilde{Q}(k) \Theta \tilde{\sigma}(k) e^{ik \cdot x} dk \\ &= (2\pi)^{-d/2} \int \tilde{Q}(\vartheta k) \tilde{\sigma}(k) e^{ik \cdot x} dk \\ &= (2\pi)^{-d/2} \int \tilde{Q}(k) \tilde{\sigma}(\vartheta k) e^{ik \cdot (\vartheta x)} dk \\ &= \left((2\pi)^{-d/2} \int \tilde{Q}(-k) \overline{\tilde{\sigma}(\vartheta k)} e^{-ik \cdot (\vartheta x)} dk \right)^* \\ &= \left((2\pi)^{-d/2} \int \tilde{Q}(k) \overline{\tilde{\sigma}(-\vartheta k)} e^{ik \cdot (\vartheta x)} dk \right)^* . \end{aligned}$$

This equals $\Phi^*(\vartheta x)$ if and only if $\overline{\tilde{\sigma}(-\vartheta k)} = \tilde{\sigma}(k)$. The equivalent condition on σ in configuration space is $\vartheta \bar{\sigma} \vartheta = \sigma$. Since $\vartheta^2 = I$ this is also equivalent to $\vartheta \sigma^T \vartheta = \sigma^*$. As D is given by (II.10),

$$\vartheta D = (\vartheta \sigma^T \vartheta) \vartheta \sigma = \sigma^* \vartheta \sigma ,$$

as claimed, and this is clearly self-adjoint. Moreover, $D\vartheta = \sigma \sigma^T \vartheta = \sigma \vartheta \sigma^*$ is also self-adjoint. \square

III.1. Time-Reflection Positivity. Consider the positive-time half-space $\mathbb{R}_+^d = [0, \infty) \times \mathbb{R}^{d-1}$; the negative-time half-space \mathbb{R}_-^d is defined similarly. Let $L_2(\mathbb{R}_\pm^d)$ denote the subspace of $L_2(\mathbb{R}^d)$ consisting of functions supported in \mathbb{R}_\pm^d . Let $\mathcal{E}_{\pm,0} \subset \mathcal{E}$ denote the subspace of finite linear combinations $A = \sum_{j=1}^N c_j e^{i\Phi(f_j)} \Omega_0^E$, for $f_j \in \mathcal{S}(\mathbb{R}_\pm^d)$, and let \mathcal{E}_\pm denote its closure in \mathcal{E} . The Osterwalder-Schrader reflection form on $\mathcal{E}_{+,0} \times \mathcal{E}_{+,0}$ (or, alternatively, on $\mathcal{E}_{-,0} \times \mathcal{E}_{-,0}$) is

$$\langle A, B \rangle_{\mathcal{H}} = \langle A, \Theta B \rangle_{\mathcal{E}} . \quad (\text{III.2})$$

This extends by continuity to $\mathcal{E}_\pm \times \mathcal{E}_\pm$. Thus, the left hand side extends to a pre-inner product on equivalence classes $[A] = A + n \in \mathcal{E}_\pm$. Here, $n \in \mathcal{E}_\pm$ is an element of the null space of the form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For simplicity denote the equivalence class $[A]$ by A . The space \mathcal{H} is the completion of the equivalence classes in this inner product. Let M denote the $N \times N$ matrix with entries

$$M_{jj'} = S(f_{j'} - \vartheta \overline{f_j}) , \quad \text{where } f_j \in \mathcal{E} \quad \text{for } j = 1, \dots, N . \quad (\text{III.3})$$

Definition III.2 (RP). Various possible formulations of RP (Osterwalder-Schrader Positivity) with respect to time reflection are:

- (i) One has $0 \leq \Theta$ on either \mathcal{E}_+ or \mathcal{E}_- . If both hold, then Θ is doubly RP.
- (ii) The functional $S(f)$ is RP with respect to ϑ , if for every choice of $N \in \mathbb{N}$ and for all functions $f_j \in \mathcal{S}(\mathbb{R}_+^d)$ the matrix M in (III.3) is positive definite. The functional $S(f)$ is also RP, if M is positive for $f_j \in \mathcal{S}(\mathbb{R}_-^d)$. If both conditions hold, then $S(f)$ is *doubly-reflection positive*.
- (iii) A symmetric operator $D = D^T$ on $L_2(\mathbb{R}^d)$ is RP with respect to ϑ , if $0 \leq \vartheta D$ on $L_2(\mathbb{R}_+^d)$. It is also RP if $0 \leq \vartheta D$ on $L_2(\mathbb{R}_-^d)$. The latter is equivalent to $0 \leq D\vartheta$ on $L_2(\mathbb{R}_+^d)$. The operator D is doubly RP if both conditions hold.

Proposition III.3. *Let Φ be a field on \mathcal{E} defined by (II.11), and assume (III.1). Then*

- (i) *The characteristic functional $S(f)$ defined in (II.2) satisfies $S(f) = \overline{S(-\vartheta \overline{f})}$.*
- (ii) *The matrix $M = M^*$ defined in (III.3) is hermitian.*
- (iii) *Statements III.2.(i) and III.2.(ii) of Definition III.2 are equivalent.*
- (iv) *All three Statements III.2.(i), III.2.(ii), and III.2.(iii) of Definition III.2 are equivalent in case that*

$$S(f) = e^{-\frac{1}{2} \langle \overline{f}, Df \rangle_{L_2}} , \quad \text{with } D = \sigma \sigma^T = D^T . \quad (\text{III.4})$$

Proof. The equivalence of Statements III.2.(i) and III.2.(ii) follows from the identity

$$\langle A, A \rangle_{\text{OS}} = \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \vartheta \bar{f}_j) = \sum_{j,j'=1}^N \bar{c}_j c_{j'} M_{jj'} , \quad (\text{III.5})$$

which is a consequence of

$$\begin{aligned} \langle A, A \rangle_{\text{OS}} &= \sum_{j,j'=1}^N \bar{c}_j c_{j'} \langle e^{i\Phi(f_j)} \Omega_0^{\text{E}}, e^{i\Phi(f_{j'})} \Omega_0^{\text{E}} \rangle_{\mathcal{E}} \\ &= \sum_{j,j'=1}^N \bar{c}_j c_{j'} \left\langle \Omega_0^{\text{E}}, e^{-i\Phi^*(\bar{f}_j)} \Theta e^{i\Phi(f_{j'})} \Omega_0^{\text{E}} \right\rangle_{\mathcal{E}} . \end{aligned}$$

As $\Theta \Omega_0^{\text{E}} = \Omega_0^{\text{E}}$, one can substitute $\Theta e^{-i\Phi^*(\bar{f}_j)} \Theta = e^{-i\Phi(\vartheta \bar{f}_j)}$ to give (III.5).

To verify that $S(f) = \overline{S(-\vartheta \bar{f})}$ is valid, compute

$$\begin{aligned} \overline{S(-\vartheta \bar{f})} &= \overline{\langle \Omega_0^{\text{E}}, e^{-i\Phi(\vartheta \bar{f})} \Omega_0^{\text{E}} \rangle_{\mathcal{E}}} = \left\langle e^{-i\Phi(\vartheta \bar{f})} \Omega_0^{\text{E}}, \Omega_0^{\text{E}} \right\rangle_{\mathcal{E}} \\ &= \left\langle \Omega_0^{\text{E}}, e^{i\Phi^*(\vartheta f)} \Omega_0^{\text{E}} \right\rangle_{\mathcal{E}} = \left\langle \Omega_0^{\text{E}}, \Theta e^{i\Phi^*(\vartheta f)} \Theta \Omega_0^{\text{E}} \right\rangle_{\mathcal{E}} \\ &= \left\langle \Omega_0^{\text{E}}, e^{i\Phi(f)} \Omega_0^{\text{E}} \right\rangle_{\mathcal{E}} = S(f) , \end{aligned}$$

Using this relationship one also sees that the matrix M is hermitian, for

$$M_{jj'} = S(f_{j'} - \vartheta \bar{f}_j) = \overline{S(-\vartheta \bar{f}_{j'} + f_j)} = \overline{M_{j'j}} .$$

Finally consider the Gaussian characteristic function (III.4). Now we show the equivalence of Statements III.2.(ii) and III.2.(iii). Take $A \in \mathcal{E}_{+,0}$ and choose $N = 2$, $f_1 = -i\lambda f$, and $f_2 = 0$. Then for

$$c_1 = \lambda^{-1} = -c_2 ,$$

one can take the limit $\lambda \rightarrow 0$ of $0 \leq \langle A, A \rangle_{\mathcal{H}}$. This is just

$$\begin{aligned} 0 &\leq \langle \Phi(f) \Omega_0^{\text{E}}, \Theta \Phi(f) \Omega_0^{\text{E}} \rangle_{\mathcal{E}} = \langle \Omega_0^{\text{E}}, \Phi^*(f) \Theta \Phi(f) \Omega_0^{\text{E}} \rangle_{\mathcal{E}} \\ &= \langle \Omega_0^{\text{E}}, \Phi(\vartheta \bar{f}) \Phi(f) \Omega_0^{\text{E}} \rangle_{\mathcal{E}} = \langle f, \vartheta Df \rangle_{L_2} . \end{aligned}$$

Hence, Statement III.2.(ii) ensures $0 \leq \vartheta D$ on $L_2(\mathbb{R}_+^d)$.

The converse is also true. To see this, use

$$\sum_{j,j'=1}^N \bar{c}_j c_{j'} M_{jj'} = \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \vartheta \bar{f}_j) = \sum_{j,j'=1}^N \bar{d}_j d_{j'} e^{\langle f_j, \vartheta Df_{j'} \rangle_{L_2}} ,$$

where

$$d_j = c_j e^{-\frac{1}{2} \langle \bar{f}_j, Df_j \rangle_{L_2}} .$$

In fact,

$$\begin{aligned}
\sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \overline{\vartheta f_j}) &= \sum_{j,j'=1}^N \bar{c}_j c_{j'} e^{-\frac{1}{2} \langle \overline{f_{j'} - \vartheta f_j}, D(f_{j'} - \vartheta f_j) \rangle_{L_2}} \\
&= \sum_{j,j'=1}^N \left(\bar{c}_j e^{-\frac{1}{2} \langle \vartheta f_j, D \vartheta f_j \rangle_{L_2}} \right) \left(c_{j'} e^{-\frac{1}{2} \langle \overline{f_{j'}}, D f_{j'} \rangle_{L_2}} \right) \times \\
&\quad \times e^{\frac{1}{2} \langle f_j, \vartheta D f_{j'} \rangle_{L_2} + \frac{1}{2} \langle \overline{f_{j'}}, D \vartheta f_j \rangle_{L_2}} \\
&= \sum_{j,j'=1}^N \bar{d}_j d_{j'} e^{\frac{1}{2} \langle f_j, \vartheta (D + D^T) f_{j'} \rangle_{L_2}} \\
&= \sum_{j,j'=1}^N \bar{d}_j d_{j'} e^{\langle f_j, \vartheta D f_{j'} \rangle_{L_2}} . \tag{III.6}
\end{aligned}$$

The second to last equality follows from using $\vartheta D^* \vartheta = D$ in

$$\langle \vartheta f_j, D \vartheta f_j \rangle_{L_2} = \overline{\langle D \vartheta f_j, \vartheta f_j \rangle_{L_2}} = \overline{\langle \vartheta f_j, D^* \vartheta f_j \rangle_{L_2}} = \overline{\langle \overline{f_j}, D f_j \rangle_{L_2}} ,$$

and from

$$\begin{aligned}
\langle \overline{f_{j'}}, D \vartheta f_j \rangle_{L_2} &= \overline{\langle D \vartheta f_j, f_{j'} \rangle_{L_2}} = \overline{\langle \vartheta f_j, D^* f_{j'} \rangle_{L_2}} \\
&= \langle \vartheta f_j, D^T f_{j'} \rangle_{L_2} = \langle f_j, \vartheta D^T f_{j'} \rangle_{L_2} .
\end{aligned}$$

The last inequality in (III.6) then is a consequence of the symmetry of D .

Thus, Statement III.2.(ii) on \mathcal{E}_+ follows from the positivity of the matrix \mathcal{K} with entries

$$\mathcal{K}_{jj'} = e^{\langle f_j, \vartheta D f_{j'} \rangle_{L_2}} \quad \text{for } f_j \in L_2(\mathbb{R}_+^d) .$$

In fact, we now see that $I \leq \mathcal{K}$. The assumption $0 \leq \vartheta D$ on $L_2(\mathbb{R}_+^d)$ means that the matrix \mathbf{k} with entries $\mathbf{k}_{jj'} = \langle f_j, \vartheta D f_{j'} \rangle_{L_2}$ has non-negative eigenvalues for $f_j \in L_2(\mathbb{R}_+^d)$. The same is true for the matrix \mathbf{k}^{on} with entries $\mathbf{k}_{jj'}^{on} = (\mathbf{k}_{jj'})^n$. In fact, the matrix \mathbf{k}^{on} equals $\mathbf{k}^{\otimes n}$ on the diagonal, and the eigenvalues $\lambda_j(\mathcal{K})$ of the matrix $\mathcal{K} = \sum_{n=0}^{\infty} n!^{-1} \mathbf{k}^{on}$ with entries $\mathcal{K}_{jj'} = e^{\mathbf{k}_{jj'}}$ equal 1 plus an eigenvalue of a positive matrix. Consequently, $1 \leq \lambda_j(\mathcal{K})$, as claimed.

A function $f \in L_2(\mathbb{R}_+^d)$, if and only if $\vartheta f \in L_2(\mathbb{R}_+^d)$. Thus, Statement III.2 on \mathcal{E}_- is equivalent to $0 \leq \vartheta^2 D \vartheta = D \vartheta$ on $L_2(\mathbb{R}_+^d)$. \square

IV. QUANTIZATION

If the time reflection Θ is reflection positive (or doubly reflection positive), then the form (III.2) defines a pre-Hilbert space (or two pre-Hilbert spaces) $\mathcal{H}_{\pm,0}$ whose elements are equivalence classes

$$\widehat{A} = \{A + n\} ,$$

where $A, n \in \mathcal{E}_+$ or $A, n \in \mathcal{E}_-$ and where n is an element of the null space of the form (III.2). In either case, the inner product on $\mathcal{H}_{\pm,0}$ is given by the form (III.2). Let \mathcal{H}_{\pm} denote the completions of the pre-Hilbert spaces $\mathcal{H}_{\pm,0}$.

In case the characteristic functional $S(f)$ is both time-translation invariant and reflection positive, one obtains a positive Hamiltonian operator from the RP inner product. This follows from the standard construction provided in [15], so we refer to this operator as the *OS-Hamiltonian*. Let the time translation group $T(t)$ act on functions as

$$(T(s)f)(t, \vec{x}) = f(t - s, \vec{x}) .$$

If $S(T(s)f) = S(f)$ for all s , then $S(f)$ is time-translation invariant. In this case, $T(t)$ acts as a unitary transformation on \mathcal{E} , and

$$T(t): \mathcal{E}_{\pm} \rightarrow \mathcal{E}_{\pm} \text{ for } 0 \leq \pm t .$$

Proposition IV.1 (OS-Hamiltonian). *Let $\hat{\cdot}$ denote the canonical projection from \mathcal{E}_{\pm} to \mathcal{H}_{\pm} resulting from the reflection-positive OS form. Then the maps*

$$R(t)_{\pm} = \widehat{T(t)} \quad \text{acting on } \mathcal{H}_{\pm}$$

are contraction semigroups with infinitesimal generators H_{\pm} , namely $R_{\pm}(t) = e^{-tH_{\pm}}$ with $0 \leq H_{\pm}$.

The proof of this proposition follows from the arguments given in [15]. In the Gaussian case, let \mathbf{h}_{\pm} denote the restriction of H_{\pm} to the one-particle subspace $\mathcal{H}_{1,\pm} \subset \mathcal{H}_{\pm}$. If the functional $S(f)$ has covariance D , the reflection positivity condition on the one-particle space requires $0 \leq \vartheta D$ on $L_2(\mathbb{R}_{\pm}^d)$. For example, if $f \in L_2(\mathbb{R}_+^d)$, then

$$\langle f, \vartheta D f \rangle_{L_2} = \langle F, F \rangle_{\mathcal{H}_{1,+}} ,$$

where $F = \int_0^{\infty} e^{-th_+} f_t dt$ and $f_t(\vec{x}) = f(t, \vec{x})$. Likewise, if $f \in L_2(\mathbb{R}_-^d)$, then one has

$$\langle f, \vartheta D f \rangle_{L_2} = \langle F, F \rangle_{\mathcal{H}_{1,-}} \quad \text{for } F = \int_{-\infty}^0 e^{th_-} f_t dt .$$

IV.1. Spatial Reflection-Positivity for the Neutral Field. In a fashion similar to time-reflection, one considers spatial reflections. The spatial reflection through a plane orthogonal to a given spatial vector $\vec{n} \in \mathbb{R}^{d-1}$ is $\pi_{\vec{n}}: (t, \vec{x}) \mapsto (t, \vec{x} - 2(\vec{n} \cdot \vec{x})\vec{n})$. Let $\pi_{\vec{n}}$ denote the action that this reflection induces as a real, self-adjoint, unitary on $L_2(\mathbb{R}^d)$.

Let $\Pi_{\vec{n}}$ denote the push-forward of $\pi_{\vec{n}}$ to a real, self-adjoint unitary on \mathcal{E} . Instead of the positive-time subspace $L_2(\mathbb{R}_+^d)$ that arose in the study of time reflection in §III, use the subspace of functions $L_2(\mathbb{R}_{\vec{n}\pm}^d)$ supported on one side of the reflection hyperplane. Let $\mathcal{E}_{\vec{n}\pm} \subset \mathcal{E}$ denote the closure of the subspace spanned by the vectors

$$\{e^{i\Phi(f)}\Omega_0^E\} \quad \text{with } f \in \mathcal{S}(\mathbb{R}_{\vec{n}\pm}^d).$$

Spatial reflection $\Pi_{\vec{n}}$ leaves Ω_0^E invariant and maps $f(E, \vec{k}) \in \mathcal{E}_1$ to $f(\pi_{\vec{n}}k)$. In Fourier space

$$\pi_{\vec{n}}(E, \vec{k}) = (E, \vec{k} - 2(\vec{k} \cdot \vec{n})\vec{n})$$

and $\pi_{\vec{n}}^2 k = k$. Spatial-reflection transforms the $\tilde{Q}(k)$'s according to $\Pi_{\vec{n}} \tilde{Q}(k) \Pi_{\vec{n}} = \tilde{Q}(\pi_{\vec{n}}k)$. Following the proof of Proposition III.1, we obtain:

Proposition IV.2 (Spatial Reflection of the Field). *Let the neutral field $\Phi(x)$ be defined by (II.11) on \mathcal{E} . Then the spatial-reflection $\Pi_{\vec{n}}$ transforms Φ according to*

$$\Pi_{\vec{n}} \Phi(x) \Pi_{\vec{n}} = \Phi^*(\pi_{\vec{n}}x), \quad (\text{IV.1})$$

if and only if

$$\pi_{\vec{n}} \sigma^T \pi_{\vec{n}} = \sigma^*. \quad (\text{IV.2})$$

If (IV.2) holds, then $\pi_{\vec{n}} D = \sigma^* \pi_{\vec{n}} \sigma = D^* \pi_{\vec{n}}$,

$$\pi_{\vec{n}} D = (\pi_{\vec{n}} D)^*, \quad \text{and} \quad D \pi_{\vec{n}} = (D \pi_{\vec{n}})^*$$

are self-adjoint operators on $L_2(\mathbb{R}^d)$.

One can formulate spatial reflection positivity by substituting $\pi_{\vec{n}}$ for ϑ and $\Pi_{\vec{n}}$ for Θ , and $\mathcal{E}_{\vec{n}\pm}$ for \mathcal{E}_{\pm} in Definition III.2. Spatial reflection positivity gives a pre-inner product

$$\langle A, B \rangle_{\mathcal{H}(\vec{n})} = \langle A, \Pi_{\vec{n}} B \rangle_{\mathcal{E}} \quad (\text{IV.3})$$

on $\mathcal{E}_{\vec{n}+}$, as well as new requirements on the transformation of the field or on D . The proof of the following result is identical to the proof of Proposition III.3.

Proposition IV.3 (Spatial-Reflection Positivity). *If one replaces ϑ , Θ , \mathbb{R}_{\pm}^d , \mathcal{E}_{\pm} in the statement of Proposition III.3 by $\pi_{\vec{n}}$, $\Pi_{\vec{n}}$, $\mathbb{R}_{\vec{n}\pm}^d$, $\mathcal{E}_{\vec{n}\pm}$, respectively, then the proposition remains valid.*

V. CHARGED FIELDS

One defines charged fields $\Phi_{\pm}(x)$ to replace the usual hermitian-conjugate fields $\Phi(x)$ and $\Phi^*(x)$.⁴ These fields are linear in the complex coordinates $\tilde{Q}_{\pm}(k)$ and the kernels σ_{\pm} , whose properties we elaborate on below. Similar to (II.11), take the charged fields to be

$$\Phi_{\pm}(x) = (2\pi)^{-d/2} \int \tilde{Q}_{\pm}(k) \tilde{\sigma}_{\pm}(k) e^{ik \cdot x} dk. \quad (\text{V.1})$$

While one might regard the charged coordinates \tilde{Q}_{\pm} , or their Fourier transforms Q_{\pm} , as linear combinations of two independent sets of neutral coordinates $Q_{\pm} = \frac{1}{\sqrt{2}}(Q_1 \pm iQ_2)$, here we do not require this.

⁴Note that we use “ \pm ” here to label charges, while in earlier sections we use this notation to label positive and negative time subspaces, *etc.* We hope that this causes no confusion.

We take independent creation and annihilation operator-valued distributions $A_{\pm}^*(k)$ and $A_{\pm}(k)$, which act on a Fock-Hilbert space \mathcal{E} and satisfy the relations

$$[A_{\pm}(k), A_{\pm'}^*(k')] = \delta_{\pm\pm'} \delta(k - k'), \quad \text{and} \quad [A_{\pm}(k), A_{\pm'}(k')] = 0.$$

One can decompose the one-particle space into a direct sum

$$\mathcal{E}_1 = \mathcal{E}_1^+ \oplus \mathcal{E}_1^-.$$

Correspondingly, the Fock space \mathcal{E} decomposes into the tensor product⁵ $\mathcal{E} = \mathcal{E}^+ \otimes \mathcal{E}^-$.

One defines the complex coordinates as

$$\tilde{Q}_+(k) = A_+^*(k) + A_-(-k),$$

and

$$\tilde{Q}_-(k) = A_-^*(k) + A_+(-k), \quad \text{so} \quad \tilde{Q}_-(k) = \tilde{Q}_+^*(-k). \quad (\text{V.2})$$

These coordinates all mutually commute,

$$[\tilde{Q}_{\pm}(k), \tilde{Q}_{\pm'}(k')] = 0 = [\tilde{Q}_{\pm}(k), \tilde{Q}_{\pm'}^*(k')].$$

As a consequence, the fields and their adjoints commute

$$[\Phi_{\pm}(x), \Phi_{\pm'}(x')] = [\Phi_{\pm}(x), \Phi_{\pm'}^*(x')] = 0.$$

The characteristic function of the charged field depends on two variables

$$S(f_+, f_-) = \langle \Omega_0^E, e^{i\Phi_+(f_+) + i\Phi_-(f_-)} \Omega_0^E \rangle_{\mathcal{E}}.$$

V.1. Twist Symmetry of the Charged Field. Define two number operators

$$N_{\pm} = \int A_{\pm}^*(k) A_{\pm}(k) dk.$$

In terms of these, the total number operator is $N = N_+ + N_-$. The charge (or vortex number) is $F = N_+ - N_-$. The vortex number F implements the twist transformation on the field:

$$e^{i\alpha F} \tilde{Q}_{\pm}(k) e^{-i\alpha F} = e^{\pm i\alpha} \tilde{Q}_{\pm}(k), \quad \alpha \in \mathbb{C}.$$

As a consequence,

$$e^{i\alpha F} \Phi_{\pm}(x) e^{-i\alpha F} = e^{\pm i\alpha} \Phi_{\pm}(x).$$

Moreover, the zero-particle vector is twist invariant, $e^{i\alpha F} \Omega_0^E = \Omega_0^E$. Unitarity of $e^{i\alpha F}$ then ensures the vanishing of the diagonal expectations

$$\langle \Omega_0^E, \Phi_+(x) \Phi_+(x') \Omega_0^E \rangle_{\mathcal{E}} = \langle \Omega_0^E, \Phi_-(x) \Phi_-(x') \Omega_0^E \rangle_{\mathcal{E}} = 0.$$

⁵The spaces \mathcal{E}^{\pm} labelled with a superscript denote spaces where each particle has positive or negative charge. The spaces \mathcal{E}_{\pm} are subspaces of vectors at positive or negative times. However, for the fields themselves, and some other associated operators, we retain the notation used elsewhere in this section to label charges by subscripts.

The remaining two-point functions do not vanish. Let

$$D(x - x') = (2\pi)^{-d} \int \tilde{\sigma}_+(k) \tilde{\sigma}_-(-k) e^{ik \cdot (x - x')} dk ,$$

be the kernel of the operator

$$D = \sigma_+ \sigma_-^T = \sigma_-^T \sigma_+ \quad \text{with transpose} \quad D^T = \sigma_- \sigma_+^T = \sigma_+^T \sigma_- . \quad (\text{V.3})$$

Then

$$\langle \Omega_0^E, \Phi_+(x) \Phi_-(x') \Omega_0^E \rangle_{\mathcal{E}} = D(x - x')$$

and

$$\langle \Omega_0^E, \Phi_-(x) \Phi_+(x') \Omega_0^E \rangle_{\mathcal{E}} = D^T(x - x') .$$

As σ_+, σ_-^T are both translation invariant, they commute. Using these relations, the characteristic function of the charged field is

$$\begin{aligned} S(f_+, f_-) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \langle \Omega_0^E, \Phi_+(f_+)^n \Phi_-(f_-)^n \Omega_0^E \rangle_{\mathcal{E}} \\ &= e^{-\langle \bar{f}_+, D f_- \rangle_{L_2}} . \end{aligned} \quad (\text{V.4})$$

Finally, note that

$$D = D^* , \quad \text{either if } \sigma_{\pm} = (\sigma_{\pm})^* \quad \text{or if } \sigma_{\pm} = \overline{\sigma_{\mp}} . \quad (\text{V.5})$$

Also

$$D = D^T , \quad \text{either if } \sigma_+ = \sigma_- \quad \text{or if } \sigma_{\pm} = \sigma_{\pm}^T .$$

Thus there are in principle four ways that $D = D^* = D^T = \overline{D}$. They are: (i) $\sigma_+ = \sigma_- = \sigma_+^* = \sigma_-^*$, or (ii) $\sigma_+ = \sigma_+^T = \overline{\sigma_+}$ along with $\sigma_- = \sigma_-^T = \overline{\sigma_-}$, or (iii) $\sigma_+ = \sigma_- = \overline{\sigma_+}$, or (iv) $\sigma_+ = \sigma_- = \sigma_+^T$.

V.2. Matrix Notation for the Charged Field. We combine the two components Φ_{\pm} of the charged field into a vector

$$\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix} \quad \text{that pairs with test functions } \mathbf{f} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} ,$$

to yield a field acting on \mathcal{E} ,

$$\Phi(\mathbf{f}) = \sum_{\alpha=\pm} \Phi_{\alpha}(f_{\alpha}) .$$

Likewise, combine the various two-point functions of the charged field into a 2×2 matrix of operators \mathbf{D} with entries indexed by $\alpha = \pm$, each acting on $L_2(\mathbb{R}^d)$. Let $\mathbf{L}_2 = L_2(\mathbb{R}^d) \oplus L_2(\mathbb{R}^d)$, and let $\bar{\mathbf{f}}$ denote complex conjugation of \mathbf{f} . Then, with D defined by (V.3), let

$$\mathbf{D} = \begin{pmatrix} 0 & D \\ D^T & 0 \end{pmatrix} .$$

One observes that \mathbf{D} is symmetric, $\mathbf{D} = \mathbf{D}^T$. The two-point function for the field Φ is

$$\langle \Omega_0^E, \Phi(\mathbf{f}) \Phi(\mathbf{g}) \Omega_0^E \rangle_{\mathcal{E}} = \langle \bar{\mathbf{f}}, \mathbf{D} \mathbf{g} \rangle_{L_2} = \langle \Omega_0^E, \Phi(\mathbf{g}) \Phi(\mathbf{f}) \Omega_0^E \rangle_{\mathcal{E}} .$$

The second identity arises from the commutativity of the field. Equivalent to commutativity is the relation

$$\langle \bar{g}, D\mathbf{f} \rangle_{L_2} = \langle D^*\bar{g}, \mathbf{f} \rangle_{L_2} = \langle \overline{D\mathbf{g}}, \mathbf{f} \rangle_{L_2} = \overline{\langle \mathbf{f}, D\mathbf{g} \rangle_{L_2}} = \langle \bar{\mathbf{f}}, D\mathbf{g} \rangle_{L_2} .$$

Note also that

$$\langle \bar{\mathbf{f}}, D\mathbf{g} \rangle_{L_2} = \langle \bar{f}_+, Dg_- \rangle_{L_2} + \langle \bar{g}_+, Df_- \rangle_{L_2} ,$$

so

$$\langle \bar{\mathbf{f}}, D\mathbf{f} \rangle_{L_2} = 2\langle \bar{f}_+, Df_- \rangle_{L_2} . \quad (\text{V.6})$$

In fact,

$$\langle \bar{f}_-, D^T g_+ \rangle_{L_2} = \overline{\langle D^T g_+, \bar{f}_- \rangle_{L_2}} = \langle D^* \bar{g}_+, f_- \rangle_{L_2} = \langle \bar{g}_+, Df_- \rangle_{L_2} .$$

Proposition V.1. *The characteristic function $S(f_+, f_-)$ of (V.4) has the standard form of a Gaussian,*

$$\begin{aligned} S(\mathbf{f}) &= \langle \Omega_0^E, e^{i\Phi(\mathbf{f})} \Omega_0^E \rangle_{\mathcal{E}} = e^{-\frac{1}{2}\langle \bar{\mathbf{f}}, D\mathbf{f} \rangle_{L_2}} \\ &= S(f_+, f_-) = e^{-\langle \bar{f}_+, Df_- \rangle_{L_2}} . \end{aligned} \quad (\text{V.7})$$

Proof. The product of $n \in \mathbb{N}$ fields with test functions $\mathbf{f}^{(j)}$ with components $f_\alpha^{(j)}$, $j = 1, \dots, n$, is

$$\Phi(\mathbf{f}^{(1)}) \dots \Phi(\mathbf{f}^{(n)}) = \sum_{\alpha_1, \dots, \alpha_n = \pm} \Phi_{\alpha_1}(f_{\alpha_1}^{(1)}) \dots \Phi_{\alpha_n}(f_{\alpha_n}^{(n)}) . \quad (\text{V.8})$$

As the fields are linear in creation and annihilation operators, the zero-particle expectation of such a product satisfies the Gaussian recursion relation

$$\begin{aligned} \left\langle \Omega_0^E, \Phi(\mathbf{f}^{(1)}) \dots \Phi(\mathbf{f}^{(n)}) \Omega_0^E \right\rangle_{\mathcal{E}} &= \sum_{j=2}^n \left\langle \bar{\mathbf{f}}^{(1)}, D\mathbf{f}^{(j)} \right\rangle_{L_2} \times \\ &\times \left\langle \Omega_0^E, \Phi(\mathbf{f}^{(2)}) \dots \Phi(\mathbf{f}^{(j)}) \dots \Phi(\mathbf{f}^{(n)}) \Omega_0^E \right\rangle_{\mathcal{E}} . \end{aligned}$$

Here, $\Phi(\mathbf{f}^{(j)})$ indicates that one omits the term with index j^{th} from the product. Moreover, the expression (V.8) is a multi-linear, symmetric function of the $\mathbf{f}^{(j)}$'s, so it is determined uniquely—using a polarization identity—as a linear combination of powers $\Phi(\mathbf{g})^n$, where \mathbf{g} is one of the 2^n functions $\pm \mathbf{f}^{(1)} \pm \dots \pm \mathbf{f}^{(n)}$.

Hence, the expectations

$$\left\langle \Omega_0^E, \Phi(\mathbf{f}^{(1)}) \dots \Phi(\mathbf{f}^{(n)}) \Omega_0^E \right\rangle_{\mathcal{E}}$$

are determined uniquely by the characteristic function $S(\mathbf{f})$. From the recursion relation, we infer that

$$\left\langle \Omega_0^E, \Phi(\mathbf{f})^n \Omega_0^E \right\rangle_{\mathcal{E}} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (2k-1)!! \langle \bar{\mathbf{f}}, D\mathbf{f} \rangle_{L_2}^k, & \text{if } n = 2k \end{cases} ,$$

so $S(\mathbf{f}) = e^{-\frac{1}{2}\langle \bar{\mathbf{f}}, \mathbf{D}\mathbf{f} \rangle_{L_2}}$. Also $S(\mathbf{f}) = S(f_+, f_-)$ as a consequence of the second identity in (V.6). \square

V.3. Charge Conjugation. We define *charge conjugation* as a unitary transformation \mathbf{U}_c on \mathcal{E} . We substitute Φ_+ for the ordinary charged field $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ representing positive charge and Φ_- for its hermitian conjugate $\varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$ representing negative charge. Thus, hermitian conjugation reverses the charge for the ordinary field, while charge conjugation of our classical field is determined by the action of the charge-conjugation matrix

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{C}^* = \mathcal{C}^T = \bar{\mathcal{C}} = \mathcal{C}^{-1}. \quad (\text{V.9})$$

Note that \mathcal{C} is a 2×2 matrix that acts on the components of the field, *i.e.*,

$$(\mathcal{C}\Phi)(x) = \begin{pmatrix} \Phi_-(x) \\ \Phi_+(x) \end{pmatrix}.$$

The unitary charge-conjugation operator \mathbf{U}_c on \mathcal{E} is defined setting by

$$\mathbf{U}_c \Phi(\mathbf{f}^{(1)}) \dots \Phi(\mathbf{f}^{(n)}) \Omega_0^E = (\mathcal{C}\Phi)(\mathbf{f}^{(1)}) \dots (\mathcal{C}\Phi)(\mathbf{f}^{(n)}) \Omega_0^E,$$

and $\mathbf{U}_c \Omega_0^E = \Omega_0^E$.

V.4. Complex Conjugation. Next we consider complex conjugation of the classical field, defined by

$$\bar{\Phi}(x) = \begin{pmatrix} \Phi_+^*(x) \\ \Phi_-^*(x) \end{pmatrix} = (\Phi^*)^T(x).$$

As the charge conjugation is connected with hermitian conjugation, it is natural that the existence of a positive measure on the fields entails that the expectation $\langle \Omega_0^E, \mathcal{C}\Phi(x) \Phi(x') \Omega_0^E \rangle_{\mathcal{E}} = (\mathcal{C}\mathbf{D})(x, x')$ is the kernel of a positive transformation

$$\mathcal{C}\mathbf{D} = \begin{pmatrix} D^T & 0 \\ 0 & D \end{pmatrix} \text{ on } L_2.$$

This will be the case, if and only if D itself is positive on L_2 . Thus, a positive measure is associated with the condition

$$0 \leq \sum_{j, j'=1}^N c_j \bar{c}_{j'} S(\mathbf{f}_j - \mathcal{C}\bar{\mathbf{f}}_{j'}),$$

for any choice of $\mathbf{f}_j \in L_2$ and $c_j \in \mathbb{C}$.

V.5. **Time Reflection.** Let Θ denote the unitary time reflection on \mathcal{E} , namely

$$\Theta = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}.$$

In order to recover the usual properties of the charged fields, we expect that they should transform under time inversion as

$$\Theta \Phi(x) \Theta = \mathcal{C} \overline{\Phi}(\vartheta x),$$

or in components

$$\Theta \Phi_{\pm}(x) \Theta = \Phi_{\mp}^*(\vartheta x). \quad (\text{V.10})$$

In fact, we have the following criterion.

Proposition V.2 (Time Reflection). *The field transforms according to (V.10) if and only if*

$$\vartheta \sigma_{\pm}^{\text{T}} \vartheta = \sigma_{\mp}^*. \quad (\text{V.11})$$

In this case,

$$(\vartheta \mathcal{C}) \mathbf{D} (\vartheta \mathcal{C}) = \mathbf{D}^* = \overline{\mathbf{D}}, \quad (\text{V.12})$$

and the operator $\vartheta \mathcal{C} \mathbf{D}$ equals

$$\vartheta \mathcal{C} \mathbf{D} = \begin{pmatrix} \vartheta D^{\text{T}} & 0 \\ 0 & \vartheta D \end{pmatrix} = \begin{pmatrix} \sigma_-^* \vartheta \sigma_- & 0 \\ 0 & \sigma_+^* \vartheta \sigma_+ \end{pmatrix}, \quad (\text{V.13})$$

which is self-adjoint on \mathbf{L}_2 .

Proof. In terms of components,

$$\begin{aligned} \Theta \Phi_{\pm}(x) \Theta &= (2\pi)^{-d/2} \int \Theta \tilde{Q}_{\pm}(k) \Theta \tilde{\sigma}_{\pm}(k) e^{ik \cdot x} dk \\ &= (2\pi)^{-d/2} \int \tilde{Q}_{\pm}(\vartheta k) \tilde{\sigma}_{\pm}(k) e^{ik \cdot x} dk \\ &= (2\pi)^{-d/2} \int \tilde{Q}_{\mp}^*(-\vartheta k) \tilde{\sigma}_{\pm}(k) e^{ik \cdot x} dk \\ &= (2\pi)^{-d/2} \int \tilde{Q}_{\mp}^*(k) \tilde{\sigma}_{\pm}(-\vartheta k) e^{-ik \cdot (\vartheta x)} dk \\ &= (2\pi)^{-d/2} \left(\int \tilde{Q}_{\mp}(k) \overline{\tilde{\sigma}_{\pm}(-\vartheta k)} e^{ik \cdot (\vartheta x)} dk \right)^*. \end{aligned}$$

Thus, the desired equivalence (V.10) is equivalent to $\tilde{\sigma}_{\pm}$ satisfying the relation $\overline{\tilde{\sigma}_{\pm}(-\vartheta k)} = \tilde{\sigma}_{\mp}(k)$. In configuration space this is equivalent to the operator relation (V.11).

Note that the self-adjoint, unitary operator on \mathbf{L}_2 , given by

$$\vartheta \mathcal{C} = \begin{pmatrix} 0 & \vartheta \\ \vartheta & 0 \end{pmatrix},$$

yields

$$(\vartheta \mathcal{C}) \mathbf{D} (\vartheta \mathcal{C}) = \begin{pmatrix} 0 & \vartheta D^{\text{T}} \vartheta \\ \vartheta D \vartheta & 0 \end{pmatrix}.$$

Hence, relation (V.12) is equivalent to $\vartheta D \vartheta = D^*$ and $\vartheta D^T \vartheta = D^{T*}$, namely to the self-adjointness of ϑD and of ϑD^T on L_2 . In the case that (V.11) holds, one can use (V.3) to infer $\vartheta D = \vartheta \sigma_+^T \sigma_+ = \sigma_+^* \vartheta \sigma_+$. Thus, ϑD is self-adjoint on L_2 . Likewise, $\vartheta D^T = \vartheta \sigma_+^T \sigma_- = \sigma_-^* \vartheta \sigma_-$ is self-adjoint on L_2 . As a consequence, one infers that

$$\vartheta \mathcal{C} \mathbf{D} = \begin{pmatrix} \vartheta D^T & 0 \\ 0 & \vartheta D \end{pmatrix} = \begin{pmatrix} \sigma_-^* \vartheta \sigma_- & 0 \\ 0 & \sigma_+^* \vartheta \sigma_+ \end{pmatrix},$$

which has the desired form (V.13) and is self-adjoint on \mathbf{L}_2 . \square

V.6. Time-Reflection Positivity for the Charged Field. To study time-reflection positivity of Θ on \mathcal{E}_+ , take the Osterwalder-Schrader form for the charged field to be

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \Theta \cdot \rangle_{\mathcal{E}}. \quad (\text{V.14})$$

Let $\mathbf{L}_{2,+} = L_2(\mathbb{R}_+^d) \oplus L_2(\mathbb{R}_+^d)$.

Definition V.3. The functional $S(\mathbf{f})$ is time-reflection positive if

$$0 \leq \sum_{j,j'=1}^N c_j \overline{c_{j'}} S(\mathbf{f}_j - \vartheta \mathcal{C} \overline{\mathbf{f}_{j'}}), \quad (\text{V.15})$$

for any choices of $\mathbf{f}_j \in \mathbf{L}_{2,+}$ and $c_j \in \mathbb{C}$, $j = 1, \dots, N$. It is also time-reflection positive if (V.15) holds for any choices of $\mathbf{f}_j \in \mathbf{L}_{2,-}$ and $c_j \in \mathbb{C}$, $j = 1, \dots, N$. In case both conditions hold, then $S(\mathbf{f})$ is doubly time-reflection positive.

Proposition V.4. *Time-reflection positivity on \mathcal{E}_+ is equivalent to the statement that*

$$0 \leq \vartheta \mathcal{C} \mathbf{D} = \begin{pmatrix} \vartheta D^T & 0 \\ 0 & \vartheta D \end{pmatrix} = \begin{pmatrix} \sigma_-^* \vartheta \sigma_- & 0 \\ 0 & \sigma_+^* \vartheta \sigma_+ \end{pmatrix} \text{ on } \mathbf{L}_{2,+}. \quad (\text{V.16})$$

Proof. As the characteristic function for the charged field is Gaussian, we infer from the proof of Proposition III.3 that positivity of the OS form (V.14) on \mathcal{E} is equivalent to positivity of the two point function

$$\langle \Phi(\mathbf{f}) \Omega_0^{\mathcal{E}}, \Phi(\mathbf{g}) \Omega_0^{\mathcal{E}} \rangle_{\mathcal{H}}$$

for $\mathbf{f} \in \mathbf{L}_{2,+} = L_2(\mathbb{R}_+^d) \oplus L_2(\mathbb{R}_+^d)$. For $\mathbf{f}, \mathbf{g} \in \mathbf{L}_{2,+}$ we claim that the putative inner product of two such vectors in \mathcal{H} is

$$\langle \Phi(\mathbf{f}) \Omega_0^{\mathcal{E}}, \Phi(\mathbf{g}) \Omega_0^{\mathcal{E}} \rangle_{\mathcal{H}} = \langle \mathbf{f}, \vartheta \mathcal{C} \mathbf{D} \mathbf{g} \rangle_{L_2}, \quad (\text{V.17})$$

where

$$\vartheta \mathcal{C} \mathbf{D} = \begin{pmatrix} \vartheta D^T & 0 \\ 0 & \vartheta D \end{pmatrix} = \begin{pmatrix} \sigma_-^* \vartheta \sigma_- & 0 \\ 0 & \sigma_+^* \vartheta \sigma_+ \end{pmatrix}.$$

From (V.17) we infer that positivity of the OS form on \mathcal{E} is equivalent to (V.16), as claimed. In fact,

$$\begin{aligned} \langle \Phi(f)\Omega_0^E, \Phi(g)\Omega_0^E \rangle_{\mathcal{H}} &= \langle \Phi(f)\Omega_0^E, \Theta \Phi(g)\Omega_0^E \rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, \Theta \Phi^*(\bar{f})\Theta \Phi(g)\Omega_0^E \rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, \Phi(\mathcal{C}\vartheta\bar{f})\Phi(g)\Omega_0^E \rangle_{\mathcal{E}} = \langle f, \vartheta\mathcal{C}Dg \rangle_{L_2} . \end{aligned}$$

In the final equality we use the fact that action of the matrix \mathcal{C} on Φ is given by (V.9). Therefore $(\mathcal{C}\Phi)(\vartheta f) = \Phi(\mathcal{C}\vartheta f)$, and

$$\langle \Omega_0^E, (\mathcal{C}\Phi)(\vartheta f)\Phi(g)\Omega_0^E \rangle_{\mathcal{E}} = \langle \bar{f}, \vartheta\mathcal{C}Dg \rangle_{L_2} . \quad (\text{V.18})$$

□

V.7. Spatial-Reflection Positivity for the Charged Field. One can formulate spatial reflection positivity by substituting $L_2(\mathbb{R}_{\bar{n}+}^d)$ for $L_2(\mathbb{R}_+^d)$, substituting $\pi_{\bar{n}}$ for ϑ , substituting $\Pi_{\bar{n}}$ for Θ , and substituting $\mathcal{E}_{\bar{n}+}$ for \mathcal{E}_+ in (V.14). Spatial reflection positivity gives a pre-inner product

$$\langle A, B \rangle_{\mathcal{H}(\bar{n})} = \langle A, \Pi_{\bar{n}}B \rangle_{\mathcal{E}} ,$$

on $\mathcal{E}_{\bar{n}+,0}$, as well as new requirements on the transformation of the field and of D .

Proposition V.5 (Spatial-Reflection). *If one replaces $\vartheta, \Theta, \Theta, \mathbb{R}_+^d, \mathcal{E}_+$ in the statement of Proposition V.2 by $\pi_{\bar{n}}, \Pi_{\bar{n}}, \Pi_{\bar{n}}, \mathbb{R}_{\bar{n}+}^d, \mathcal{E}_{\bar{n}+}$, respectively, then the proposition remains valid.*

Proposition V.6 (Spatial-Reflection Positivity). *If one replaces $\vartheta, \Theta, \Theta, \mathbb{R}_+^d, \mathcal{E}_+$ in the statement of Proposition V.4 by $\pi_{\bar{n}}, \Pi_{\bar{n}}, \Pi_{\bar{n}}, \mathbb{R}_{\bar{n}+}^d, \mathcal{E}_{\bar{n}+}$, respectively, then the proposition remains valid.*

VI. COMPACTIFICATION

In this section we show that reflection positivity carries over when we compactify one coordinate $x_j \in \mathbb{R}$ on the line to a corresponding coordinate on a circle $x_j \in S^1$.

We consider spacetimes of the general form $\mathbf{X} = X_1 \times \cdots \times X_d$, where each factor X_i either equals \mathbb{R} (the real line) or S^1 (a circle of length ℓ_j). Denote the first coordinate by $t = x_0$ and let $x = (t, \vec{x})$. *Compactification* of a coordinate $x_j \in X_j$ means replacing the coordinate $x_j \in \mathbb{R}$ by a corresponding coordinate $x_j \in S^1$. One denotes this compactification as

$$\begin{aligned} \mathbf{X} &= X_1 \times \cdots \times X_{j-1} \times \mathbb{R} \times X_{j+1} \times \cdots \times X_d \longrightarrow \\ &\longrightarrow \mathbf{X}^{c_j} = X_1 \times \cdots \times X_{j-1} \times S^1 \times X_{j+1} \times \cdots \times X_d . \end{aligned}$$

Of course one can take \mathbf{X}^{c_j} as a new \mathbf{X} , and continue to compactify. The minimally compactified space is \mathbb{R}^d ; the maximally compactified space is the torus \mathbb{T}^d , with periods $\beta = \ell_0, \ell_1, \dots, \ell_{d-1}$.

Parameterize the compactified coordinate x_j as

$$x_j \in S^1 = [-\frac{1}{2}\ell_j, \frac{1}{2}\ell_j],$$

and the positive-subspace of X_j by

$$X_{j+} = (S^1_+)_j = [0, \frac{1}{2}\ell_j], \quad \text{or} \quad X_{j+} = \mathbb{R}_+ = [0, \infty).$$

Likewise, denote the j -positive subspace of \mathbf{X} by

$$\mathbf{X}_{j+} = X_1 \times \cdots \times X_{j+} \times \cdots \times X_d. \quad (\text{VI.1})$$

The reflection of the coordinate x_j is the transformation

$$\pi_j: x \rightarrow \pi_j x, \text{ where } (\pi_j x)_i = \begin{cases} x_i, & \text{if } i \neq j \\ -x_j, & \text{if } i = j \end{cases}.$$

We denote ϑ by π_0 , so we treat time reflection and spatial reflection on an equal footing. The Fock space \mathcal{E} over \mathcal{K} and the j -positive subspaces \mathcal{E}_{j+} will now refer to the one-particle space $\mathcal{K} = L_2(\mathbf{X})$, and the j -positive subspace $\mathcal{K}_{j+} = L_2(\mathbf{X}_{j+})$.

Definition VI.1. The operator D is doubly-reflection-positive with respect to π_j if both

$$0 \leq \pi_j D, \quad \text{and} \quad 0 \leq D\pi_j \quad \text{on} \quad L_2(\mathbf{X}_{j+}). \quad (\text{VI.2})$$

This is equivalent to both

$$0 \leq \pi_j D, \quad \text{and} \quad 0 \leq D\pi_j \quad \text{on} \quad L_2(\mathbf{X}_{j-}), \quad (\text{VI.3})$$

or to

$$0 \leq \pi_j D \quad \text{on both} \quad L_2(\mathbf{X}_{j\pm}). \quad (\text{VI.4})$$

Recall that symmetry of the operator D with integral kernel $D(x, x')$ means $D(x, x') = D(x', x)$. Moreover, a covariance for classical fields that is reflection-positive with respect to the reflection π must be symmetric and satisfy $\pi D \pi = D^*$.

Proposition VI.2. *Suppose D is symmetric on $L_2(\mathbf{X})$, and*

$$\pi D \pi = D^*$$

for a reflection π . Then double-reflection-positivity and reflection-positivity with respect to π on $L_2(\mathbf{X}_+)$ are equivalent.

Proof. Double RP is a stronger condition, so one only needs to show that RP ensures double RP. Since $\pi X_{\pm} = X_{\mp}$ and π is self-adjoint and unitary, the condition (VI.4) is equivalent to the other two conditions. Thus, it is sufficient to show that $0 \leq \pi D$ on $L_2(X_+)$ implies $0 \leq D\pi$ on the same subspace.

Using $\pi D \pi = D^*$, one infers

$$D(x, x') = (\pi D^* \pi)(x, x') = (D^*)(\pi x, \pi x') = \overline{D(\pi x', \pi x)},$$

so symmetry ensures that $D(x, x') = \overline{D(\pi x, \pi x')}$. This is the operator relation $D = \overline{\pi D \pi} = \pi \overline{D} \pi$, as π is real, so also $\overline{D \pi} = \pi D$. Therefore

$$\langle f, D\pi f \rangle_{L_2(\mathbf{X})} = \overline{\langle f, \overline{D\pi} \overline{f} \rangle_{L_2(\mathbf{X})}} = \overline{\langle f, \pi D \overline{f} \rangle_{L_2(\mathbf{X})}}. \quad (\text{VI.5})$$

Complex conjugation leaves $L_2(\mathbf{X}_+)$ invariant, so reflection-positivity ensures $0 \leq \langle \overline{f}, \pi D \overline{f} \rangle_{L_2(\mathbf{X})}$. Therefore $0 \leq \langle f, D\pi f \rangle_{L_2(\mathbf{X})}$ for $f \in L_2(\mathbf{X}_+)$, as claimed. \square

Let e_j denote a unit vector in the j^{th} coordinate direction. Define T_j as the unitary translation operator on spacetime that translates by one period ℓ_j in the coordinate direction j , namely

$$(T_j f)(x) = f(x - \ell_j e_j),$$

with $T_j : L_2(\mathbf{X}_{j+}) \rightarrow L_2(\mathbf{X}_{j+})$. In addition,

$$\left(\pi_j T_j^{-1/2} f \right) (x) = \left(T_j^{-1/2} f \right) (\pi_j x) = f(\pi_j x + \frac{1}{2} \ell_j e_j).$$

Now assume that $D(x - x')$ decreases sufficiently rapidly so that the sum

$$D^{cj}(x, x') = \sum_{n=-\infty}^{\infty} (T_j^{-n} D)(x - x') = \sum_{n=-\infty}^{\infty} D(x - x' + n \ell_j e_j) \quad (\text{VI.6})$$

converges absolutely. Then (VI.6) defines the operator D^{cj} on the space \mathbf{X}^{cj} .

Proposition VI.3 (Compactification of Reflection Positivity).

Let D be translation-invariant and symmetric on $L_2(\mathbf{X})$ and let D be doubly-reflection-positive with respect to π_j . Define D^{ci} by the integral kernel (VI.6). Then D^{ci} is doubly-reflection-positive with respect to π_j .

Remark VI.4. If $i = j$, the proposition states that reflection positivity for a coordinate $x_j \in \mathbb{R}$ extends to the case of a compactified coordinate $x_j \in S^1$. However, for $i \neq j$ the proposition says that reflection positivity in the j^{th} -direction remains unaffected by the compactification of spacetime along a different coordinate direction x_i .

Proof. The Case $i = j$. We first show that $0 \leq \langle f, \pi_j D^{cj} f \rangle_{L_2(\mathbf{X}^{cj})}$ for all $f \in L_2(\mathbf{X}_{j+}^{cj})$. Here, f depends on the coordinate $x_j \in [-\frac{1}{2}\ell_j, \frac{1}{2}\ell_j]$ and is supported in the positive half interval. Imbed $L_2(\mathbf{X}^{cj})$ in $L_2(\mathbf{X})$ in the natural way, so that translations T_j on $L_2(\mathbf{X})$ translate the support of f by $\ell_j e_j$. Then

$$\begin{aligned} \langle f, \pi_j D^{cj} f \rangle_{L_2(\mathbf{X}^{cj})} &= \langle f, \pi_j D^{cj} f \rangle_{L_2(\mathbf{X})} \\ &= \sum_{n=-\infty}^{\infty} \langle f, \pi_j T_j^{-n} D f \rangle_{L_2(\mathbf{X})}. \end{aligned} \quad (\text{VI.7})$$

For the moment, let us suppress the variables x_i for $i \neq j$. Thus, we write

$$\begin{aligned} \langle f, \pi_j T_j^{-n} Df \rangle_{L_2(\mathbf{X})} &= \int_0^{\frac{1}{2}\ell_j} \int_0^{\frac{1}{2}\ell_j} \overline{f(x_j)} D(-x_j - x'_j + n\ell_j) f(x'_j) dx_j dx'_j \\ &= \left\langle T_j^{-n/2} f, \pi_j D T_j^{-n/2} f \right\rangle_{L_2(\mathbf{X})}. \end{aligned} \quad (\text{VI.8})$$

We now show that each term in the sum (VI.7) is positive. These terms are of the form given in (VI.8). In case $n \leq 0$, the operator $T_j^{-n/2}$ maps $L_2(\mathbf{X}_{j+})$ into itself. Thus, reflection positivity of $\pi_j D$ ensures positivity, *i.e.*,

$$0 \leq \langle f, \pi_j T_j^{-n} Df \rangle_{L_2(\mathbf{X})} \quad \text{for } n \leq 0. \quad (\text{VI.9})$$

On the other hand, if $n \geq 1$, then $T_j^{-n/2}$ maps $L_2(\mathbf{X}_{j+})$ into $L_2(\mathbf{X}_{j-})$. Hence, $\pi_j T_j^{-n/2}$ maps $L_2(\mathbf{X}_{j+})$ into itself. Thus, for $f \in L_2(\mathbf{X}_{j+})$,

$$\begin{aligned} \langle f, \pi_j T_j^{-n} Df \rangle_{L_2(\mathbf{X})} &= \left\langle T_j^{n/2} \pi_j f, D T_j^{-n/2} f \right\rangle_{L_2(\mathbf{X})} \\ &= \left\langle \pi_j T_j^{-n/2} f, (D\pi_j) \pi_j T_j^{-n/2} f \right\rangle_{L_2(\mathbf{X})}. \end{aligned}$$

The positivity of $D\pi_j$ on $L_2(\mathbf{X}_{j+})$ now ensures that

$$0 \leq \langle f, \pi_j T_j^{-n} Df \rangle_{L_2(\mathbf{X})} \quad \text{for } 1 \leq n. \quad (\text{VI.10})$$

The relations (VI.9) and (VI.10) show that (VI.7) is a sum of non-negative terms. Thus, $0 \leq \pi_j D^{cj}$ on $L_2(\mathbf{X}_{j+}^{cj})$, as claimed.

One can reduce the proof of positivity of $D^{cj} \pi_j$ on $L_2(\mathbf{X}_{j+}^{cj})$ to the previous case. In fact, one can replace n by $-n$ in (VI.6) and write

$$\begin{aligned} \langle f, D^{cj} \pi_j f \rangle_{L_2(\mathbf{X})} &= \sum_{n=-\infty}^{\infty} \langle f, T_j^n D \pi_j f \rangle_{L_2(\mathbf{X})} \\ &= \sum_{n=-\infty}^0 \left\langle T_j^{-n/2} f, (D\pi_j) T_j^{-n/2} f \right\rangle_{L_2(\mathbf{X})} \\ &\quad + \sum_{n=1}^{\infty} \left\langle \pi_j T_j^{-n/2} f, (\pi_j D) \pi_j T_j^{-n/2} f \right\rangle_{L_2(\mathbf{X})}. \end{aligned}$$

We have already shown that these matrix elements are positive. Thus, $0 \leq D^{cj} \pi_j$ on $L_2(\mathbf{X}_{j+}^{cj})$.

The Case $i \neq j$. For $f \in L_2(\mathbf{X}_{j+}^{ci})$ one has, for any $n' \in \mathbb{Z}$,

$$\begin{aligned} \langle f, \pi_j D^{ci} f \rangle_{L_2(\mathbf{X}^{cj})} &= \sum_{n=-\infty}^{\infty} \langle f, \pi_j T_i^{-n} D f \rangle_{L_2(\mathbf{X})} \\ &= \sum_{n=-\infty}^{\infty} \langle f, \pi_j T_i^{-n+n'} D f \rangle_{L_2(\mathbf{X})} . \end{aligned}$$

Using the fact that T_i commutes with both π_j and D , one arrives at

$$\begin{aligned} \langle f, \pi_j D^{ci} f \rangle_{L_2(\mathbf{X}^{cj})} &= \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} \sum_{n'=-N}^N \langle T_i^{-n'} f, \pi_j D T_i^{-n} f \rangle_{L_2(\mathbf{X})} \\ &= \lim_{N \rightarrow \infty} \langle g_N, \pi_j D g_N \rangle_{L_2(\mathbf{X})} , \end{aligned} \quad (\text{VI.11})$$

where

$$g_N = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^N T_i^{-n} f \in L_2(\mathbf{X}_{j+}) . \quad (\text{VI.12})$$

Observe that the translations T_i act in the i^{th} -coordinate direction, orthogonal to the j^{th} -coordinate direction, so they map $L_2(\mathbf{X}_{j+})$ into itself. As $\pi_j D$ is positive on $L_2(\mathbf{X}_{j+})$, one concludes that

$$0 \leq \langle g_N, \pi_j D g_N \rangle_{L_2(\mathbf{X})} \rightarrow \langle f, \pi_j D^{ci} f \rangle_{L_2(\mathbf{X}^{cj})} .$$

Thus, $0 \leq \pi_j D^{ci}$ on $L_2(\mathbf{X}_{j+})$, as claimed. The argument to show that $0 \leq D^{ci} \pi_j$ on $L_2(\mathbf{X}_{j+})$ is similar, so we omit the details. \square

VII. SUMMARY OF POSITIVITY CONDITIONS

We now summarize the various positivity conditions that we have discussed in this paper in the case $X = \mathbb{R}^d$. We state the conditions the characteristic function $S(f)$ of the field is subject to, and—for the Gaussian case—the conditions the covariance of the characteristic function has to obey.

VII.1. Neutral Fields. We use the unitary time-reflection operator ϑ , the unitary reflection $\pi_{\vec{n}}$ in the plane orthogonal to \vec{n} , and in the Gaussian case the covariance D .

Measure Positivity: The condition

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \bar{f}_j) , \quad (\text{VII.1})$$

leads to the existence of a positive measure as the Fourier transform of $S(f)$. In the Gaussian case $S(f) = e^{-\frac{1}{2} \langle \bar{f}, D f \rangle_{L_2}}$, and condition (VII.1) is equivalent to

$$0 \leq D \quad \text{on} \quad L_2(\mathbb{R}^d) . \quad (\text{VII.2})$$

Time-Reflection Positivity: The condition of time-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \vartheta \bar{f}_j) \quad \text{for } f_j \in L_2(\mathbb{R}_+^d). \quad (\text{VII.3})$$

In the Gaussian case this is equivalent to

$$0 \leq \vartheta D \quad \text{on } L_2(\mathbb{R}_+^d). \quad (\text{VII.4})$$

Alternative Time-Reflection Positivity: The alternative condition of time-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \vartheta \bar{f}_j) \quad \text{for } f_j \in L_2(\mathbb{R}_-^d). \quad (\text{VII.5})$$

In the Gaussian case this is equivalent to

$$0 \leq D \vartheta \quad \text{on } L_2(\mathbb{R}_+^d). \quad (\text{VII.6})$$

Spatial-Reflection Positivity: The condition for spatial-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \pi_{\vec{n}} \bar{f}_j) \quad \text{for } f_j \in L_2(\mathbb{R}_{\vec{n}_+}^d). \quad (\text{VII.7})$$

In the Gaussian case this is equivalent to

$$0 \leq \pi_{\vec{n}} D \quad \text{on } L_2(\mathbb{R}_{\vec{n}_+}^d). \quad (\text{VII.8})$$

Alternative Spatial-Reflection Positivity: The alternative condition for spatial-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(f_{j'} - \pi_{\vec{n}} \bar{f}_j) \quad \text{for } f_j \in L_2(\mathbb{R}_{\vec{n}_-}^d). \quad (\text{VII.9})$$

In the Gaussian case this is equivalent to

$$0 \leq D \pi_{\vec{n}} \quad \text{on } L_2(\mathbb{R}_{\vec{n}_+}^d). \quad (\text{VII.10})$$

VII.2. Charged Fields. In the case of the charged field we use the charge conjugation operator \mathcal{C} , the unitary time-reflection operator ϑ , and the unitary reflection $\pi_{\vec{n}}$ in the plane orthogonal to \vec{n} . In the Gaussian case we also use the matrix covariance \mathbf{D} .

Measure Positivity: The positivity condition

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(\mathbf{f}_{j'} - \mathcal{C} \bar{\mathbf{f}}_j), \quad (\text{VII.11})$$

leads to the existence of a positive measure as the Fourier transform of $S(\mathbf{f})$. In the Gaussian case this positivity is equivalent to

$$0 \leq \mathcal{C} \mathbf{D} \quad \text{on } \mathbf{L}_2. \quad (\text{VII.12})$$

Time-Reflection Positivity: The condition of time-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(\mathbf{f}_{j'} - \vartheta \mathcal{C} \bar{\mathbf{f}}_j) \text{ for } \mathbf{f}_j \in \mathbf{L}_{2+}. \quad (\text{VII.13})$$

In the Gaussian case this is equivalent to

$$0 \leq \vartheta \mathcal{C} \mathbf{D} \quad \text{on } \mathbf{L}_{2+}. \quad (\text{VII.14})$$

Alternative Time-Reflection Positivity: The alternative condition of time-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(\mathbf{f}_{j'} - \mathcal{C} \vartheta \bar{\mathbf{f}}_j) \quad \text{for } \mathbf{f}_j \in \mathbf{L}_{2-}. \quad (\text{VII.15})$$

In the Gaussian case the alternative condition is equivalent to

$$0 \leq \mathbf{D} \vartheta \mathcal{C} \quad \text{on } \mathbf{L}_{2+}. \quad (\text{VII.16})$$

Spatial-Reflection Positivity: The condition for spatial-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(\mathbf{f}_{j'} - \pi_{\bar{n}} \mathcal{C} \bar{\mathbf{f}}_j) \quad \text{for } \mathbf{f}_j \in \mathbf{L}_{2,\bar{n}+}. \quad (\text{VII.17})$$

Here, $\mathbf{L}_{2,\bar{n}+} = L_2(\mathbb{R}_{\bar{n}+}^d) \oplus L_2(\mathbb{R}_{\bar{n}+}^d)$ and $\mathbf{L}_{2,\bar{n}-} = \pi_{\bar{n}} \mathbf{L}_{2,\bar{n}+}$. In the Gaussian case (VII.17) is equivalent to

$$0 \leq \pi_{\bar{n}} \mathcal{C} \mathbf{D} \quad \text{on } \mathbf{L}_{2,\bar{n}+}. \quad (\text{VII.18})$$

Alternative Spatial-Reflection Positivity: The alternative condition for spatial-reflection positivity is

$$0 \leq \sum_{j,j'=1}^N \bar{c}_j c_{j'} S(\mathbf{f}_{j'} - \pi_{\bar{n}} \mathcal{C} \bar{\mathbf{f}}_j) \quad \text{for } \mathbf{f}_j \in \mathbf{L}_{2,\bar{n}-}. \quad (\text{VII.19})$$

In the Gaussian case this is equivalent to

$$0 \leq \mathcal{C} \mathbf{D} \pi_{\bar{n}} \quad \text{on } \mathbf{L}_{2,\bar{n}+}. \quad (\text{VII.20})$$

REFERENCES

- [1] Tadeusz Balaban, Joel Feldman, Horst Knörrer, and Eugene Trubowitz, The temporal ultraviolet limit, in “Quantum Theory from Small to Large Scales” Lecture Notes of the Les Houches Summer School: Volume 95, August 2010. Edited by Jürg Fröhlich, Manfred Salmhofer, Vieri Mastropietro, Wojciech De Roeck, and Leticia F. Cugliandolo, Oxford University Press, 2012.
- [2] Marek Biskup, Reflection positivity and phase transitions for lattice models, in “Methods of Contemporary Mathematical Statistical Physics,” Roman Kotecny, Editor, Lecture Notes in Mathematics, Springer 2009.
- [3] Hans Borchers and Jakob Yngvason, Necessary and sufficient conditions for integral representations of Wightman functionals at Schwinger points, *Comm. Math. Phys.* **47** (1976) 197–213.

- [4] David C. Brydges and John Z. Imbrie, Green's function for a hierarchical self-avoiding walk in four dimensions, *Comm. Math. Phys.* **239** (2003) 549–584.
- [5] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions. Volume 4*, Academic Press (HBJ Publishers), New York, 1964.
- [6] James Glimm and Arthur Jaffe, *Quantum Physics, A Functional Point of View*, Springer (1981).
- [7] James Glimm, Arthur Jaffe, and Thomas Spencer, The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model, *Ann. of Math.* **100** (1974) 585–632.
- [8] Edward P. Heifets and Edward P. Osipov, The energy-momentum spectrum in the $P(\varphi)_2$ quantum field theory, *Comm. Math. Phys.* **56** (1977) 161–172.
- [9] Arthur Jaffe, Christian Jäkel, and Roberto Martinez, Complex Classical Fields and Partial Wick Rotations,
- [10] Palle E.T. Jørgensen and Gestur Ólafsson, Unitary representations and Osterwalder-Schrader duality, published in *The Mathematical Legacy of Harish-Chandra: A Celebration of Representation Theory and Harmonic Analysis* (R. Doran and V. Varadarajan, eds.), Proc. Sympos. Pure Math., Vol. **68**, American Mathematical Society, Providence, R.I. (2000) 333–401.
- [11] R. A. Minlos, Generalized stochastic processes and their extension to a measure, *Trudy Mosk. Mat. Obs.* **8** (1959) 497–518.
- [12] Edward Nelson, Construction of quantum fields from Markoff fields, *J. Funct. Anal.* **12** (1973) 97–112.
- [13] Edward Nelson, The free Markoff field, *J. Funct. Anal.* **12** (1973), 211–227.
- [14] Edward Nelson, Analytic Vectors, *Annals of Math.* **70** (1959) 572–615.
- [15] Konrad Osterwalder and Robert Schrader, Axioms for Euclidean Green's functions, *Comm. Math. Phys.* **31** (1973) 83–112.
- [16] Konrad Osterwalder and Robert Schrader, Axioms for Euclidean Green's functions II. *Comm. Math. Phys.* **42** (1975) 281–305.
- [17] Konrad Osterwalder and Robert Schrader, Euclidean Fermi fields and a Feynman-Kac formula for boson-fermion interactions, *Helv. Phys. Acta* **46** (1973), 227–302.
- [18] Kurt Symanzik, A modified model of Euclidean quantum field theory, Courant Institute of Mathematical Sciences Report IMM-NYU 327, June 1964, and Euclidean quantum field theory, in *Local Quantum Theory*, Varenna Lectures 1968, pp. 152–226, R. Jost Editor, Academic Press, New York (1969).
- [19] Jakob Yngvason, Euclidean invariant integral representations for Schwinger functionals, *J. Math. Phys.* **27**(1) (1986) 311–320.
- [20] Yury Zinoviev, Equivalence of Euclidean and Wightman field theories, *Comm. Math. Phys.* **174** (1995) 1–27.

HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: arthur_jaffe@harvard.edu

SCHOOL OF MATHEMATICS, CARDIFF UNIVERSITY, WALES
E-mail address: christian.jaekel@mac.com

HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: remartin@fas.harvard.edu